Symmetries and Similarity Reductions of (2+1)-Dimensional Equal Width Wave Equation

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Abstract. We have employed the symmetries and similarity reductions of (2+1)-dimensional equal width wave equation

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1. Introduction

A simple model equation is the Korteweg-de Vries (KdV) equation

\[ \psi_t + 6 \psi \psi_x + \delta \psi_{xxx} = 0 \]  

which describe the long waves in shallow water. Its modified version is,

\[ \psi_t - 6 \psi \psi_x + \psi_{xxx} = 0 \]  

And again there is Miura transformation

\[ \psi = \psi_t + \psi_x, \]  

between the KdV equation (1.1) and its modified version (1.2).

In 2002, Liu and Yang studied the bifurcation properties of generalized KdV equation (GKdVE)

\[ U_t + au^a u_x + u_{xx} = 0, \ a \in \mathbb{R}, \ n \in \mathbb{Z} \]  

Gungor and Winternitz transformed the Generalized Kadomtsev-Petviashvili Equation (GKPE)

\[ (u_t + p(t)u_x + q(t)u_{xxx}) + \sigma(y,t)u_{yy} + a(y,t)u_y + b(y,t)u_{xy} + c(y,t)u_{xx} + e(y,t)u_x + f(y,t)u + h(y,t) \]  

to its canonical form and established conditions on the coefficient functions under which (1.5) has an infinite dimensional symmetry group having a Kac-Moody-Virasoro structure.

In this chapter, we discuss the symmetry reductions of the (2+1)-dimensional Equal Width Wave equation as,

\[ u_t + uu_x - \mu(u_{xx} + u_{yy}). \]  

2. The symmetry group and lie algebra of equal width wave equation

If (1.6) is invariant under a one parameter Lie group of Point transformations (Bluman and Kumei, Olver)
Then the third prolongation \( p^3(v) \) of the corresponding Vector field

\[
\begin{align*}
V &= \v v_1 \frac{\partial}{\partial x} + \v v_2 \frac{\partial}{\partial y} + \v v_3 \frac{\partial}{\partial t} + \v v_4 \frac{\partial}{\partial u} \\
&= \v v_1 \frac{\partial}{\partial x} + \v v_2 \frac{\partial}{\partial y} + \v v_3 \frac{\partial}{\partial t} + \v v_4 \frac{\partial}{\partial u}
\end{align*}
\]

(1.11)

Satisfies

\[
\left. p^3(v) \Omega(x,y,t;u) \right|_{u=0} = 0
\]

(1.12)

The determining equations are obtained from (1.12) and solved for the infinitesimals \( \v v_1, \v v_2, \v v_3 \) and \( \v v_4 \). They are as follows

\[
\begin{align*}
\v v_1 &= \k_1, \\
\v v_2 &= \k_2, \\
\v v_3 &= \k_3 + t \k_4, \\
\v v_4 &= -u \k_4.
\end{align*}
\]

(1.13) - (1.16)

At this stage, we construct the symmetry generators corresponding to each of the constants involved.

Totally there are four generators given by

\[
\begin{align*}
V_1 &= \frac{\partial}{\partial x}, \\
V_2 &= \frac{\partial}{\partial y}, \\
V_3 &= \frac{\partial}{\partial t}, \\
V_4 &= t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}
\end{align*}
\]

(1.17)

The symmetry generators found in Eq. (1.17) form a closed Lie Algebra whose commutation table is shown below.

**Table 1:** Commutator Table

<table>
<thead>
<tr>
<th></th>
<th>V_1</th>
<th>V_2</th>
<th>V_3</th>
<th>V_4</th>
</tr>
</thead>
<tbody>
<tr>
<td>V_1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>V_2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>V_3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>V_1</td>
</tr>
<tr>
<td>V_4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The commutation relations of the Lie algebra, determined by \( V_1, V_2, V_3 \) and \( V_4 \) are shown in the above table.

3. Reductions of (2+1)-dimensional equal width wave equation by one-dimensional subalgebras

**Case 1:** \( v_1 = \frac{\partial}{\partial x} \)

The characteristic equation associated with this generator is

\[
\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}
\]

(1.18)

We integrate the characteristic equation to get three similarity variables, \( y=r, t=s, \) and \( u=W(r,s) \).

Using these similarity variables in Eq.(1.6) can be recast in the form

\[
W_s - \mu(W_{rrs}) = 0
\]

(1.19)
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Case 2: \( v_2 = \partial_y s \)
The characteristic equation associated with this generator is

\[
\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0}
\]

Following the standard procedure we integrate the characteristic equation to get three similarity variables,

\[
X = r, \quad y = s \quad \text{and} \quad u = W(r, s) \quad (1.20)
\]

Using these similarity variables in Eq.(1.6) can be recast in the form

\[
W_s + WW_r - \mu(WW_{rr}) = 0 \quad (1.21)
\]

Case 3: \( v_3 = \partial_t \)
The characteristic equation associated with this generator is

\[
\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}
\]

Following the standard procedure we integrate the characteristic equation to get three similarity variables,

\[
X = r, \quad y = s \quad \text{and} \quad u = W(r, s) \quad (1.22)
\]

Using these similarity variables in Eq.(1.6) can be recast in the form

\[
WW_r = 0 \quad (1.23)
\]

Case 4: \( v_4 = t\partial_t - u\partial_u \)
The characteristic equation associated with this generator is

\[
\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{-u}
\]

Following the standard procedure we integrate the characteristic equation to get three similarity variables,

\[
X = r, \quad t = s \quad \text{and} \quad u = W(r, s) t^{-1} \quad (1.24)
\]

4. Reductions of (2+1)-dimensional equal width wave equation by two-dimensional abelian subalgebras

Case I: Reduction under \( v_1 \) and \( v_2 \).
The transformed \( v_2 \) is

\[
\tilde{v}_2 = \partial_t
\]

The characteristic equation for \( \tilde{v}_2 \) is

\[
\frac{dr}{1} = \frac{ds}{0} = \frac{dW}{0}
\]

Integrating this equation as before leads to new variables

\[
s = \zeta \quad \text{and} \quad W = R(\zeta)
\]

which reduce Eq. (1.19) to

\[
R(\zeta) = 0 \quad (1.25)
\]

Case II: Reduction under \( V_1 \) and \( V_3 \).
The transformed \( V_3 \) is

\[
\tilde{V}_3 = \partial_s
\]

The characteristic equation for \( \tilde{V}_3 \) is
Integrating this equation as before leads to new variable $r = \zeta$ and $W = R(\zeta)$.

which satisfies Eq. (1.19)

**Case III: Reduction under $V_1$ and $V_4$**

The transformed $V_1$ is

$\tilde{V}_1 = \partial_r$

The characteristic equation for $\tilde{V}_1$ is

$$
\frac{dr}{0} = \frac{ds}{1} = \frac{dW}{0}
$$

Integrating this equation as before leads to new variables $s = \zeta$ and $W = R(\zeta)$.

which reduce eq. (1.21) to

$$R - \mu R\zeta = 0 \quad (1.26)$$

**Case IV: Reduction under $V_2$ and $V_3$**

The transformed $V_3$ is

$\tilde{V}_3 = \partial_s$

The characteristic equation for $\tilde{V}_3$ is

$$
\frac{dr}{0} = \frac{ds}{1} = \frac{dW}{0}
$$

Integrating this equation as before leads to new variables $r = \zeta$ and $W = R(\zeta)$.

which reduce Eq.(1.21) to

$$RR\zeta = 0. \quad (1.27)$$

**Case V: Reduction under $V_2$ and $V_4$**

The transformed $V_2$ is

$\tilde{V}_2 = \partial_s$

The characteristic equation for $\tilde{V}_2$ is

$$
\frac{dr}{0} = \frac{ds}{1} = \frac{dW}{0}
$$

Integrating this equation as before leads to new variables $r = \zeta$ and $W = R(\zeta)$.

which reduce Eq. (1.23) to

$$R - RR\zeta - \mu R\zeta = 0 \quad (1.28)$$

5. **Conclusions**

In this paper,

(i) A $(2+1)$-dimensional Equal width wave equation, $u_t + uu_x - \mu (u_{xxt} + u_{yyt}) = 0$

where $\mu \in \mathbb{R}$ is subjected to Lie’s classical method.

(ii) Equation (1.6) admits a four-dimensional symmetry group.

(iii) It is established that the symmetry generators form a closed Lie algebra.

(iv) Classification of symmetry algebra of (1.6) into one-and two-dimensional sub-algebras is carried out.
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(v) Systematic reduction to (1+1)-dimensional PDE and then to first order ODEs are performed using one dimensional and two dimension solvable Abelian sub-algebras.

REFERENCES