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Entropy Measure of Temporal Intuitionistic Fuzzy Sets

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Abstract. In this paper, entropy measure of temporal intuitionistic fuzzy sets (TIFSs) is introduced along with its axiomatic definition. The relationship among entropy, similarity and distance measures of TIFSs are studied.

Keywords: Temporal intuitionistic fuzzy sets, Entropy, Distance measures, Similarity measures.

AMS Mathematics Subject Classification (2010): 28E10

1. Introduction

The distance measure, similarity measure and entropy of fuzzy sets and IFSs gained importance since their introduction by Zadeh [12] and Atanassov [1] respectively. In [2], the membership and the non-membership functions depend not only on the elements but also on the time moments. Similarity of two IFSs is indicated by similarity measure. Szmidt and Kacprzyk in [9] and Xu in [10] proposed similarity measures of IFSs and Hung presented a similarity measure of IFSs based on Hausdorff metric in [4]. The entropy of IFSs measures the degree of fuzziness of the IFSs due to imperfect information. Fuzzy entropy was first introduced by Zadeh in 1969. In 1972, De Luca and Termini [5] proposed the axiomatic definition of fuzzy entropy and several researches have been done on these measures for FSs and IFSs by different authors in different views. Similarity measures and distance measures of TIFSs were discussed by the author in [6]. Burillo and Bustince in [3], Szmidt and Kacprzyk in [8] and Zhang, Xing and Wu in [7] putforth few IFS entropy formulae.

In this paper, entropy for TIFSs is defined and few of its properties are studied. The paper is organized as follows: In Section 2, some basic notions of IFSs and TIFSs are reviewed. In Section 3, axiomatic definition and formulae of entropy of TIFSs are established and illustrated with suitable example. In Section 4, a detailed study is made on the relationship of entropy with distance and similarity measures of TIFSs with numerical illustrations. Section 5 concludes the paper.

2. Preliminaries and notations

Definition 2.1. *[2] Let E be the universal set and T be a non-empty set of time moments. Then, a temporal intuitionistic fuzzy set (TIFS) is an object having the form*

$$A(T)=\{<x,\mu_A(x,t),\nu_A(x,t)>/(x,t)\in E\times T\}$$

where

1. $A\subset E$ is a fixed set.
2. $\mu_A(x,t)$ and $\nu_A(x,t)$ denote the degrees of membership and non-membership respectively of the element (x,t) such that $0\le\mu_A(x,t)+\nu_A(x,t)\le 1$ for every $(x,t)\in E\times T$.

Definition 2.2. *[7] A real function $E:\phi(X)\to[0,1]$ where $\phi(X)$ is the set of all IFSs defined on X, is named as an entropy of IFSs on finite universe X if E satisfies the following properties:*

1. $E(A)=0$ if A is a crisp set.
2. $E(A)=1$ iff $\mu_A(x_i)=\nu_A(x_i)$, for every $x_i\in X$.
3. $E(A)\le E(B)$ if A is less fuzzy than B, denoted by $A\prec\prec B$, is defined as
   $\mu_A(x_i)\le\mu_B(x_i),\nu_A(x_i)\ge\nu_B(x_i)$ for $\mu_B(x_i)\le\nu_B(x_i)$ and
   $\mu_A(x_i)\ge\mu_B(x_i),\nu_A(x_i)\le\nu_B(x_i)$ for $\mu_B(x_i)\ge\nu_B(x_i)$.
4. $E(A)=E(\overline{A})$.

The following are two entropy formulae of IFSs which satisfy the above conditions $(e1)-(e4)$.

$$E_1(A)=1-\sum_{i=1}^{n}\frac{|\mu_A(x_i)-\nu_A(x_i)|}{n}$$

$$E_2(A)=1-\sqrt{\sum_{i=1}^{n}\frac{|\mu_A(x_i)-\nu_A(x_i)|^2}{n}}$$

Definition 2.3. *[6] Let $\phi(X,T)$ denotes the set of all TIFSs defined on X and T and let $A(T),B(T),C(T)\in\phi(X,T)$. Then $S:\phi(X,T)\times\phi(X,T)\to[0,1]$ is called a similarity measure of TIFSs if it satisfies the following conditions :*

1. $0\le S(A(T),B(T))\le 1$. In particular, $S(A(T),\overline{A(T)})=0$ if $A(T)$ is a crisp set.
2. $S(A(T),B(T))=1$ iff $A(T)=B(T)$.
3. $S(A(T),B(T))=S(B(T),A(T))$.
4. If $A(T)\subset B(T)\subset C(T)$ where $A(T),B(T),C(T)\in\phi(X,T)$, then $S(A(T),C(T))\le S(A(T),B(T))$ and $S(A(T),C(T))\le S(B(T),C(T))$.

Definition 2.4. *[6] Let $\phi(X,T)$ denotes the set of all TIFSs defined on the finite universe X and the time domain T. Let $d:\phi(X,T)\times\phi(X,T)\to[0,1]$. Then, the distance measure between $A(T)$ and $B(T)$, $d(A(T),B(T))$ satisfies the following conditions: [(D1)]*

1. $0 \le d(A(T), B(T)) \le 1$.
2. $d(A(T), B(T)) = 0$ if and only if $A(T) = B(T)$.
3. $d(A(T), B(T)) = d(B(T), A(T))$.
4. If $A(T) \subset B(T) \subset C(T)$ where $A(T), B(T), C(T) \in \phi(X, T)$, then $d(A(T), C(T)) \ge d(A(T), B(T))$ and $d(A(T), C(T)) \ge d(B(T), C(T))$.

Note : $S(A(T), B(T)) = 1 - d((A(T), B(T))$ where d and S are respectively the distance and similarity measures of TIFSs.

Definition 2.5. *[6] Let TIFSs $A(T)$ and $B(T)$ be defined on the finite universe $X = \{x_1, x_2, \ldots, x_n\}$ and on the time domain $T = \{t_1, t_2, \cdots t_m\}$. Let $(x_i, t_j) \in X \times T$. Then, the Hamming distance* is defined as

$$d_H(A(T), B(T)) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}[|\mu_A(x_i, t_j) - \mu_B(x_i, t_j)| + |\nu_A(x_i, t_j) - \nu_B(x_i, t_j)| + |\pi_A(x_i, t_j) - \pi_B(x_i, t_j)|]$$

the normalized Hamming distance is defined as

$$d_{NH}(A(T), B(T)) = \frac{1}{2mn}\sum_{i=1}^{n}\sum_{j=1}^{m}[|\mu_A(x_i, t_j) - \mu_B(x_i, t_j)| + |\nu_A(x_i, t_j) - \nu_B(x_i, t_j)| + |\pi_A(x_i, t_j) - \pi_B(x_i, t_j)|]$$

the Euclidean distance is defined as

$$d_E(A(T), B(T)) = [\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}[|\mu_A(x_i, t_j) - \mu_B(x_i, t_j)|^2 + |\nu_A(x_i, t_j) - \nu_B(x_i, t_j)|^2 + |\pi_A(x_i, t_j) - \pi_B(x_i, t_j)|^2]]^{1/2}$$

and *the normalized Euclidean distance* is defined as

$$d_{NE}(A(T), B(T)) = [\frac{1}{2mn}\sum_{i=1}^{n}\sum_{j=1}^{m}[|\mu_A(x_i, t_j) - \mu_B(x_i, t_j)|^2 + |\nu_A(x_i, t_j) - \nu_B(x_i, t_j)|^2 + |\pi_A(x_i, t_j) - \pi_B(x_i, t_j)|^2]]^{1/2}$$

Notations: Throughout this paper, the following notations are implied unless otherwise specified:

Let $X = \{x_1, x_2, \ldots, x_n\}$ be the finite universe, $T = \{t_1, t_2, \ldots, t_m\}$ be the finite time domain and $(x, t) \in X \times T$. Also, let $\phi(X, T)$ be the set of all TIFSs defined on X and T and $A(T)$ and $B(T)$ are any two TIFSs defined on X and T.

3. Entropy of TIFSs

Entropy, a measure of fuzziness which features the lack of sharpness in boundaries is introduced by Zadeh in 1965 [12] and it is defined for IFSs by Szmidt [8]. Later, entropy of

IFSs is discussed by various authors and several entropy measures were given which satisfy the De Luca and Termini axioms formulated for FSs [5]. In this section, entropy axioms and entropy formulae of TIFSs are presented.

Definition 3.1. *Let* $\phi(X,T)$ *denotes the set of all TIFSs on the finite universe* $X=\{x_1,x_2,\ldots,x_n\}$ *and* $T=\{t_1,t_2,\ldots,t_m\}$. *A real function* $E:\phi(X,T)\to[0,1]$ *is called an entropy of TIFSs, if* E *satisfies the following properties : [(e1)]*

1. $E(A(T))=0$ if $A(T)$ is a crisp set.
2. $E(A(T))=1$ iff $\mu_A(x,t)=\nu_A(x,t),$ for every $(x,t)\in X\times T.$
3. $E(A(T))\le E(B(T))$ if $A(T)$ is less fuzzy than $B(T)$ which is

defined as

$\mu_A(x,t)\le\mu_B(x,t),\nu_A(x,t)\ge\nu_B(x,t)$ for $\mu_B(x,t)\le\nu_B(x,t)$ or
$\mu_A(x,t)\ge\mu_B(x,t),\nu_A(x,t)\le\nu_B(x,t)$ for $\mu_B(x,t)\ge\nu_B(x,t).$

4. $E(A(T))=E(\overline{A(T)}).$

Definition 3.2. *The entropy formulae of a TIFS* $A(T)$, *which satisfy the entropy properties of TIFSs* $(e1)-(e4)$ *are given as follows :*

$$(i)E_1(A(T))=1-\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}|\mu_A(x,t)-\nu_A(x,t)|$$

$$(ii)E_2(A(T))=1-\sqrt{\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}|\mu_A(x,t)-\nu_A(x,t)|^2}$$

Remark 3.1. *The entropy for IFSs and hence for FSs can be derived from* (i) *and* (ii).

Remark 3.2. $E_\lambda(A(T))=1-\sqrt{\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}|\mu_A(x,t)-\nu_A(x,t)|^\lambda}\quad \lambda\in[1,\infty).$

Definition 3.3. *Let the universe of discourse be defined on* $[a,b]$ *and time domain be defined on* $[c,d]$, *then the entropy formulae* $E_1(A(T))$ *and* $E_2(A(T))$ *which satisfy the entropy properties* $(e1)-(e4)$ *are defined as follows:*

$$(i)E_1(A(T))=1-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d|\mu_A(x,t)-\nu_A(x,t)|\,dxdt$$

$$(ii)E_2(A(T))=1-\sqrt{\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d|\mu_A(x,t)-\nu_A(x,t)|^2\,dxdt}$$

Example 3.1. *Suppose* $A(T),B(T)$ *are any two TIFSs defined on* $X=\{1,2\}$ *and* $T=\{t_1,t_2\}$ *as follows:*

$A(T)=\{(1,0)/1+(0,1)/2 \text{ at } t=t_1,(0.2,0.5)/1+(0.4,0.5)/2 \text{ at } t=t_2\}$ and

$B(T) = \{(0.5,0.3)/1 + (0.2,0.6)/2 \text{ at } t = t_1, (0.4,0.3)/1 + (0.1,0.9)/2 \text{ at } t = t_2\}$.

The entropies for the TIFSs $A(T)$ and $B(T)$ are calculated using Definition 3.2 as $E(A(T)) = 0.4$ and $E(B(T)) = 0.63$

It is worth noticing that the entropy for $B(T)$ is comparatively higher than $A(T)$ which coincides with the observation results of the considered TIFSs $A(T)$ and $B(T)$.

4. Relationship among entropy, distance and similarity measures of TIFSs

Relationship among entropy, similarity and distance measures for IFSs is presented in [7]. In this section, a study with numerical examples is made on the relationship within entropy, distance and similarity measures of TIFSs which, in turn, provide interesting formulae for the entropy and similarity measures of TIFSs.

Theorem 4.1. *Assume* d *is the distance measure of TIFSs and* $A(T) \in \phi(X,T)$. *If* f *is a monotonic decreasing function from* $[0,1]$ *to* $[0,1]$, *then*

$$E(A(T)) = \frac{f(d(A(T)), \overline{A(T)}) - f(1)}{f(0) - f(1)}$$ *is an entropy of TIFS* $A(T)$.

Proof: It is enough to prove that $E(A(T))$ satisfies the axiomatic properties $(e1)-(e4)$ of entropy measure.

1. If $A(T)$ is a crisp set, then by the distance property of TIFSs,

$d(A(T), \overline{A(T)}) = 1 \Rightarrow E(A(T)) = 0.$

2. If $\mu_A(x,t) = \nu_A(x,t)$ for every $(x,t) \in X \times T$

$\Rightarrow \mu_{\bar{A}}(x,t) = \nu_A(x,t) = \mu_A(x,t)$ and $\nu_{\bar{A}}(x,t) = \mu_A(x,t) = \nu_A(x,t)$

$\Rightarrow \overline{A(T)} = A(T)$

$\Rightarrow d(A(T), \overline{A(T)}) = 0$ (by distance property)

$\Rightarrow E(A(T)) = 1.$

3. Let $A(T) \prec\prec B(T)$. That is, $A(T)$ is less fuzzy than $B(T)$.

There arises two cases:

Case (i):

$\mu_A(x,t) \le \mu_B(x,t), \nu_A(x,t) \ge \nu_B(x,t)$ for $\mu_B(x,t) \le \nu_B(x,t)$

$\Rightarrow \mu_A(x,t) \le \mu_B(x,t) \le \nu_B(x,t) \le \nu_A(x,t)$

$\Rightarrow A(T) \subseteq B(T) \subseteq \overline{B(T)} \subseteq \overline{A(T)}$

$\Rightarrow d(A(T), \overline{A(T)}) \ge d(B(T), \overline{A(T)}) \ge d(B(T), \overline{B(T)})$

$$\Rightarrow E(A(T)) = \frac{f(d(A(T), \overline{A(T)})) - f(1)}{f(0) - f(1)} \le \frac{f(d(B(T), \overline{B(T)})) - f(1)}{f(0) - f(1)} = E(B(T))$$

which proves entropy property $(e3)$.

Case (ii):

$\mu_A(x,t) \geq \mu_B(x,t), \nu_A(x,t) \leq \nu_B(x,t)$ for $\mu_B(x,t) \geq \nu_B(x,t)$

$\Rightarrow A(T) \supseteq B(T) \supseteq \overline{B(T)} \supseteq \overline{A(T)}$

$\Rightarrow d(A(T), \overline{A(T)}) \geq d(B(T), \overline{B(T)})$

$\Rightarrow E(A(T)) \leq E(B(T))$

4. By using distance measure of TIFSs,

$d(A(T), \overline{A(T)}) = d(\overline{A(T)}, A(T))$

Then, $E(A(T)) = E(\overline{A(T)})$.

Remark 4.1. *The entropy formula presented here corresponds to the entropy formulae* $E_1(A(T))$ *and* $E_2(A(T))$ *given in Definition 3.2 by taking* $f(x) = 1 - x$.

Example 4.1. *Suppose TIFS* $A(T)$ *is considered as in Example* 3.1 *and* $f(x) = 1 - x$.

Then by Theorem 4.1 and Definition $3.2(i)$, $d(A(T), \overline{A(T)}) = 0.6$ and $E(A(T)) = 0.4$.

Theorem 4.2. *Let* S *be a similarity measure for FSs and* $A(T) \in \phi(X,T)$. *Then,* $S(\overline{\mu_A}, \overline{\nu_A})$ *is an entropy of TIFS* $A(T)$, *where* $\overline{\mu_A} = 1 - \mu_A$ *and* $\overline{\nu_A} = 1 - \nu_A$.

Proof : All the properties of entropy of TIFSs $(e1)-(e4)$ are to be proved.

1. Let $A(T)$ be a crisp set.

Then, $\mu_A(x,t) = 1$ and $\nu_A(x,t) = 0$ (or) $\mu_A(x,t) = 0$ and $\nu_A(x,t) = 1$

$\Rightarrow \nu_A(x,t) = \mu_{\overline{A}}(x,t)$ and $\nu_{\overline{A}}(x,t) = \mu_A(x,t)$ for every $(x,t) \in X \times T$

$\Rightarrow S(\overline{\mu_A}, \overline{\nu_A}) = S(\overline{\mu_A}, \mu_A) = 0 \Rightarrow E(A(T)) = 0$

2. $S(\overline{\mu_A}, \overline{\nu_A}) = 1 \Leftrightarrow \overline{\mu_A} = \overline{\nu_A}$

$\Leftrightarrow 1 - \mu_A = 1 - \nu_A \Leftrightarrow \mu_A(x,t) = \nu_A(x,t)$ for every $(x,t) \in X \times T$

That is, $E(A(T)) = 1$ iff $\mu_A(x,t) = \nu_A(x,t)$ for every $(x,t) \in X \times T$

3. Let $A(T) \prec\prec B(T)$.

The result is derived from two cases:

Case (i):

$\mu_A(x,t) \leq \mu_B(x,t), \nu_A(x,t) \geq \nu_B(x,t)$ for $\mu_B(x,t) \leq \nu_B(x,t)$

$\Rightarrow \mu_A(x,t) \leq \mu_B(x,t) \leq \nu_B(x,t) \leq \nu_A(x,t)$

$\Rightarrow \mu_A \subseteq \mu_B \subseteq \nu_B \subseteq \nu_A$

$\Rightarrow S(\mu_A, \nu_A) \leq S(\mu_A, \nu_B) \leq S(\mu_B, \nu_B)$ (by similarity properties)

$\Rightarrow E(A(T)) \leq E(B(T))$ when $A(T) \prec\prec B(T)$.

Case (ii):

$\mu_A(x,t) \geq \mu_B(x,t), \nu_A(x,t) \leq \nu_B(x,t)$ for $\mu_B(x,t) \geq \nu_B(x,t)$,

$\Rightarrow E(A(T)) \leq E(B(T))$.

4. Let $A = <\mu_A, \nu_A>$, then $A' = <\nu_A, \mu_A>$

$\Rightarrow E(A'(T)) = S(\overline{\mu_{A'}}, \overline{\nu_{A'}}) = S(\overline{\nu_A}, \overline{\mu_A}) = S(\overline{\mu_A}, \overline{\nu_A}) = E(A(T))$

Example 4.2. *Suppose TIFS* $A(T)$ *is considered as in Example* 3.1. *Also, let* $\overline{\mu_A}$, $\overline{\nu_A}$ *and similarity measure for FSs* S *are defined as in Theorem 4.2. Then,*
$S(\overline{\mu_A}, \overline{\nu_A}) = 0.4 = E(A(T))$.

Example 4.3. *The entropy formulae for TIFSs due to the similarity measures of FSs [10] are as follows:*

$$E_1^*(A(T)) = 1 - \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} |\mu_A(x_i,t_j) - \nu_A(x_i,t_j)|$$

$$E_2^*(A(T)) = 1 - \sqrt{\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} |\mu_A(x_i,t_j) - \nu_A(x_i,t_j)|^2}$$

$$E_3^*(A(T)) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\{1 - \mu_A(x_i,t_j) \vee \nu_A(x_i,t_j)\}}{\sum_{i=1}^{n}\sum_{j=1}^{m}\{1 - \mu_A(x_i,t_j) \wedge \nu_A(x_i,t_j)\}}$$

$$E_4^*(A(T)) = \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\{1 - \mu_A(x_i,t_j) \vee \nu_A(x_i,t_j)\}}{\{1 - \mu_A(x_i,t_j) \wedge \nu_A(x_i,t_j)\}}$$

where $E_1^*(A(T))$ and $E_2^*(A(T))$ are same as $E_1(A(T))$ and $E_2(A(T))$ in Definition 3.2.

Theorem 4.3. *Assume* $A(T)$ *and* $B(T)$ *are any two TIFSs defined on* $X = \{x_1, x_2, \ldots, x_n\}$ *and* $T = \{t_1, t_2, \ldots, t_m\}$. *Let* $A(T) \subseteq B(T)$ *or* $B(T) \subseteq A(T)$ *for any* $(x,t) \in X \times T$. *Then,*
$E_i^*(A(T)) + E_i^*(B(T)) = E_i^*(A(T) \cup B(T)) + E_i^*(A(T) \cap B(T))$ *for* $i = 1,2$.

Proof :

Case (i): The result is proved for $i = 1$ as follows:

$$E_1^*(A(T) \cup B(T)) = 1 - \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} |\mu_{A\cup B}(x_i,t_j) - \nu_{A\cup B}(x_i,t_j)|$$

Let $A(T) \subseteq B(T) \Rightarrow \mu_A(x,t) \le \mu_B(x,t)$ and $\nu_A(x,t) \ge \nu_B(x,t)$

$$\Rightarrow E_1^*(A(T) \cup B(T)) = 1 - \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} |\mu_B(x_i,t_j) - \nu_B(x_i,t_j)|.$$

(since $\mu_{A\cup B} = \max(\mu_A, \mu_B)$ and $\nu_{A\cup B} = \min(\nu_A, \nu_B)$)

Also, $E_1^*(A(T) \cap B(T)) = 1 - \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} |\mu_A(x,t) - \nu_A(x,t)|$

Thus, $E_1^*(A(T)\cup B(T))+E_1^*(A(T)\cap B(T))=E_1^*(B(T))+E_1^*(A(T))$

Case (ii) :For $i=2$,

$$E_2^*(A(T)\cup B(T))=1-\sqrt{\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}|\mu_{A\cup B}(x_i,t_j)-\mu_{A\cup B}(x_i,t_j)|^2}$$

Let $A(T)\subseteq B(T)\Rightarrow \mu_A(x,t)\le\mu_B(x,t)$ and $\nu_A(x,t)\ge\nu_B(x,t)$

$$\Rightarrow E_2^*(A(T)\cup B(T))=1-\sqrt{\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}|\mu_B(x_i,t_j)-\nu_B(x_i,t_j)|^2}$$

(since $\mu_{A\cup B}=\max(\mu_A,\mu_B)$ and $\nu_{A\cup B}=\min(\nu_A,\nu_B)$)

Also, $E_2^*(A(T)\cap B(T))=1-\sqrt{\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}|\mu_A(x_i,t_j)-\nu_A(x_i,t_j)|^2}$

Thus, $E_2^*(A(T)\cup B(T))+E_2^*(A(T)\cap B(T))=E_2^*(B(T))+E_2^*(A(T))$

Remark 4.2. *The result can be verified for* E_3^* *ad* E_4^*.

Example 4.4. *Let* $A(T),B(T)$ *are any two TIFSs with* $A(T)\subseteq B(T)$ *defined on* $X=\{1,2\}$ *and* $T=\{t_1,t_2\}$ as follows:

$A(T)=\{(0.2,0.4)/1+(0.3,0.6)/2 \text{ at } t=t_1,(0.8,0.2)/1+(0.7,0.2)/2 \text{ at } t=t_2\}$ and

$B(T)=\{(0.8,0.2)/1+(0.5,0.1)/2 \text{ at } t=t_1,(0.9,0)/1+(0.8,0.1)/2 \text{ at } t=t_2\}$.

The entropies are calculated using Example 4.3 and the above result is verified for $i=1,2,3,4$.

$$E_1^*(A(T))+E_1^*(B(T))=0.6+0.35=E_1^*(A(T)\cup B(T))+E_1^*(A(T)\cap B(T))$$
$$E_2^*(A(T))+E_2^*(B(T))=0.57+0.33=E_2^*(A(T)\cup B(T))+E_2^*(A(T)\cap B(T))$$
$$E_3^*(A(T))+E_3^*(B(T))=0.48+0.27=E_3^*(A(T)\cup B(T))+E_3^*(A(T)\cap B(T))$$

and $E_4^*(A(T))+E_4^*(B(T))=0.49+0.28=E_4^*(A(T)\cup B(T))+E_4^*(A(T)\cap B(T))$.

Definition 4.1. *Let* $A(T),B(T)\in\phi(X,T)$ *be any two TIFSs. Then, a Hausdorff based averaging temporal intuitionistic fuzzy set* $\chi(A(T),B(T))\in\phi(X,T)$ *is defined from* $A(T)$ *and* $B(T)$ *as follows:*

$$\chi(A(T),B(T))=\{<x,\mu_{\chi(A,B)}(x,t),\nu_{\chi(A,B)}(x,t)>/(x,t)\in X\times T\}\textit{where}$$
$$\mu_{\chi(A,B)}(x,t)=\frac{1}{2}[1+\min\{|\mu_A(x,t)-\mu_B(x,t)|,|\nu_A(x,t)-\nu_B(x,t)|\}]$$
$$\nu_{\chi(A,B)}(x,t)=\frac{1}{2}[1-\max\{|\mu_A(x,t)-\mu_B(x,t)|,|\nu_A(x,t)-\nu_B(x,t)|\}]$$

with $\mu_{\chi(A,B)}(x,t)$ and $\nu_{\chi(A,B)}(x,t)$ denote the degrees of membership and non-membership respectively of the element (x,t) such that

$0 \le \mu_{\chi(A,B)}(x,t) + \nu_{\chi(A,B)}(x,t) \le 1$ for every $(x,t) \in X \times T$.

Theorem 4.4. *Let* E *be an entropy of TIFS and* $A(T), B(T) \in \phi(X,T)$. *If* $\chi(A(T), B(T))$ *is defined as above, then* $E(\chi(A(T)), B((T)))$ *is a similarity measure of the TIFSs* $A(T)$ *and* $B(T)$.

Proof : [(s1)]

1. Let $A(T)$ be a crisp set.

$\Rightarrow \mu_A(x_i,t_j) = 1$ and $\nu_A(x_i,t_j) = 0$ (or) $\mu_A(x_i,t_j) = 0$ and $\nu_A(x_i,t_j) = 1$.

$\Rightarrow \mu_{\bar{A}}(x_i,t_j) = 0$ and $\nu_{\bar{A}}(x_i,t_j) = 1$ (or) $\mu_{\bar{A}}(x_i,t_j) = 1$ and $\nu_{\bar{A}}(x_i,t_j) = 0$

$\Rightarrow \mu_{\chi(A,\bar{A})}(x_i,t_j) = \dfrac{1+\min\{1,1\}}{2} = 1$ and $\nu_{\chi(A,\bar{A})}(x_i,t_j) = \dfrac{1-\max\{1,1\}}{2} = 0$

for every $(x_i,t_j) \in X \times T$.

$\Rightarrow \chi(A(T), \overline{A(T)}) = \{< x_i, 1, 0 > / x_i \in X\}$ is a crisp set in X .

$\Rightarrow E[\chi(A(T), \overline{A(T)})] = 0$

2. Let $E(\chi(A(T), B(T))) = 1$

$\Leftrightarrow \mu_{\chi(A,B)} = \nu_{\chi(A,B)}$ (by the entropy property of TIFSs)

$\Leftrightarrow 1 + \min\{|\mu_A(x_i,t_j) - \mu_B(x_i,t_j)|, |\nu_A(x_i,t_j) - \nu_B(x_i,t_j)|\}$

$= 1 - \max\{|\mu_A(x_i,t_j) - \mu_B(x_i,t_j)|, |\nu_A(x_i,t_j) - \nu_B(x_i,t_j)|\}$

$\Leftrightarrow \min\{|\mu_A(x_i,t_j) - \mu_B(x_i,t_j)|, |\nu_A(x_i,t_j) - \nu_B(x_i,t_j)|\} +$

$\max\{|\mu_A(x_i,t_j) - \mu_B(x_i,t_j)|, |\nu_A(x_i,t_j) - \nu_B(x_i,t_j)|\} = 0$

$\Leftrightarrow |\mu_A(x_i,t_j) - \mu_B(x_i,t_j)| + |\nu_A(x_i,t_j) - \nu_B(x_i,t_j)| = 0$

$\Leftrightarrow |\mu_A(x_i,t_j) - \mu_B(x_i,t_j)| = 0$ and $|\nu_A(x_i,t_j) - \nu_B(x_i,t_j)| = 0$

$\Leftrightarrow \mu_A(x_i,t_j) = \mu_B(x_i,t_j)$ and $\nu_A(x_i,t_j) = \nu_B(x_i,t_j)$ for every $(x_i,t_j) \in X \times T$

$\Leftrightarrow A(T)) = B(T)$

3. Assume $\mu_{\chi(A,B)}(x_i,t_j) = \mu_{\chi(B,A)}(x_i,t_j)$ and $\nu_{\chi(A,B)}(x_i,t_j) = \nu_{\chi(B,A)}(x_i,t_j)$ for every $(x_i,t_j) \in X \times T$

$\Rightarrow \chi(A(T), B(T)) = \chi(B(T), A(T))$

$\Rightarrow E(\chi(A(T), B(T))) = E(\chi(B(T), A(T)))$

4. If $A(T) \subseteq B(T) \subseteq C(T)$, then for $(x_i,t_j) \in X \times T$,

$\mu_A(x_i,t_j) \le \mu_B(x_i,t_j) \le \mu_C(x_i,t_j)$ and $\nu_A(x_i,t_j) \ge \nu_B(x_i,t_j) \ge \nu_C(x_i,t_j)$

$\Rightarrow |\mu_A(x_i,t_j) - \mu_C(x_i,t_j)| \ge |\mu_A(x_i,t_j) - \mu_B(x_i,t_j)|$ for every $(x_i,t_j) \in X \times T$ and $|\nu_A(x_i,t_j) - \nu_C(x_i,t_j)| \ge |\nu_A(x_i,t_j) - \nu_B(x_i,t_j)|$

Thus, $\mu_{\chi(A,C)}(x_i,t_j)\geq\mu_{\chi(A,B)}(x_i,t_j)$ and
$\nu_{\chi(A,C)}(x_i,t_j)\leq\nu_{\chi(A,B)}(x_i,t_j)$ for every $(x_i,t_j)\in X\times T$.
Also, $\mu_{\chi(A,B)}(x_i,t_j)\geq\nu_{\chi(A,B)}(x_i,t_j)$ for every $(x_i,t_j)\in X\times T$.
$\Rightarrow\chi(A(T),C(T))\prec\prec\chi(A(T),B(T))$
Thus by entropy property, $E(\chi(A(T),C(T)))\leq E(\chi(A(T),B(T)))$
Similarly, $E(\chi(A(T),C(T)))\leq E(\chi(B(T),C(T)))$ can be proved.
Hence, $E(\chi(A(T),C(T))\leq E(\chi(A(T),(B(T))\wedge E(\chi(B(T),(C(T))$

Remark 4.3. $E(\overline{\chi(A(T),B(T))})$ *between TIFSs* $A(T)$ *and* $B(T)$ *is also a similarity measure.*

Remark 4.4. *For any* $\alpha\in[1,\infty)$, *TIFS* $\varphi(A(T),B(T))$ *is given as a generalization of Definition 4.1 as follows:*

$$\phi(A(T),B(T))=\{<x,\mu_{\varphi(A,B)}(x,t),\nu_{\varphi(A,B)}(x,t)>/(x,t)\in E\times T\}$$

where

$$\mu_{\varphi(A,B)}(x_i,t_j)=\frac{1}{2}[1+\min\{|\mu_A(x_i,t_j)-\mu_B(x_i,t_j)|^\alpha,|\nu_A(x_i,t_j)-\nu_B(x_i,t_j)|^\alpha\}]$$

$$\nu_{\varphi(A,B)}(x_i,t_j)=\frac{1}{2}[1-\max\{|\mu_A(x_i,t_j)-\mu_B(x_i,t_j)|^\alpha,|\nu_A(x_i,t_j)-\nu_B(x_i,t_j)|^\alpha\}]$$

Then, $E(\varphi(A(T),B(T)))$ and $E(\overline{\varphi(A(T),B(T))})$ are both similarity measures of TIFSs A(T) and B(T).

Example 4.5. *Suppose TIFSs* $A(T)$ *and* $B(T)$ *are considered as in Example* 3.1 *and* $\chi(A(T),B(T))$ *is defined as in Definition* 4.1.

Then,

$\chi(A(T),B(T))=\{(0.65,0.25)/1+(0.6,0.3)/2 \text{ at } t=t_1,(0.6,0.4)/1+(0.65,0.3)/2 \text{ at } t=t_2\}$
and $E(\chi(A(T),B(T))=0.69=S(A(T),B(T))$.

Example 4.6. *Let* $A(T)\in\phi(X,T)$ *and*

$E_1(A(T))=1-\frac{1}{mn}\sum_{i=1}^n\sum_{j=1}^m|\mu_A(x_i,t_j)-\nu_A(x_i,t_j)|$ *is an entropy of* $A(T)$.

Therefore, from Theorem 4.4,

$$S(A(T),B(T))=E(\chi(A(T),B(T))$$

$$\Leftrightarrow 1-\frac{1}{mn}\sum_{i=1}^n\sum_{j=1}^m|\mu_{(A,B)}(x_i,t_j)-\nu_{(A,B)}(x_i,t_j)|$$

$$\Leftrightarrow 1-\frac{1}{2mn}\sum_{i=1}^n\sum_{j=1}^m[[1+min|\mu_A(x_i,t_j)-\mu_B(x_i,t_j)|,|\nu_A(x_i,t_j)-\nu_B(x_i,t_j)|]-$$
$$[1-max|\mu_A(x_i,t_j)-\mu_B(x_i,t_j)|,|\nu_A(x_i,t_j)-\nu_B(x_i,t_j)|]]$$

$$\Leftrightarrow 1-\frac{1}{2mn}\sum_{i=1}^{n}\sum_{j=1}^{m}[|\mu_{A(x_i,t_j)}-\mu_{B(x_i,t_j)}|+|\nu_{A(x_i,t_j)}-\nu_{B(x_i,t_j)}|]$$
$$\Leftrightarrow 1-d(A(T),B(T))$$

Definition 4.2. *Let* $A(T)\in\phi(X,T)$. *Then, the averaging distance based temporal intuitionistic fuzzy sets* $f(A(T))$ *and* $g(A(T))$ *are defined from TIFS* $A(T)$ *as follows:*

$$f(A(T))=\{<x,\mu_{f(A)}(x,t),\nu_{f(A)}(x,t)>/(x,t)\in X\times T\}$$

where

$$\mu_{f(A)}(x,t)=\frac{1+|\mu_A(x,t)-\nu_A(x,t)|^2}{2} \text{ and}$$
$$\nu_{f(A)}(x,t)=\frac{1-|\mu_A(x,t)-\nu_A(x,t)|}{2}.$$
$$g(A(T))=\{<x,\mu_{g(A)}(x,t),\nu_{g(A)}(x,t)>/(x,t)\in X\times T\}$$

where

$$\mu_{g(A)}(x,t)=\frac{1-|\mu_A(x,t)-\nu_A(x,t)|}{2} \text{ and}$$
$$\nu_{g(A)}(x,t)=\frac{1+|\mu_A(x,t)-\nu_A(x,t)|^2}{2}.$$

with $\mu(x,t)$ and $\nu(x,t)$ denote the degrees of membership and non-membership respectively of the element (x,t) such that $0\le\mu(x,t)+\nu(x,t)\le 1$ for every $(x,t)\in X\times T$.

Theorem 4.5. *Let* $A(T)\in\phi(X,T)$ *and* S *be a similarity measure of TIFSs. Define the TIFSs* $f(A(T))$ *and* $g(A(T))$ *as in Definition 4.2. Then,* $S(f(A(T)),g(A(T)))$ *is an entropy of* $A(T)$.

Proof: The conditions of entropy property of TIFSs $(e1)-(e4)$ are to be satisfied. [(e1)]

1. Let $A(T)$ be a crisp set. Then, for every $(x_i,t_j)\in X\times T$,

$\Rightarrow \mu_A(x_i,t_j)=1,\nu_A(x_i,t_j)=0$ (or) $\mu_A(x_i,t_j)=0,\nu_A(x_i,t_j)=1$.

$\Rightarrow \mu_{f(A)}(x_i,t_j)=1,\nu_{f(A)}(x_i,t_j)=0$ (or) $\mu_{g(A)}(x_i,t_j)=0,\nu_{g(A)}(x_i,t_j)=1$.

$\Rightarrow f(A(T))=X$ and $g(A(T))=\phi$

$\Rightarrow S(f(A(T)),g(A(T)))=S(X,\phi)=0$.

2. $E(A(T))=1$

$\Leftrightarrow S(f(A(T)),g(A(T)))=1$

$\Leftrightarrow f(A(T))=g(A(T))$

$\Leftrightarrow \mu_{f(A)}(x_i,t_j)=\mu_{g(A)}(x_i,t_j)$ and $\nu_{f(A)}(x_i,t_j)=\nu_{g(A)}(x_i,t_j)$ for every $(x_i,t_j)\in X\times T$

$\Leftrightarrow [|\mu_A(x_i,t_j)-\nu_A(x_i,t_j)|]\,[|\mu_A(x_i,t_j)-\nu_A(x_i,t_j)|+1]=0$

$\Leftrightarrow \mu_A(x_i,t_j)=\nu_A(x_i,t_j)$ for every $(x_i,t_j)\in X\times T$

3. Let $A(T)\prec\prec B(T)$. There arises two cases:

Case (i):

$\Rightarrow \mu_A(x_i,t_j)\le\mu_B(x_i,t_j), \nu_A(x_i,t_j)\ge\nu_B(x_i,t_j)$ for $\mu_B(x_i,t_j)\le\nu_B(x_i,t_j)$

$\Rightarrow \mu_A(x_i,t_j)\le\mu_B(x_i,t_j)\le\nu_B(x_i,t_j)\le\nu_A(x_i,t_j)$

$\Rightarrow |\mu_A(x_i,t_j)-\nu_A(x_i,t_j)|\ge|\mu_B(x_i,t_j)-\nu_B(x_i,t_j)|$

Hence, $g(A(T))\subseteq g(B(T))\subseteq f(B(T))\subseteq f(A(T))$

Thus, by using similarity measure property,

$E(A(T))=S(f(A(T)),g(A(T)))\le S(f(B(T)),g(A(T)))$

$\le S(f(B(T)),g(B(T)))=E(B(T))$

Case (ii):

$\mu_A(x_i,t_j)\ge\mu_B(x_i,t_j), \nu_A(x_i,t_j)\le\nu_B(x_i,t_j)$ for $\mu_B(x_i,t_j)\ge\nu_B(x_i,t_j)$,

$\Rightarrow \mu_A(x_i,t_j)\ge\mu_B(x_i,t_j)\ge\nu_B(x_i,t_j)\ge\nu_A(x_i,t_j)$

$\Rightarrow |\mu_A(x_i,t_j)-\nu_A(x_i,t_j)|\ge|\mu_B(x_i,t_j)-\nu_B(x_i,t_j)|$

Hence, $g(A(T))\subseteq g(B(T))\subseteq f(B(T))\subseteq f(A(T))$

$\Rightarrow E(A(T))=S(f(A(T)),g(A(T)))\le S(f(B(T)),g(A(T)))$

$\le S(f(B(T)),g(B(T)))=E(B(T))$

4. By the definition of $f(A(T))$ and $g(A(T))$,

$f(A(T))=f(\overline{A(T)})$ and $g(A(T))=g(\overline{A(T)})$

$\Rightarrow E(\overline{A(T)})=S(f(\overline{A(T)}),g(\overline{A(T)}))=S(f(A(T)),g(A(T)))=E(A(T))$

Remark 4.5. $S(\overline{f(A(T))},\overline{g(A(T))})$ *is an entropy of TIFS* $A(T)$.

Example 4.7. *Suppose TIFS* $A(T)$ *is considered as in Example* 3.1. *TIFSs* $f(A(T))$ *and* $g(A(T))$ *are defined using Definition* 4.2 *as follows:*

$$f(A(T))=\{(1,0)/1+(1,0)/2 \text{ at } t=t_1, (0.5,0.35)/1+(0.5,0.45)/2 \text{ at } t=t_2\}$$

and

$$g(A(T))=\{(0,1)/1+(0,1)/2 \text{ at } t=t_1, (0.35,0.5)/1+(0.45,0.5)/2 \text{ at } t=t_2\}$$

Then, $S(f(A(T)),g(A(T)))=0.4=E(A(T))$.

Example 4.8. *Consider any two TIFSs* $A(T),B(T)\in\phi(X,T)$ *and the normalized Euclidean distance measure. Also,* $S(A(T),B(T))=1-d(A(T),B(T))$. *Then, by Theorem 4.5, entropy of A(T) can be drawn as follows:*

$S(A(T),B(T))$

$$= 1 - \frac{1}{2mn}\sum_{i=1}^{n}\sum_{j=1}^{m}[|\mu_A(x_i,t_j) - \mu_B(x_i,t_j)|^2 + |\nu_A(x_i,t_j) - \nu_B(x_i,t_j)|^2]^{\frac{1}{2}}$$

$$S(f(A(T)), g(B(T)))$$

$$= 1 - [\frac{1}{2mn}\sum_{i=1}^{n}\sum_{j=1}^{m} 2(\frac{1}{2})^2 |\mu_A(x_i,t_j) - \nu_A(x_i,t_j)|^2 + |\mu_A(x_i,t_j) - \nu_A(x_i,t_j)|^2]^{\frac{1}{2}}$$

$$= 1 - \frac{1}{4mn}\sum_{i=1}^{n}\sum_{j=1}^{m}[|\mu_A(x_i,t_j) - \nu_A(x_i,t_j)|^4 + |\mu_A(x_i,t_j) - \nu_A(x_i,t_j)|^2$$

$$+ 2|\mu_A(x_i,t_j) - \nu_A(x_i,t_j)|^3]^{\frac{1}{2}}$$

is an entropy of TIFSs $A(T)$.

5. Conclusion

In this paper, entropy measure of TIFSs and relationship among entropy, distance and similarity measures are studied. Thus, different entropy and similarity measure formulae are obtained and are demonstrated by suitable illustrations. It is further proposed to work on the relationship of inclusion measures of TIFSs with distance, similarity and entropy measures.

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