Intern. J. Fuzzy Mathematical Archive
Vol. 15, No. 2, 2018, 235-244
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 30 April 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/ijfma.v15n2a15

International Journal of
Fuzzy Mathematical
Archive

Perishable Inventory Control in Retrial Service Facility-Semi Markov Decision Process

***S.Krishnakumar*[1] and *C.Elango*[2]**

Research Department of Mathematical Sciences
Cardamom Planters' Association College, Bodinayakanur – 625513
Tamil Nadu, India.
Email: [1]krishmathew@gmail.com [2]chellaelango@gmail.com

Received 4 March 2018; accepted 21 April 2018

Abstract. This article deals the problem of optimally controlling the perishable inventory with exponential perishable rate and exponential lead time in a finite capacity retrial service facility system. Arrival of demands to the system is assumed as Poisson and service times are assumed to follows an exponential distribution. Here, the customers are not allowed to form a queue. A customer who sees the server busy joins the orbit and reattempts the system with exponential distributed time. For the given values of maximum inventory and reorder level, we determine the optimal ordering policy at various instants of time. The system is formulated as a Semi-Markov Decision Process and the optimum inventory control to be employed by using linear programming method so that the long–run expected cost rate is minimized. Numerical examples are provided to illustrate the model.

Keywords: Single server, Service facility, Retrial queue, Inventory system, Perishable items, Semi-Markov Decision Process

AMS Mathematics Subject Classification (2010): 90B05

1. Introduction

The analysis of perishable inventory systems has been the theme of many articles due to its potential applications in sectors like food, chemicals, pharmaceuticals, photography and blood bank management. Most of these models deal with either the periodic review systems with fixed life times or continuous review systems with instantaneous supply of reorders.

In last two decades, many researchers in the field of retrial queuing system contributed many results. For example, Elcan [6], Arivudainambi et al. [1], Dragieva [4], and Dudin et al. [5] discussed a single server retrial queue with returning customers examined by balking or Bernoulli vacations and derived the analysis part and solution technique using Matrix method or generating function or Truncation method using level dependent quasi-birth-and-death process (LDQBD).

Paul et al. [12] and Krishnamoorthy et al. [10, 11] analyzed a continuous review inventory system at a service facility and retrial of customers. In all these systems, arrival of customers form a Poisson process and service times are exponentially distributed.

They investigate the systems to compute performance measures and construct suitable cost functions.

The main contribution of this article is to derive the optimum control rule for adjusted inventory replenishment process in retrial service facility system maintaining inventory for service. We consider a service facility system and the orbit with finite waiting space. For the given values of maximum waiting space, maximum inventory, reorder level s, lead times and perishable rate with adjusted inventory replenishment, the system is formulated as a Semi-Markov Decision Process and the optimum inventory policy to be employed is obtained using linear programming method so that the long – run expected cost rate is minimized.

The rest of the paper is organized as follows. Preliminary concepts of retrial queues is given in section 1. A brief account of Markov process with continuous time space is described in section 2. We provide a formulation of our Semi-Markov Decision model in the next section 3. In section 4, we present a procedure to implement long–run expected cost rate criteria to get the optimal vales of the system parameters.

2. Problem formulation

In this paper we assume the following:

- A customer arrives to the system according to a Poisson process with rate $\lambda(>0)$.
- When the server is idle the arriving customer directly enters the server gets service and leaves the system.
- An arriving customer who finds a server busy is obliged to leave the service area and repeats his request from a virtual space (orbit). A reattempt made by a customer after a random time for the service from the virtual space (orbit) is called *retrial*.
- The capacity of orbit is limited to the maximum of N.
- Customer’s retrials for service from an orbit follow an exponential distribution with rate δ (> 0). (if there are j customers stay in the orbit the retrial rate is jδ). The capacity of orbit is assumed to be finite.

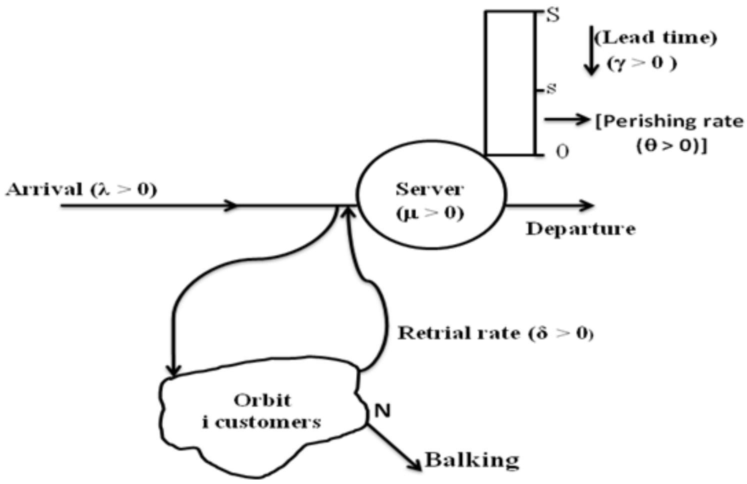

Figure 1: Perishable inventory control in retrial service facility system

- Service times of customers are independent of each other and have a common exponential distribution with parameter μ (> 0).
- One (unit) of item is served to each customer during service. The items in stock are of perishable nature with perishing rate θ.
- The maximum capacity of the inventory is fixed as S. Whenever the inventory level reaches to a prefixed level s ($0 \leq s < S$), decision for ordering is taken for each state lies between 0 and s. The lead time follows an exponential distribution with parameter γ.
- The size of the order is adjusted at the time of replenishment so that immediately after replenishment the inventory level becomes S.
- Order decision is made at each level less than or equal to the reorder level s. Whenever the inventory level reaches to zero, the arriving customers enter the orbit.

3. Analysis of system

Let X(t), Y(t) and I(t) denotes the status of the server, number of customers in the *orbit* and inventory level at time t, respectively.

Then $\{(X(t),Y(t),I(t)): t \geq 0\}$ is a three dimensional continuous time Markov process with state space $E_1 \times E_2 \times E_3$,where, $E_1 = \{0,1\}$; $E_2 = \{0,1,2,\ldots,N\}$; and $E_3 = \{0,1,2,\ldots,S\}$.

The infinitesimal generator A of the Markov process has entries of the form $\left(a_{(i,j,k)}^{(l,m,n)}\right)$.

Some of the state transitions are given below:

From state (0, j, k) only transitions into the following states are possible:

(i) (1, j, k) with rate λ (direct customer arrival).

(ii) (1, j–1, k) with rate $j\delta$ (Customer arrival from orbit).

Here, j = 0, 1, 2…, N and k = 1, 2… S.

From state (1, j-1, k) only transitions into the following states are possible:

(i) (1, j, k) with rate λ for $1 \leq j \leq N$ and $0 \leq k \leq S$ (direct customer arrival).

(ii) (0, j-1, k-1) with rate μ for $1 \leq j \leq N$ and $1 \leq k \leq S$ (Service completion).

From states (0, j, 0) transitions are possible only to the states (0,j +1,0) for $0 \leq j \leq N-1$.

3.1. MDP formulation

Now, we formulate the MDP by considering the following five components:

Decision epochs: The decision epochs for the system are taken as random points of time say the *service completion and Perishing times.*

State space: $E_1 \times E_2 \times E_3 = E$ is considered as the state space.

Action set: The reordering decisions (0- no order; 1- order; 2 –compulsory order) taken at each state of the system $(i, j, k) \in E$. The compulsory order for S items is made when inventory level is zero. Let A_r (r =1, 2, 3) denotes the set of possible actions. Where, $A_1 = \{0\}$, $A_2 = \{0, 1\}$, $A_3 = \{2\}$ and $A = A_1 \cup A_2 \cup A_3$.

The set of all possible actions are at $r \in E$.

$$A_r = \begin{cases} \{0\}, & s+1 \le k \le S \\ \{0,1\}, & 1 \le k \le s \\ \{2\}, & k = 0 \end{cases}, \quad A = \bigcup_{r \in E} A_r .$$

Suppose the policy π (sequence of decisions) is defined as a function π : E → A, given by

π (i, j, k) = {(a): (i, j, k) ∈ E_r, a∈ A_r, r = 1, 2, 3}

Transition probability: $p_{(i,j,k)}^{(l,m,n)}(a)$ denote the transition probability from state (i, j, k) to state (l, m, n) when decision a is made at state (i, j, k).

Cost: $C_{(i,j,k)}(a)$ denote the cost occurred in the system when action 'a' is taken at state (i, j, k).

3.2. Steady state analysis

Let R denote the stationary policy, which is deterministic time invariant and Markovian Policy (MD). From our assumptions it can be seen that $\{(X(t), N(t), I(t)) : t \ge 0\}$ is denoted as the controlled process $\{(X^R(t), N^R(t), I^R(t)) : t \ge 0\}$ when policy R is adopted. The above process is completely Ergodic, if every stationary policy gives rise to an irreducible Markov chain. It can be seen that for every stationary policy π, $\{X^\pi, Y^\pi, I^\pi\}$ is completely Ergodic and also the optimal stationary policy R^* exists, because the state and action spaces are finite.

If d_t is the Markovian deterministic decision, the expected reward satisfies the transition probability relations.

$$p_t\left((l,m,n) \mid (i,j,k), d_t(i,j,k)\right) = \sum_{a \in A_s} p_t\left((l,m,n) \mid (i,j,k), a\right) p_{d_t(i,j,k)}(a).$$

and $$r_t(i,j,k), d_t(i,j,k) = \sum_{a \in A_s} r_t(i,j,k,a) p_{d_t(i,j,k)}(a).$$

For Deterministic Markovian Policy $\Pi \in \Pi^{MD}$, where, Π^{MD} denotes the space of Deterministic Markovian policy. Under this policy Π an action $a \in A(r)$ is chosen with probability $\Pi_a(r)$, whenever the process is in state $k \in E$ .Whenever $\Pi_a(r) = 0 or 1$, the stationary Markovian policy Π reduces to a familiar stationary policy.

Then the controlled process $\{X^R, Y^R, I^R\}$, where, R is the deterministic Markovian policy is a Markov process. Under the policy Π , the expected long run total cost rate is given by

$$C^\Pi = h\bar{I}^\Pi + c_1\bar{w}^\Pi + c_2\alpha_a{}^\Pi + c_3\alpha_b{}^\Pi + p\alpha_c{}^\Pi + g\alpha_d{}^\Pi . \tag{1}$$

where, h -holding cost / unit item / unit time, c_1 – waiting cost / customer, c_2 – reordering cost / order, c_3 – service cost / customer, p – perishing cost / unit item, g – balking cost | customer, $\bar{I}^\Pi$ - mean inventory level, $\bar{w}^\Pi$ - expected number of customers in the orbit, α_a^Π - reordering rate, $\alpha_b{}^\Pi$ - service completion rate, $\alpha_c{}^\Pi$ - expected perishing rate, $\alpha_d{}^\Pi$ - balking rate.

Our objective here is to find an optimal policy Π^* for which $C^{\Pi^*} \le C^{\Pi}$ for every MD policy in Π^{MD}

For any fixed MD policy $\Pi \in \Pi^{MD}$ and $(i, \mathrm{j}, \mathrm{k}), (\mathrm{l}, \mathrm{m}, \mathrm{n}) \in E,$ define

$$P_{ijk}^{\Pi}(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{t}) = P\mathrm{r}\left\{X^{\Pi}(t) = l, \mathrm{Y}^{\Pi}(t) = m, \mathrm{I}^{\Pi}(t) = n \mid \mathrm{X}^{\Pi}(0) = i, \mathrm{Y}^{\Pi}(0) = j, \mathrm{I}^{\Pi}(0) = k\right\}$$

$(i, \mathrm{j}, \mathrm{k}), (\mathrm{l}, \mathrm{m}, \mathrm{n}) \in E.$

Now $P_{ijk}^{\Pi}(l, m, n, t)$ satisfies the Kolmogorov forward differential equation $P'(\mathrm{t}) = \mathrm{P}(\mathrm{t})\,\mathrm{A}$, where, A is an infinitesimal generator of the Markov process $\{(\mathrm{X}^{\Pi}(\mathrm{t}), \mathrm{Y}^{\Pi}(\mathrm{t}), \mathrm{I}^{\Pi}(\mathrm{t})) : \mathrm{t} \ge 0\}$.

For each MD policy π, we get an irreducible Markov chain with the state space E and actions space A which are finite,

$P^{\Pi}(\mathrm{l}, \mathrm{m}, \mathrm{n}) = \lim_{t \to \infty} P_{ijk}^{\Pi}(\mathrm{l}, \mathrm{m}, \mathrm{n}; t)$ exists and is independent of initial state conditions.

Now the system of equations obtained can be written as follows:

$$\left(\lambda + S\theta\right) P^{\Pi}(0,0,\mathrm{S}) = \gamma \sum_{k=0}^{s} \mathrm{P}^{\Pi}(0,0,\mathrm{k}) \tag{2}$$

$$(\lambda + \mathrm{j}\delta + S\theta)\mathrm{P}^{\Pi}(0,\mathrm{j},\mathrm{S}) = \gamma \sum_{k=0}^{s} \mathrm{P}^{\Pi}(0,\mathrm{j},\mathrm{k}),\ 1 \le j \le N \tag{3}$$

$$(\lambda + \mu + S\theta)\mathrm{P}^{\Pi}(1,0,\mathrm{S}) = \lambda P^{\Pi}(0,0,\mathrm{S}) + \delta P^{\Pi}(0,1,\mathrm{S}) + \gamma \sum_{k=1}^{s} \mathrm{P}^{\Pi}(1,0,\mathrm{k}), \tag{4}$$

$$(\lambda + \mu + S\theta)\mathrm{P}^{\Pi}(1,\mathrm{j},\mathrm{S}) = \lambda \sum_{i=0}^{1} P^{\Pi}(\mathrm{i},\mathrm{j}-\mathrm{i},\mathrm{S}) + (\mathrm{j}+1)\delta P^{\Pi}(0,\mathrm{j}+1,\mathrm{S}) + \gamma \sum_{k=1}^{s} \mathrm{P}^{\Pi}(1,\mathrm{j},\mathrm{k}),\ 1 \le j \le N-1 \tag{5}$$

$$\left(\mu + S\theta\right)\mathrm{P}^{\Pi}(1,\mathrm{N},\mathrm{S}) = \lambda \sum_{i=0}^{1} P^{\Pi}(\mathrm{i},\mathrm{N}-\mathrm{i},\mathrm{k}) + \gamma \sum_{k=1}^{s} \mathrm{P}^{\Pi}(1,\mathrm{N},\mathrm{k}), \tag{6}$$

$$\left(\lambda + k\theta\right) P^{\Pi}(0,0,\mathrm{k}) = \mu \mathrm{P}^{\Pi}(1,0,\mathrm{k}+1) + (\mathrm{k}+1)\theta \mathrm{P}^{\Pi}(0,0,\mathrm{k}+1),\ s+1 \le k \le S-1 \tag{7}$$

$$(\lambda + \mathrm{j}\delta + k\theta)\mathrm{P}^{\Pi}(0,\mathrm{j},\mathrm{k}) = \mu \mathrm{P}^{\Pi}(1,\mathrm{j},\mathrm{k}+1) + (\mathrm{k}+1)\theta \mathrm{P}^{\Pi}(0,j,\mathrm{k}+1),\ 1 \le \mathrm{j} \le \mathrm{N},\ s+1 \le k \le S-1 \tag{8}$$

$$(\lambda + \mu + k\theta)\mathrm{P}^{\Pi}(1,0,\mathrm{k}) = \lambda \mathrm{P}^{\Pi}(0,0,\mathrm{k}) + \delta \mathrm{P}^{\Pi}(0,1,\mathrm{k}) + (\mathrm{k}+1)\theta \mathrm{P}^{\Pi}(1,0,\mathrm{k}+1),\ s+1 \le k \le S-1 \tag{9}$$

$$(\lambda + \mu + k\theta)\mathrm{P}^{\Pi}(1,\mathrm{j},\mathrm{k}) = \lambda \sum_{i=0}^{1} P^{\Pi}(\mathrm{i},\mathrm{j}-\mathrm{i},\mathrm{k}) + (\mathrm{j}+1)\delta \mathrm{P}^{\Pi}(0,\mathrm{j}+1,\mathrm{k}) + (\mathrm{k}+1)\theta \mathrm{P}^{\Pi}(1,\mathrm{j},\mathrm{k}+1),\ 1 \le \mathrm{j} \le \mathrm{N}-1,\ s+1 \le k \le S-1 \tag{10}$$

$$\left(\mu + k\theta\right)\mathrm{P}^{\Pi}(1,\mathrm{N},\mathrm{k}) = \lambda \sum_{i=0}^{1} P^{\Pi}(\mathrm{i},\mathrm{N}-\mathrm{i},\mathrm{k}) + (\mathrm{k}+1)\theta \mathrm{P}^{\Pi}(1,\mathrm{N},\mathrm{k}+1),\ s+1 \le k \le S-1 \tag{11}$$

$$(\lambda+\gamma+k\theta)P^{\Pi}(0,0,k)=\mu P^{\Pi}(1,0,k+1)+(k+1)\theta P^{\Pi}(0,0,k+1),\ 1\le k\le s \qquad (12)$$

$$(\lambda+j\delta+\gamma+k\theta)P^{\Pi}(0,j,k)=\mu P^{\Pi}(1,j,k+1)+(k+1)\theta P^{\Pi}(0,j,k+1),1\le j\le N,1\le k\le s \qquad (13)$$

$$(\lambda+\mu+\gamma+k\theta)P^{\Pi}(1,0,k)=\lambda P^{\Pi}(0,0,k)+\delta P^{\Pi}(0,1,k)+(k+1)\theta P^{\Pi}(1,0,k+1),\ 1\le k\le s \qquad (14)$$

$$(\lambda+\mu+\gamma+k\theta)P^{\Pi}(1,j,k)=\lambda\sum_{i=0}^{1}P^{\Pi}(i,j-i,k)+(j+1)\delta P^{\Pi}(0,j+1,k)+(k+1)\theta P^{\Pi}(1,j,k+1),1\le j\le N-1,1\le k\le s \qquad (15)$$

$$(\mu+\gamma+k\theta)P^{\Pi}(1,N,k)=\lambda\sum_{i=0}^{1}P^{\Pi}(i,N-i,k)+(k+1)\theta P^{\Pi}(1,N,k+1),\ 1\le k\le s \qquad (16)$$

$$(\lambda+\gamma)P^{\Pi}(0,0,0)=\mu P^{\Pi}(1,0,0)+\theta P^{\Pi}(0,0,1) \qquad (17)$$

$$(\lambda+\gamma)P^{\Pi}(0,j,0)=\mu P^{\Pi}(1,j,1)+\lambda P^{\Pi}(0,j-1,0)+\theta P^{\Pi}(0,j,1),1\le j\le N-1 \qquad (18)$$

$$\gamma P^{\Pi}(0,N,0)=\lambda P^{\Pi}(0,N-1,0)+\mu P^{\Pi}(1,N,1)+\theta P^{\Pi}(0,N,1) \qquad (19)$$

$$(\lambda+\gamma)P^{\Pi}(1,0,0)=\theta P^{\Pi}(1,0,1) \qquad (20)$$

$$(\lambda+\gamma)P^{\Pi}(1,j,0)=\lambda P^{\Pi}(1,j-1,0)+\theta P^{\Pi}(1,j,1),1\le j\le N-1 \qquad (21)$$

$$\gamma P^{\Pi}(1,N,0)=\theta P^{\Pi}(1,N,1)+\lambda P^{\Pi}(1,N-1,0) \qquad (22)$$

Together with the above set of equations, the total probability condition

$$\sum_{(i,j,k)\in E} P^{\pi}(i,j,k)=1, \qquad (23)$$

gives steady state probabilities $\{P^{\pi}(i, j, k), (i, j, k) \in E\}$ uniquely.

3.3. System Performance Measures.

The average inventory level in the system is given by

$$\overline{I}^{\pi}=\sum_{i=0}^{1}\sum_{j=0}^{N}\sum_{k=1}^{S}kP^{\pi}(i,j,k). \qquad (24)$$

Expected number of customers in the orbit is given by

$$\overline{W}^{\pi}=\sum_{i=0}^{1}\sum_{j=1}^{N}\sum_{k=0}^{S}jP^{\pi}(i,j,k). \qquad (25)$$

The reorder rate is given by

$$\alpha_a^{\pi}=\mu\sum_{j=0}^{N}\sum_{k=1}^{s+1}P^{\pi}(1,j,k)+\sum_{i=0}^{1}\sum_{j=0}^{N}\sum_{k=1}^{s+1}(k\theta)P^{\pi}(i,j,k). \qquad (26)$$

The service completion rate is given by

$$\alpha_b^{\pi}=\mu\sum_{i=0}^{1}\sum_{j=1}^{N}\sum_{k=1}^{S}P^{\pi}(i,j,k). \qquad (27)$$

The expected perishable rate is given by

$$\alpha_c^{\pi}=\sum_{i=0}^{1}\sum_{j=0}^{N}\sum_{k=1}^{S}k\theta P^{\pi}(i,j,k). \qquad (28)$$

The balking rate is given by

$$\alpha_d^{\pi} = \sum_{i=0}^{1} \sum_{k=0}^{S} \lambda P^{\pi}(\mathrm{i}, \mathrm{N}, \mathrm{k}) . \tag{29}$$

Now the long run expected cost rate is given by

$$C^{\pi} = h\sum_{i=0}^{1}\sum_{j=0}^{N}\sum_{k=1}^{S} kP^{\pi}(\mathrm{i},\mathrm{j},\mathrm{k}) + c_1\sum_{i=0}^{1}\sum_{j=1}^{N}\sum_{k=0}^{S} jP^{\pi}(\mathrm{i},\mathrm{j},\mathrm{k})$$
$$+ c_2\left(\mu\sum_{j=0}^{N}\sum_{k=1}^{s+1} P^{\pi}(1,\mathrm{j},\mathrm{k}) + \sum_{i=0}^{1}\sum_{j=0}^{N}\sum_{k=1}^{s+1}(k\theta)P^{\pi}(\mathrm{i},\mathrm{j},\mathrm{k})\right)$$
$$+ c_3\mu\sum_{i=0}^{1}\sum_{j=1}^{N}\sum_{k=1}^{S} P^{\pi}(\mathrm{i},\mathrm{j},\mathrm{k}) + p\sum_{i=0}^{1}\sum_{j=0}^{N}\sum_{k=1}^{S} k\theta P^{\pi}(\mathrm{i},\mathrm{j},\mathrm{k})$$
$$+ g\sum_{i=0}^{1}\sum_{k=0}^{S}\lambda P^{\pi}(\mathrm{i},\mathrm{N},\mathrm{k}) \tag{30}$$

4. Linear programming problem

4.1. Formulation of LPP

In this section we propose a LPP model within a MDP framework. First we define the variables, D (i, j, k, *a*) as a conditional probability expression

D (i, j, k, *a*) = Pr {decision is *a* | state is (i, j, k)}. (31)

Since 0 ≤ D (i, j, k, *a*) ≤ 1, this is compatible with the deterministic time invariant Markovian policies. Here, the Semi–Markovian decision problem can be formulated as a linear programming problem. Hence,

0 ≤ D (i, j, k, *a*) ≤ 1 and $\sum_{a\in A=\{0,1,2\}} D(i,\ j,\ k,\ a) = 1$, i = 0, 1; 0 ≤ j ≤ N; 0 ≤ k ≤ M.

For the reformulation of the MDP as LPP, we define another variable y (i, j, k, *a*) as follows.

$$y(i,\ j,\ k,\ a) = \mathrm{D}(i,\ j,\ k,\ a)\,\mathrm{P}^{\pi}(i,\ j,\ k). \tag{32}$$

From the above definition of the transition probabilities

$$P^{\pi}(\mathrm{i},\mathrm{j},\mathrm{k}) = \sum_{a\in A} y(i,\ j,\ k,\ a),\ (\mathrm{i},\mathrm{j},\mathrm{k})\in \mathrm{E},\ a\in \mathrm{A} = \{0,1,2\} \tag{33}$$

Expressing $P^{\pi}(\mathrm{i},\mathrm{j},\mathrm{k})$ in terms of $y(\mathrm{i},\mathrm{j},\mathrm{k},a)$, the expected total cost rate function (30) is obtained and the LPP formulation is of the form Minimize

$$C^{\pi} = h\sum_{a=\{0,1,2\}}\sum_{i=0}^{1}\sum_{k=1}^{S}k\sum_{j=0}^{N}P^{\pi}(\mathrm{i,j,k},a) + c_1\sum_{a=\{0,1,2\}}\sum_{i=0}^{1}\sum_{j=1}^{N}\sum_{k=0}^{S}j.P^{\pi}(\mathrm{i,j,k},a)$$

$$+ c_2\sum_{a=\{0,1,2\}}\left(\mu\sum_{j=0}^{N}\sum_{k=1}^{s+1}P^{\pi}(1,\mathrm{j,k},a) + \sum_{i=0}^{1}\sum_{j=0}^{N}\sum_{k=1}^{s+1}(k\theta)P^{\pi}(\mathrm{i,j,k},a)\right)$$

$$+ c_3\mu\sum_{a=\{0,1,2\}}\sum_{i=0}^{1}\sum_{j=1}^{N}\sum_{k=1}^{S}P^{\pi}(\mathrm{i,j,k},a) + p\sum_{a=\{0,1,2\}}\sum_{i=0}^{1}\sum_{j=0}^{N}\sum_{k=1}^{S}k\theta P^{\pi}(\mathrm{i,j,k},a)$$

$$+ g\sum_{a=\{0,1,2\}}\sum_{i=0}^{1}\sum_{k=1}^{S}\lambda P^{\pi}(\mathrm{i,N,k},a) \qquad (34)$$

subject to the constraints,

(1) $y(i,j,k,a) \geq 0,\ (\mathrm{i,j,k}) \in \mathrm{E},\ a \in \mathrm{A}_l,\ l = 0,1,2$

(2) $\sum_{l=0}^{2}\sum_{(\mathrm{i,j,k})\in \mathrm{E}_l}\sum_{a\in A_l} y(\mathrm{i,j,k,a}) = 1,$

and the balance equations (2) – (22) are obtained by replacing $P^{\pi}(\mathrm{i,j,k})$ by $\sum_{a\in A} y(\mathrm{i,j,k,a})$.

Lemma 4.2. The optimal solution of the above Linear Programming Problem yields a Markovian deterministic (MD) policy.

Proof: From the equations

$$y(\mathrm{i,j,k,a}) = \mathrm{D}(\mathrm{i,j,k,a})\mathrm{P}^{\pi}(\mathrm{i,j,k}) \qquad (35)$$

and

$$P^{\pi}(\mathrm{i,j,k}) = \sum_{a\in A} y(\mathrm{i,j,k,a}),\ \forall(\mathrm{i,j,k}) \in \mathrm{E}. \qquad (36)$$

We have, $$D(\mathrm{i,j,k},a) = \frac{y(\mathrm{i,j,k},a)}{\sum_{a=0}^{2} y(\mathrm{i,j,k},a)} \qquad (37)$$

Since the decision problem is completely ergodic every basic feasible solution to the above linear programming problem has the property that for each $(\mathrm{i,j,k}) \in \mathrm{E},\ y(\mathrm{i,j,k},a) > 0$ for exactly one $a \in$ A. Hence, for each (i, j, k) $\in$ E, D(i, j, k, *a*) is 1for at least one value of *a* and zero for all other values of *a*. Thus, given the amount of inventory on – hand and the number of customers in the orbit, we have to choose the order of inventory for which D(i, j, k, a) is 1. Hence any basic feasible solution of the linear programming yields a deterministic policy.

5. Numerical illustration and discussion

In this section we consider a service facility system maintaining inventory with positive lead time and the size of the order is adjusted at the time of replenishment will illustrate the stochastic model described in section 4, through numerical examples. We have implemented TORA software to solve LPP by simplex algorithm.

Consider the MDP problem with the following parameters:

$S = 3$, $s = 1$, $\lambda = 2$, $\mu = 3$, $\delta=3$, $\theta = 0.7$, $\gamma = 1$, $p = 0.8$, $h = 0.1$, $c_j = 2j$; $j = 1, 2, 3$, $g = 1$.
Optimum cost for N = 2 is 15.95203. Optimum cost for N = 3 is 18.17716.
Optimum cost for N = 4 is 20.06195. Optimum cost for N = 5 is 21.86752.

Optimum policy for N = 2 is

{X(t), N(t), I(t)}	(0,0,3)	(0,1,3)	(0,2,3)	(1,0,3)	(1,1,3)	(1,2,3)
Action	0	0	0	0	0	0
{X(t), N(t), I(t)}	(0,0,2)	(0,1,2)	(0,2,2)	(1,0,2)	(1,1,2)	(1,2,2)
Action	0	0	0	0	0	0
{X(t), N(t), I(t)}	(0,0,1)	(0,1,1)	(0,2,1)	(1,0,1)	(1,1,1)	(1,2,1)
Action	0	0	0	0	0	0
{X(t), N(t), I(t)}	(0,0,0)	(0,1,0)	(0,2,0)	(1,0,0)	(1,1,0)	(1,2,0)
Action	2	2	2	2	2	2

Optimum policy for N = 3 is

{X(t), N(t), I(t)}	(0,0,3)	(0,1,3)	(0,2,3)	(0,3,3)	(1,0,3)	(1,1,3)	(1,2,3)	(1,3,3)
Action	0	0	0	0	0	0	0	0
{X(t), N(t), I(t)}	(0,0,2)	(0,1,2)	(0,2,2)	(0,3,2)	(1,0,2)	(1,1,2)	(1,2,2)	(1,3,2)
Action	0	0	0	0	0	0	0	0
{X(t), N(t), I(t)}	(0,0,1)	(0,1,1)	(0,2,1)	(0,3,1)	(1,0,1)	(1,1,1)	(1,2,1)	(1,3,1)
Action	0	0	0	0	1	0	0	0
{X(t), N(t), I(t)}	(0,0,0)	(0,1,0)	(0,2,0)	(0,3,0)	(1,0,0)	(1,1,0)	(1,2,0)	(1,3,0)
Action	2	2	2	2	2	2	2	2

Optimum policy for N = 4 and 5 is

{X(t), N(t),I(t)}	(0,0,3)	(0,1,3)	(0,2,3)	(0,3,3)	(0,4,3)	(1,0,3)	(1,1,3)	(1,2,3)	(1,3,3)	(1,4,3)
Action	0	0	0	0	0	0	0	0	0	0
{X(t), N(t),I(t)}	(0,0,2)	(0,1,2)	(0,2,2)	(0,3,2)	(0,4,2)	(1,0,2)	(1,1,2)	(1,2,2)	(1,3,2)	(1,4,2)
Action	0	0	0	0	0	0	0	0	0	0
{X(t), N(t), (t)}	(0,0,1)	(0,1,1)	(0,2,1)	(0,3,1)	(0,4,1)	(1,0,1)	(1,1,1)	(1,2,1)	(1,3,1)	(1,4,1)
Action	1	0	0	0	0	1	0	0	0	0
{X(t), N(t),I(t)}	(0,0,0)	(0,1,0)	(0,2,0)	(0,3,0)	(0,4,0)	(1,0,0)	(1,1,0)	(1,2,0)	(1,3,0)	(1,4,0)
Action	2	2	2	2	2	2	2	2	2	2

6. Conclusion

In most of previous works optimal ordering policies or system performance measures are determined. We approached the problem in new style using Semi–Markov Decision Process to control optimally with adjusted inventory replenishment. The optimum control policy to be employed is found using linear programming method so that the long–run expected cost rate is minimized. In future we like to extend this model to non – adjusted inventory replenishment in single server-retrial service facility system with inventory maintenance.

REFERENCES

1. D.Arivudainambi, I.Averbakh and O.Berman, Stationary analysis of a single server retrial queue with priority and vacation, *International Journal of Operational Research*, 5 (2009) 26 - 47.
2. J.R.Artalejo and A.Gomez-Corral, *Retrial Queuing Systems: A Computational Approach*, Springer, Berlin, (2008).
3. J.R.Artalejo, V.Rajagopalan and R.Sivasamy, On finite Markovian queues with repeated attempts, *Investigación Operativa*, 9 (2000) 83-94.
4. V.I.Dragieva, A finite source retrial queue: number of retrials, *Communications in Statistics - Theory and Methods*, 42 (2013) 812-829.
5. A.N.Dudin, R.Manzo and R.Piscopo, Single server retrial queue with group admission of customers, *Computers & Operations Research*, 61 (2015) 89-99.
6. A.Elcan, Asymptotic bounds for an optimal state-dependent retrial rate of the m/m/l queue with returning customers, *Mathematical and Computer Modelling*, 30 (1999) 129-140.
7. G.I.Falin and J.G.C.Templeton, Retrial Queues, *Chapman and Hall, London* (1997).
8. C.Henk Tijms, A First Course in Stochastic Models, *John Wiley & Sons Inc.*, (2003).
9. S.Krishnakumar and C.Elango, Inventory control in retrial service facility system – semi markov decision process, *Annals of Pure and Applied Mathematics*, 15(1) (2017) 41-49.
10. A.Krishnamoorthy and K.P.Jose, An (s, S) inventory system with positive lead time, loss and retrial of customers, *Stochastic Modelling and Applications*, 8(2) (2005) 56–71.
11. A.Krishnamoorthy and K.P.Jose, Comparison of inventory systems with service positive lead time, less and retrial of customers, *Journal of Applied Mathematics and Stochastic Analysis*, doi:10.1155/2007/37848.
12. M.Paul, B.Sivakumar and G.Arivarignan, Service facility inventory system with impatient customers, *International Journal of Mathematics, Game Theory and Algebra*, 15(4) (2006) 355–367.
13. M.L.Puterman, *Markov Decision Processes: Discrete Stochastic Dynamic Programming,* Wiley Inder science Publications Inc., (1994).
14. M.Schwarz, C.Sauer, H.Daduna, R.Kulik and R.Szekli, M /M /1 queueing system with inventory, *Queueing Systems, Theory Appl.*, 1(54) (2006) 55–78.