Intern. J. Fuzzy Mathematical Archive
Vol. 16, No. 1, 2018, 21-31
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 3 October 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/ijfma.v16n1a4

International Journal of
Fuzzy Mathematical Archive

Fuzzy Ideals of C-Algebras

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Received 1 August 2018; accepted 2 October 2018

Abstract. In this paper, we introduce the notion of fuzzy ideals in C-algebras and investigate some of their properties. Mainly, we give an algebraic characterization for fuzzy ideals generated by fuzzy sets. Furthermore, it is proved that the class of fuzzy ideals of a C −algebra forms an algebraic lattice.

Keywords: C-algebras; ideals of C-algebras; fuzzy ideals

AMS Mathematics Subject Classification (2010): 03G25, 03G05, 08A72

1. Introduction

The theory of fuzzy sets was introduced by Zadeh [16] in 1965. Rosenfeld [9] use the idea of fuzzy sets and developed the concept of fuzzy subgroups. Since then many authors have been studying fuzzy subalgebras of several algebraic structures (see [2,5,6,10]).

On the other hand, Guzman and Squier [4] introduced the variety of C-algebras as the variety generated by the three-element algebra $C = \{T, F, U\}$ with the operations "$\wedge$"; "$\vee$" and "$'$" of type $(2,2,1)$, which is the algebraic form of the three-valued conditional logic. Following this work, many more results have been appeared on the structure of C −algebras (see [7,8,11-14]). In this paper, we consider the fuzzification of ideals in C-algebra. Mainly, we characterize fuzzy ideals generated by fuzzy sets from the the algebraic point of view. Finally, it is proved that the class of all fuzzy ideals of a C −algebra forms an algebraic lattice.

2. Preliminaries

In this section, we recall some definitions and basic results on C −algebras from [4,13].

Definition 2.1. *An algebra* $(A,\vee,\wedge,')$ *of type* $(2,2,1)$ *is called a*C*-algebra, if it satisfies the following axioms:*

1. $a'' = a$
2. $(a \wedge b)' = a' \vee b'$
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
4. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
5. $(a \vee b) \wedge c = (a \wedge c) \vee (a' \wedge b \wedge c)$

6. $a \vee (a \wedge b) = a$
7. $(a \wedge b) \vee (b \wedge a) = (b \wedge a) \vee (a \wedge b)$ for all $a, b, c \in A$

Example 2.2. *The three element algebra $C = \{T, F, U\}$ with the operations given by the following tables is aC-algebra.*

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

x	x'
T	F
F	T
U	U

Note 2.1. The identities 2.1(1), 2.1(2) imply that the variety of C-algebras satisfies all the dual statements of 2.1(2) to 2.1(7).

Definition 2.3. *An element z of a C-algebra A is called a left zero for $\wedge$ if $z \wedge x = z$ for all $x \in A$.*

Definition 2.4. *A nonempty subset I of a C-algebra A is called an ideal of A, if*

1. $a, b \in I \Rightarrow a \vee b \in I$ and
2. $a \in I \Rightarrow x \wedge a \in I$, for each $x \in A$.

It is observed that $a \wedge b \in I$ if and only if $b \wedge a \in I$ for all $a, b \in A$. For any subset $S \subseteq A$, the smallest ideal of A containing S is called the ideal of A generated by S and is denoted by $\langle S]$. Note that:

$$\langle S] = \{\vee_i (y_i \wedge x_i) : y_i \in A, x_i \in S, i = 1, \ldots, n \text{ for some } n \in Z_+\}$$

If $S = \{a\}$ then we write $\langle a]$ for $\langle S]$. In this case $\langle a] = \{x \wedge a : x \in A\}$. Moreover it is observed in [13] that the set $I_0 = \{x \wedge x' : x \in A\}$ is the smallest ideal in A.

3. Fuzzy ideals

In this section we define fuzzy ideals in $C-$ algebra and we give several characterizations. Throughout this note we simply write A to say a C $-$algebra $(A, \vee, \wedge, ')$. By a fuzzy subset of A we mean a mapping μ of A into the unit interval $[0,1]$. For each $\alpha \in [0,1]$ the set

$$\mu_\alpha = \{x \in A : \mu(x) \geq \alpha\}$$

is called the level subset of μ at α and the set

$$supp(\mu) = \{x \in A : \mu(x) > 0\}$$

is called the support set of μ [16]. For numbers α and β in [0,1] we write $\alpha \wedge \beta$ (respectively $\alpha \vee \beta$) instead of $min\{\alpha, \beta\}$ (respectively $max\{\alpha, \beta\}$).

Definition 3.1. *A fuzzy subset μ of A is called a fuzzy subalgebra of A if:*

$$\mu(x \vee y) \wedge \mu(x \wedge y) \geq \mu(x) \wedge \mu(y) \quad \text{for all } x, y \in A$$

Definition 3.2. *A fuzzy subset μ of A is called a fuzzy ideal of A if:*

1. $\mu(a) = 1$, whenever a is a left zero for $\wedge$
2. $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$
3. $\mu(x \wedge y) \geq \mu(y)$

for all $x, y \in A$.

Note that the condition (3) in the above definition can be replaced by:

$$y \wedge x = y \Rightarrow \mu(y) \geq \mu(x)$$

for all $x, y \in A$. We denote the class of all fuzzy ideals of A by $\mathcal{FI}(A)$.

Lemma 3.3. *Let μ be a fuzzy ideal of A. Then the following holds for all $a, b \in A$.*

1. $\mu(a \wedge b) = \mu(b \wedge a)$
2. $\mu(a \wedge x \wedge b) \geq \mu(a \wedge b)$ for each $x \in A$
3. $\mu(a) \geq \mu(a \vee b)$ and hence $\mu(a) \wedge \mu(b) = \mu(a \vee b) \wedge \mu(b \vee a)$
4. If $x \in \langle a]$, then $\mu(x) \geq \mu(a)$.
5. For a nonempty subset S of A;

$$x \in \langle S] \Rightarrow \mu(x) \geq \bigwedge_{i=1}^{n} \mu(a_i)$$

for some $a_1, a_2, \ldots, a_n \in S$.

In the following we give the most natural characterization of fuzzy ideals using their level sets.

Lemma 3.4. *A fuzzy subset μ of A is a fuzzy ideal of A if and only if each α $-$level set μ_α is an ideal of A. In particular; A nonempty subset I of A is an ideal of A if and only if its characteristic function χ_I is a fuzzy ideal of A.*

It is routine to verify that the intersection of any family of fuzzy ideals of A is a fuzzy ideal. So that for any fuzzy subset μ of A, there exists a smallest fuzzy ideal containing μ. But the union of a family of fuzzy ideals of A is not in general a fuzzy ideal of A. Moreover, if we define binary operations $\circ_\wedge$ and $\circ_\vee$ on the class $[0,1]^A$ as follows; for each $\mu, \sigma \in [0,1]^A$ and all $x \in A$:

$$(\mu \circ_\wedge \sigma)(x) = Sup\{\mu(y) \wedge \sigma(z) : y \wedge z = x\}$$

and

$$(\mu \circ_\vee \sigma)(x) = Sup\{\mu(y) \wedge \sigma(z) : y \vee z = x\}$$

If μ and σ are fuzzy ideals, then $\mu \circ_\wedge \sigma$ is a fuzzy ideal of A and $\mu \circ_\wedge \sigma = \mu \cap \sigma = \mu \wedge \sigma$. But it is not true in general that $\mu \circ_\vee \sigma$ is a fuzzy ideal of A.

Theorem 3.5. *A fuzzy subset μ of A with $\mu(z) = 1$ for all $z \in I_0$, is a fuzzy ideal of A if and only if:*

1. $\mu \circ_\vee \mu \subseteq \mu$ and
2. $\eta \circ_\wedge \mu \subseteq \mu$ for all $\eta \in [0,1]^A$

Definition 3.6. *Let μ be fuzzy subset of A. The smallest fuzzy ideal of A containing μ is called a fuzzy ideal of A generated by μ and is denoted by $\langle\mu]$.*

Lemma 3.7. *Let S be any subset of A and χ_S its characteristic function. Then $\langle\chi_S] = \chi_{\langle S]}$.*

Proof: We show that $\chi_{\langle S]}$ is the smallest fuzzy ideal of A containing χ_S. Clearly $\chi_S \subseteq \chi_{\langle S]}$. Also from lemma 3.4 we have $\chi_{\langle S]}$ is a fuzzy ideal of A. Let θ be any fuzzy ideal of A containing χ_S. Then $\theta(x) = 1$ for each $x \in S$. It remain to show that $\chi_{\langle S]} \subseteq \theta$. If $y \in \langle S]$, then $y = \vee_{i=1}^{n} (y_i \wedge x_i)$, where $y_i \in A$, $x_i \in S$. Then consider:

$$\begin{aligned}\theta(y) &= \theta(\bigvee_{i=1}^{n} (y_i \wedge x_i))\\ &\geq Inf\{\theta(y_i \wedge x_i): i = 1,2,\ldots,n\}\\ &\geq Inf\{\theta(x_i): i = 1,2,\ldots,n\}\\ &\geq 1\end{aligned}$$

So that $\chi_{\langle S]} \subseteq \theta$.

For any fuzzy subset μ of A, it is clear that

$$\mu(x) = Sup\{\alpha \in [0,1]: x \in \mu_\alpha\} \text{ for all } x \in A$$

In the following theorem we characterize a fuzzy ideal generated by a fuzzy set in terms of its level ideals.

Theorem 3.8. *For a fuzzy subset μ of A let $\hat{\mu}$ be defined by:*

$$\hat{\mu}(x) = Sup\{\alpha \in [0,1]: x \in \langle\mu_\alpha]\} \text{for all } x \in A$$

Then $\hat{\mu} = \langle\mu]$.

Proof: It is enough if we show that $\hat{\mu}$ is the smallest fuzzy ideal of A containing μ.

We first show that $\hat{\mu}$ is a fuzzy ideal of A. For; for any $x, y \in A$ consider:

$$\begin{aligned}\hat{\mu}(x) \wedge \hat{\mu}(y) &= Sup\{\alpha \in [0,1]: x \in \langle\mu_\alpha]\} \wedge Sup\{\beta \in [0,1]: y \in \langle\mu_\beta]\}\\ &= Sup\{min\{\alpha,\beta\}: x \in \langle\mu_\alpha], y \in \langle\mu_\beta]\}\end{aligned}$$

If we put $\lambda = min\{\alpha,\beta\}$, where $x \in \langle\mu_\alpha]$ and $y \in \langle\mu_\beta]$, then $x, y \in \langle\mu_\lambda]$. So that $x \vee y \in \langle\mu_\lambda]$. Therefore

$$\begin{aligned}\hat{\mu}(x) \wedge \hat{\mu}(y) &= Sup\{min\{\alpha,\beta\}: x \in \langle\mu_\alpha], y \in \langle\mu_\beta]\}\\ &\leq Sup\{\lambda \in [0,1]: x \vee y \in \langle\mu_\lambda]\}\\ &= \hat{\mu}(x \vee y).\end{aligned}$$

Also, for any $\alpha \in [0,1]$,

$$y \in \langle\mu_\alpha] \Rightarrow x \wedge y \in \langle\mu_\alpha]$$

for all $x, y \in A$ which implies that $\hat{\mu}(x \wedge y) \geq \hat{\mu}(y)$. So that $\hat{\mu}$ is a fuzzy ideal of A. It is also clear that $\mu \subseteq \hat{\mu}$. Let γ be any fuzzy ideal of A such that $\mu \subseteq \gamma$, then $\mu_\alpha \subseteq \gamma_\alpha$ for all $\alpha \in [0,1]$. As γ is a fuzzy ideal of A, γ_α is an ideal of A for all $\alpha \in [0,1]$. That is, γ_α is an ideal of A containing μ_α. Then $\langle\mu_\alpha] \subseteq \gamma_\alpha$ for all $\alpha \in [0,1]$

Now for any $x \in A$ consider;

$$\begin{aligned}\hat{\mu}(x) &= Sup\{\alpha \in [0,1]: x \in \langle\mu_\alpha]\}\\ &\leq Sup\{\alpha \in [0,1]: x \in \gamma_\alpha\}\\ &= \gamma(x)\end{aligned}$$

Hence the result holds.

Corollary 3.9. *For any fuzzy subset μ of A.*

$$\langle \mu_t] = \langle \mu]_t for\ all\ t \in [0,1]$$

Notation 3.10. *We write $F \subset\subset A$ to say that F is a finite subset of A.*

Theorem 3.11. *Let μ be a fuzzy subset of A. Then a fuzzy subset $\overline{\mu}$ of A defined by:*

$$\overline{\mu}(x) = Sup\{\underset{a\in F}{\wedge} \mu(a): x \in \langle F], F \subset\subset A\} forall x \in A$$

is a fuzzy ideal of A.

Proof: For any $x, y \in A$ consider:

$$\overline{\mu}(x) \wedge \overline{\mu}(y) = Sup\left\{\underset{a\in E}{\wedge} \mu(a): x \in \langle E], E \subset\subset A\right\} \wedge$$
$$Sup\{\underset{b\in F}{\wedge} \mu(b): y \in \langle F], F \subset\subset A\}$$
$$= Sup\{(\underset{a\in E}{\wedge} \mu(a)) \wedge (\underset{b\in F}{\wedge} \mu(b)): x \in \langle E], y \in \langle F]\}$$
$$= Sup\{\underset{c\in E\cup F}{\wedge} \mu(c): x \vee y \in \langle E \cup F], E \cup F \subset\subset A,\}$$
$$\leq Sup\{\underset{c\in G}{\wedge} \mu(c): x \vee y \in \langle G], G \subset\subset A,\}$$
$$= \overline{\mu}(x \vee y)$$

Also;

$$\overline{\mu}(y) = Sup\{\underset{a\in F}{\wedge} \mu(a): y \in \langle F], F \subset\subset A\}$$
$$\leq Sup\{\underset{a\in F}{\wedge} \mu(a): x \wedge y \in \langle F], F \subset\subset A\}$$
$$= \overline{\mu}(x \wedge y)$$

Therefore $\overline{\mu}$ is a fuzzy ideal of A.
In the following theorem we give an algebraic characterization for fuzzy ideals generated by fuzzy sets.

Theorem 3.12. *For any fuzzy subset μ of A, $\overline{\mu} = \langle \mu]$; where $\overline{\mu}$ is as given in Theorem 3.11.*

Proof: Clearly $\mu \subseteq \overline{\mu}$. Let γ be fuzzy ideal of A such that $\mu \subseteq \gamma$. Let $x \in A$ such that $x \in \langle F]$ for some $F \subset\subset A$. Then by (5) of Lemma 3.3

$$\gamma(x) \geq \overset{n}{\underset{i=1}{\wedge}} \gamma(a_i) \text{forsome } a_1, a_2, \ldots, a_n \in F$$

Now

$$\underset{a\in F}{\wedge} \mu(a) \leq \underset{a\in F}{\wedge} \gamma(a) \leq \overset{n}{\underset{i=1}{\wedge}} \gamma(a_i) \leq \gamma(x)$$

Therefore $\overline{\mu}(x) \leq \gamma(x)$ for all $x \in A$. So $\overline{\mu}$ is the smallest fuzzy ideal of A containing μ. That is $\overline{\mu} = \langle \mu]$.

Lemma 3.13. *A fuzzy subset μ_0 of A defined by:*

$$\mu_0 = \begin{cases} 1 & if\ \ x\ is\ a\ left\ zero\ for\ \wedge \\ 0 & otherwise \end{cases}$$

for all $x \in A$, is the smallest fuzzy ideal of A.

Theorem 3.14. *The class $\mathcal{FJ}(A)$ of all fuzzy ideals of A forms a complete bounded distributive lattice where the infimum and supremum of any family $\{\mu_\alpha: \alpha \in \Delta\}$ of fuzzy ideals is given by:*

$$\wedge_\alpha \mu_\alpha = \cap \mu_\alpha, \vee_\alpha \mu_\alpha = \langle \cup \mu_\alpha]$$

μ_0 is its zero element and 1_A (the fuzzy subset of A with constant value 1) is its unit element.

For each $x \in A$ and $\alpha \in (0,1]$ remember from [15] that, the fuzzy subset x_α of A given by:

$$x_\alpha(z) = \begin{cases} \alpha & if\ z = x \\ 0 & otherwise \end{cases}$$

is called a fuzzy point of A. In this case x is called the support of x_α and α its value.

Theorem 3.15. *For each fuzzy point x_α of A, the fuzzy set μ_α^x given by:*

$$\mu_\alpha^x(z) = \begin{cases} 1 & if\ z \in I_0 \\ \alpha & if z \in \langle x] - I_0 \\ 0 & otherwise \end{cases}$$

is the fuzzy ideal of A generated by the fuzzy point x_α, or equivalently $\mu_\alpha^x = \langle x_\alpha]$.

Lemma 3.16. *For any fuzzy points x_α and y_β of A, $\langle x_\alpha] = \langle y_\beta]$, if and only if $\langle x] = \langle y]$ and $\alpha = \beta$.*

Corollary 3.17. *Let x_α, y_β and z_γ be fuzzy points of A. If $\langle x_\alpha] = \langle y_\beta]$, then $\langle z_\gamma \circ_\vee x_\alpha] = \langle z_\gamma \circ_\vee y_\beta]$.*

Theorem 3.18. *The following statements are equivalent for any C −algebra A:*

1. A is a Boolean algebra
2. For any fuzzy points x_α and y_β of A; $x_\alpha \circ_\vee y_\beta = y_\beta \circ_\vee x_\alpha$
3. For any fuzzy points x_α and y_β of A; $\langle x_\alpha \circ_\vee y_\beta] = \langle y_\beta \circ_\vee x_\alpha]$
4. For any fuzzy points x_α and y_β of A; $\langle x_\alpha \circ_\vee y_\beta] = \langle \mu_\gamma^{\{x,y\}}]$;
   where $\gamma = min\{\alpha, \beta\}$ and $\mu_\gamma^{\{x,y\}}$ is given by:
   $$\mu_\gamma^{\{x,y\}}(z) = \begin{cases} \gamma & if\ \ z \in \{x, y\} \\ 0 & otherwise \end{cases}$$
5. For any fuzzy points x_α, y_β and z_γ of A. If $\langle x_\alpha] = \langle y_\beta]$,
   then $\langle x_\alpha \circ_\vee z_\gamma] = \langle y_\beta \circ_\vee z_\gamma]$

4. Homomorphisms and fuzzy ideals

Theorem 4.1. *Let $f: A \to B$ be a surjective homomorphism of C-algebras, μ a fuzzy ideal of A and ν a fuzzy ideal of B, then*

1. $f(\mu)$ is a fuzzy ideal of B
2. $f^{-1}(\nu)$ is a fuzzy ideal of A

Proof: Since f is given to be surjective, $f^{-1}(z) \neq \emptyset$ for all $z \in B$. Let $z_1, z_2 \in B$. Then consider:

$$\begin{aligned} f(\mu)(z_1) \wedge f(\mu)(z_2) &= Sup\{\mu(x): x \in f^{-1}(z_1)\} \wedge Sup\{\mu(y): y \in f^{-1}(z_1)\} \\ &= Sup\{\mu(x) \wedge \mu(y): x \in f^{-1}(z_1), y \in f^{-1}(z_1)\} \\ &\leq Sup\{\mu(x \vee y): f(x) = z_1, f(y) = z_1\} \\ &\leq Sup\{\mu(a): a \in f^{-1}(z_1 \vee z_2)\} \\ &= f(\mu)(z_1 \vee z_2) \end{aligned}$$

Also let $z_1, z_2 \in B$ and $x \in A$ such that $f(x) = z_1$. Then consider:

$$
\begin{aligned}
f(\mu)(z_2) &= sup\{\mu(y) : y \in f^{-1}(z_2)\} \\
&\le sup\{\mu(x \wedge y) : f(x) = z_1, f(y) = z_2\} \\
&= sup\{\mu(x \wedge y) : x \wedge y \in f^{-1}(z_1 \wedge z_2)\} \\
&\le sup\{\mu(a) : a \in f^{-1}(z_1 \wedge z_2)\} \\
&= f(\mu)(z_1 \wedge z_2)
\end{aligned}
$$

Thus $f(\mu)$ is a fuzzy ideal of B. Similarly one can easily verify that $f^{-1}(\nu)$ is a fuzzy ideal of A.

Theorem 4.2. *Let $f: A \to B$ be a homomorphism, μ and ν fuzzy ideals of A, σ and θ fuzzy ideals of B then*

1. $f(\mu \vee \nu) = f(\mu) \vee f(\nu)$
2. $f(\mu \wedge \nu) = f(\mu) \wedge f(\nu)$
3. $f^{-1}(\sigma \vee \theta) = f^{-1}(\sigma) \vee f^{-1}(\theta)$
4. $f^{-1}(\sigma \wedge \theta) = f^{-1}(\sigma) \wedge f^{-1}(\theta)$

Proof: We prove the first part only. For this, we show that $f(\mu \vee \nu)$ is the smallest fuzzy ideal of B containing both $f(\mu)$ and $f(\nu)$. By theorem 4.1 $f(\mu \vee \nu)$ is a fuzzy ideal of B. Also for any $y \in B$ consider:

$$
\begin{aligned}
f(\mu)(y) &= sup\{\mu(x) : x \in f^{-1}(y)\} \\
&\le sup\{(\mu \vee \nu)(x) : x \in f^{-1}(y)\} \\
&= f(\mu \vee \nu)(y)
\end{aligned}
$$

So that $f(\mu) \subseteq f(\mu \vee \nu)$. Similarly, we get $f(\nu) \subseteq f(\mu \vee \nu)$. Now for any fuzzy ideal η of B:

$$
\begin{aligned}
f(\mu) \subseteq \eta, f(\nu) \subseteq \eta &\Rightarrow f^{-1}(f(\mu)) \subseteq f^{-1}(\eta), f^{-1}(f(\nu)) \subseteq f^{-1}(\eta) \\
&\Rightarrow \mu \subseteq f^{-1}(\eta), \nu \subseteq f^{-1}(\eta) \\
&\Rightarrow \mu \vee \nu \subseteq f^{-1}(\eta) \\
&\Rightarrow f(\mu \vee \nu) \subseteq f(f^{-1}(\eta)) = \eta
\end{aligned}
$$

Thus $f(\mu \vee \nu) = f(\mu) \vee f(\nu)$. The others can be proved using similar arguments.

5. Product of fuzzy ideals

Theorem 5.1. *If μ_1 and μ_2 are fuzzy ideals of C-algebras A_1 and A_2 respectively, then $\mu_1 \times \mu_2$ is a fuzzy ideal of $A_1 \times A_2$.*

Proof: Suppose that μ_1 and μ_2 are fuzzy ideals A_1 and A_2 respectively. Remember that $\mu_1 \times \mu_2$ is a fuzzy subset of $A_1 \times A_2$ defined as:

$$(\mu_1 \times \mu_2)(a_1, a_2) = min\{\mu_1(a_1), \mu_2(a_2)\}$$

for all $a_1 \in A_1$ and $a_2 \in A_2$. Now consider:

$$
\begin{aligned}
(\mu_1 \times \mu_2)((a_1, a_2) \vee (b_1, b_2)) &= \mu_1 \times \mu_2((a_1 \vee b_1), (a_2 \vee b_2)) \\
&= min\{\mu_1(a_1 \vee b_1), \mu_2(a_2 \vee b_2)\} \\
&\ge min\{min\{\mu_1(a_1), \mu_1(b_1)\}, min\{\mu_2(a_2), \mu_2(b_2)\}\} \\
&= min\{min\{\mu_1(a_1), \mu_2(a_2)\}, min\{\mu_1(b_1), \mu_2(b_2)\}\} \\
&= min\{(\mu_1 \times \mu_2)(a_1, a_2), (\mu_1 \times \mu_2)(b_1, b_2)\}
\end{aligned}
$$

Also

$$
\begin{aligned}
(\mu_1 \times \mu_2)((a_1, a_2) \wedge (b_1, b_2)) &= \mu_1 \times \mu_2((a_1 \wedge b_1), (a_2 \wedge b_2)) \\
&= min\{\mu_1(a_1 \wedge b_1), \mu_2(a_2 \wedge b_2)\} \\
&\ge min\{\mu_1(b_1), \mu_2(b_2)\} \\
&= (\mu_1 \times \mu_2)(b_1, b_2)
\end{aligned}
$$

So that $\mu_1 \times \mu_2$ is a fuzzy ideal of $A_1 \times A_2$. But it is not in general true that any fuzzy ideal of $A_1 \times A_2$ is of the form $\mu_1 \times \mu_2$ for some fuzzy ideals μ_1 and μ_2 of A_1 and A_2 respectively.

Definition 5.2. *[2] For a fuzzy subset μ of $A_1 \times A_2$;*

$$pr_1(\mu)(x) = sup\{\mu(x,y): y \in A_2\}$$

and

$$pr_2(\mu)(y) = sup\{\mu(x,y): x \in A_1\}$$

are called the projections of μ on A_1 and A_2 respectively.

Lemma 5.3. *If μ is a fuzzy ideal of $A_1 \times A_2$, then $pr_1(\mu)$ (respectively $pr_2(\mu)$) is a fuzzy ideal of A_1 (respectively A_2).*

Proof: It is enough if we show that, $pr_1(\mu)$ is a fuzzy ideal of A_1. For; let $x_1, x_2 \in A_1$. Then consider:

$$\begin{aligned} pr_1(\mu)(x_1) \wedge pr_1(\mu)(x_2) &= Sup\{\mu(x_1,y_1): y_1 \in A_2\} \wedge Sup\{\mu(x_2,y_2): y_2 \in A_2\} \\ &= Sup\{\mu(x_1,y_1) \wedge \mu(x_2,y_2): y_1, y_2 \in A_2\} \\ &\leq Sup\{\mu(x_1 \vee x_2, y_1 \vee y_2): y_1, y_2 \in A_2\} \\ &\leq Sup\{\mu(x_1 \vee x_2, z): z \in A_2\} \\ &= pr_1(\mu)(x_1 \vee x_2) \end{aligned}$$

Also consider:

$$\begin{aligned} pr_1(\mu)(x_2) &= Sup\{\mu(x_2,y_2): y_2 \in A_2\} \\ &\leq Sup\{\mu[(x_1,y_1) \wedge (x_2,y_2)]: y_2 \in A_2\} \quad \forall x_1 \in A_1, y_1 \in A_2 \\ &= Sup\{\mu(x_1 \wedge x_2, y_1 \wedge y_2): y_1, y_2 \in A_2\} \\ &\leq Sup\{\mu(x_1 \wedge x_2, z): z \in A_2\} \\ &= pr_1(\mu)(x_1 \wedge x_2) \quad \forall x_1 \in A_1 \end{aligned}$$

So that $pr_1(\mu)$ is a fuzzy ideal of A_1. Similarly it can be verified that $pr_2(\mu)$ is a fuzzy ideal of A_2.

Definition 5.4. *[2] Let μ be a fuzzy subset of $A_1 \times A_2$, $a \in A_2$ and $b \in A_1$, the marginal fuzzy subsets of μ(with respect to a and b) are $\mu_1^{(a)} \in [0,1]^{A_1}$ and $\mu_2^{(b)} \in [0,1]^{A_2}$ defined by:*

$$\mu_1^{(a)}(x) = \mu(x,a) \text{and} \mu_2^{(b)}(y) = \mu(b,y)$$

for all $x \in A_1$ and $y \in A_2$.

Lemma 5.5. *If μ is a fuzzy ideal of $A_1 \times A_2$, then $\mu_1^{(a)}$ is a fuzzy ideal of A_1 and $\mu_2^{(b)}$ is a fuzzy ideal of A_2 for all $a \in A_2$ and $b \in A_1$.*

Theorem 5.6. *If A_1 and A_2 are $C-$algebras with meet identity T, then any fuzzy ideal of $A_1 \times A_2$ is necessarily of the form $\mu_1 \times \mu_2$ for some fuzzy ideals μ_1 and μ_2 of A_1 and A_2 respectively.*

Proof: Suppose that μ is a fuzzy ideal of $A_1 \times A_2$. Take

$$\mu_1 = pr_1(\mu) \text{and } \mu_2 = pr_2(\mu)$$

Since each p_1 and p_2 are homomorphisms, by applying theorem 4.1 we get that μ_1 and μ_2 are fuzzy ideals of A_1 and A_2 respectively. Also for each $a_1 \in A_1$ and $a_2 \in A_2$ consider:

$$\mu_1 \times \mu_2(a_1,a_2) = min\{\mu_1(a_1), \mu_2(a_2)\}$$

$$= min\{p_1(\mu)(a_1), p_2(\mu)(a_2)\}$$
$$= min\{Sup\{\mu(x_1,x_2): (x_1,x_2) \in p_1^{-1}(a_1)\},$$
$$Sup\{\mu(x_1,x_2): (x_1,x_2) \in p_2^{-1}(a_2)\}\}$$
$$= min\{Sup\{\mu(a_1,x_2): x_2 \in A_2\}, Sup\{\mu(x_1,a_2): x_1 \in A_1\}\}$$
$$\geq min\{\mu(x_1,a_2), \mu(a_1,x_2)\} \quad \forall x_1 \in A_1, x_2 \in A_2$$

In particular $(\mu_1 \times \mu_2)(a_1,a_2) \geq \mu(a_1,a_2)$. So that $\mu \subseteq \mu_1 \times \mu_2$. On the other hand, to show that $\mu_1 \times \mu_2 \subseteq \mu$ let $a_1 \in A_1$ and $a_2 \in A_2$ such that $(\mu_1 \times \mu_2)(a_1,a_2) = \alpha$. Then $\mu_1(a_1) \geq \alpha$ and $\mu_2(a_2) \geq \alpha$. It follows from the definition of μ_1 and μ_2 that there exists $x_1 \in A_1$ and $x_2 \in A_2$ such that $\mu(a_1,x_2) \geq \alpha$ and $\mu(x_1,a_2) \geq \alpha$. Put $b_1 = (a_1,x_2)$, $b_2 = (x_1,a_2)$, $y_1 = (T,F)$, $y_2 = (F,T)$, $c_1(a_1,F)$ and $c_2(F,a_2)$. Then we have $c_1 = y_1 \wedge b_1$ and $c_2 = y_2 \wedge b_2$. Since $(a_1,a_2) = c_1 \vee c_2$, consider the following:

$$\mu(a_1,a_2) = \mu(c_1 \vee c_2)$$
$$\geq \mu(c_1) \wedge \mu(c_2)$$
$$= \mu(y_1 \wedge b_1) \wedge \mu(y_2 \wedge b_2)$$
$$\geq \mu(b_1) \wedge \mu(b_2)$$
$$= \mu(a_1,x_2) \wedge \mu(x_1,a_2)$$
$$\geq \alpha$$

Thus $(\mu_1 \times \mu_2) \subseteq \mu$ and hence the result holds.

6. The lattice of fuzzy ideals in C-algebras

It is proved in section (3) that the class $\mathcal{FI}(A)$ of all ideals of A forms a complete lattice. In this section we further prove that the lattice $\mathcal{FI}(A)$ is an algebraic lattice.

Lemma 6.1. *For any two fuzzy ideals μ and σ of A, their supremum $\mu \vee \sigma$ is given by:*

$$(\mu \vee \sigma)(x) = sup\{\bigwedge_{i=1}^{n} [\mu(a_i \vee \sigma(a_i))]: x = \bigvee_{i=1}^{n} a_i, a_i \in A\}$$

Corollary 6.2. *For any family $\{\mu_\alpha\}_{\alpha\in\Delta}$ of fuzzy ideals of A:*

$$\langle\bigcup_{\alpha\in\Delta} \mu_\alpha](x) = sup\{\bigwedge_{i=1}^{n} [sup_{\alpha\in\Delta}\{\mu_\alpha(a_i)\}: x = [\bigvee_{i=1}^{n} a_i], a_i \in A\}$$

Note 6.3. *For a fuzzy subset μ of A, by the cardinality of μ, we mean the cardinality of the support set of μ.*

Theorem 6.4. *The lattice $\mathcal{FI}(A)$ of all fuzzy ideals of A is an algebraic lattice in which the compact elements are precisely the finitely generated fuzzy ideals.*

Proof: We first show that $\langle\lambda]$ is a compact element in the class $\mathcal{FI}(A)$ for a fuzzy subset λ with finite cardinality. Let $card(\lambda) = n$. That is, the $supp(\lambda)$ has exactly n elements let say $x_1, x_2, \ldots, x_n$. Put $t_i = \lambda(x_i)$ for all $1 \leq i \leq n$. Suppose that $\{\mu_\alpha\}_{\alpha\in\Delta}$ be a family of fuzzy ideals of A such that:

$$\langle\lambda] \leq \bigvee_{\alpha\in\Delta} \mu_\alpha \Rightarrow \langle\lambda]_{t_i} \leq [\bigvee_{\alpha\in\Delta} \mu_\alpha]_{t_i}, \quad \forall i, 1 \leq i \leq n$$
$$\Rightarrow \langle\lambda_{t_i}] \leq \bigvee_{\alpha\in\Delta} [\mu_\alpha]_{t_i}, \quad \forall i, 1 \leq i \leq n$$

Since the class of all ideals is an algebraic lattice and finitely generated ideals are compact, for each $i, 1 \leq i \leq n$, there exists $\alpha_{i_j} \in \Delta$, where each $j, 1 \leq j \leq m_i$ such that

$$\langle \lambda_{t_i}] \leq \bigvee_{1\leq j\leq m_i} [\mu_{\alpha_{i_j}}]_{t_i}$$

which implies that

$$\langle \lambda_{t_i}] \leq \bigvee_{1\leq i\leq n, 1\leq j\leq m_i} [\mu_{\alpha_{i_j}}]_{t_i}, \quad \forall i, 1\leq i\leq n$$

Since λ has n distinct level sets, $\langle \lambda]$ has also n distinct level sets. So that

$$\langle \lambda] \leq \bigvee_{1\leq i\leq n, 1\leq j\leq m_i} \mu_{\alpha_{i_j}}$$

Thus $\langle \lambda]$ is compact. Conversely we show that any compact element in $\mathcal{FI}(A)$ should necessarily be of the form $\langle \lambda]$ where λ is of finite cardinal. For; suppose that μ is compact in $\mathcal{FI}(A)$. For each $\alpha \in (0,1]$ and any $x \in A$ consider a fuzzy subset α_x of A given by

$$\alpha_x(y) = \begin{cases} \alpha & if\ y = x \\ 0 & otherwise \end{cases} \qquad \forall y \in A$$

It is clear to see that:

$$\mu = \bigvee_{\mu(x)\geq\alpha} \langle \alpha_x](1)$$

As μ is compact, there exists $x_1, x_2, \ldots, x_n \in A$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in (0,1]$ with $\mu(x_i) \geq \alpha_i$ such that:

$$\mu = \bigvee_{1\leq i\leq n} \langle (\alpha_i)_{(x_i)}]$$

If we define a fuzzy subset λ of A by:

$$\lambda(y) = \begin{cases} \alpha_i & if\ y = x_i \\ 0 & otherwise \end{cases} \qquad \forall y \in A$$

Then λ is of finite cardinal such that

$$\langle \lambda] = \bigvee_{1\leq i\leq n} \langle (\alpha_i)_{(x_i)}] = \mu$$

Therefore the compact elements in $\mathcal{FI}(A)$ are precisely fuzzy ideals of A generated by fuzzy sets with finite cardinality. So that μ is generated by a fuzzy subset of finite cardinality. Also it follows from (1) that any fuzzy ideal of A is compactly generated and hence $\mathcal{FI}(A)$ is an algebraic lattice.

7. Conclusion

The results presented in this note indicate that many of the basic concepts in fuzzy normal subgroups (respectively fuzzy ideals) of the well known structures; groups (respectively rings) can readily be extended to fuzzy ideals of C −algebras. Moreover, this paper lays a ground for further studies on fuzzy ideal theories in C −algebras like: prime fuzzy ideals, fuzzy congruence relations and other fuzzy structures in C −algebras.

Acknowledgements. We express our heartfelt thanks to the Chief-Editor for accepting our paper and suggestion for improvement of the paper.

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