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Fuzzy Measure–Structural Characteristics

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Abstract. In this paper, some structural characteristics of fuzzy measure such as null-additivity, autocontinuity, uniform autocontinuity, subadditivity, and superadditivity are defined with examples. Some theorems are given about the characteristics and finite fuzzy measure.

Keywords: Fuzzy measure, null-additivity, autocontinuity, uniform autocontinuity, and exhaustive.

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1. Introduction

Fuzzy measure has been introduced by Sugeno [2] and further studied by several authors in different point of view [3, 4, 5, 6, 7, 10, 11].

As the fuzzy measures lose additivity, some additional conditions are required to develop theory of fuzzy measures. Concepts of structural characteristics of fuzzy measure like null-additivity, autocontinuity, uniform autocontinuity, and subadditivity have been introduced by Wang [8, 9] and there by a series of new results have been obtained in the fuzzy measure theory.

In this paper, we define the structural characteristics such as null-additivity, autocontinuity, uniform autocontinuity, subadditivity, and superadditivity for the fuzzy measure defined in [1] and we give some theorems which relates the characteristics and on finite fuzzy measure.

2. Fuzzy measure

Definition 2.1. Fuzzy measure [1] Let X be a non empty set, Ω be a non empty class of subsets of X and (X, Ω) be a measurable space. A fuzzy relation $m : \Omega \to [0, \infty]$ is said to be a fuzzy measure, if the following conditions are satisfied

(i) $\mu_m(\phi, 0) = 1$

(ii) For any two sets A and B in Ω, $A \subseteq B$ and A is a nonempty set then

$$\sup_{m(A)=x} x \le \sup_{m(B)=y} y \quad \text{and} \quad \mu_m\left(A, \left(\sup_{m(A)=x} x\right)\right) \le \mu_m\left(B, \left(\sup_{m(B)=y} y\right)\right)$$

(iii) For a sequence of non empty sets $\{A_n\} \subset \Omega$, $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4 \subseteq \ldots\ldots$ and

$$\bigcup_{n=1}^{\infty} A_n \in \Omega \quad \Rightarrow \quad \lim_n \left(\sup_{m(A_n)=x} x\right) = \sup_{m\left(\bigcup_{n=1}^{\infty} A_n\right)=y} y$$

$$\text{and} \quad \lim_n \mu_m\left(A_n, \left(\sup_{m(A_n)=x} x\right)\right) = \mu_m\left(\bigcup_{n=1}^{\infty} A_n, \sup_{m\left(\bigcup_{n=1}^{\infty} A_n\right)=y} y\right)$$

(iv) For a sequence of non empty sets $\{A_n\} \subset \Omega$, $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \ldots\ldots$, $m(A_1) < \infty$

$$\text{and} \quad \bigcap_{n=1}^{\infty} A_n \in \Omega \quad \Rightarrow \quad \lim_n \left(\sup_{m(A_n)=x} x\right) = \sup_{m\left(\bigcap_{n=1}^{\infty} A_n\right)=y} y$$

$$\text{and} \quad \lim_n \mu_m\left(A_n, \left(\sup_{m(A_n)=x} x\right)\right) = \mu_m\left(\bigcap_{n=1}^{\infty} A_n, \sup_{m\left(\bigcap_{n=1}^{\infty} A_n\right)=y} y\right)$$

Remark: In the above definition, m is called a lower or upper semicontinuous fuzzy measure if it satisfies the above conditions (i), (ii), and (iii) or (i), (ii), and (iv), respectively. Both of them are simply called as semicontinuous fuzzy measure.

3. Structural characteristics of fuzzy measure

Definition 3.1. Null–additive fuzzy measure

A fuzzy measure m is called null-additive if

$$\sup_{m(A\cup B)=x} x = \sup_{m(A)=y} y \quad \text{whenever } A, B \in \Omega, \left(\sup_{m(B)=x} x\right) = 0.$$

Example 3.2. Consider the example given in [1].

Let $X = \{1, 2, 3, \ldots n\}$, n is a finite value, and $\Omega = P(X)$. Let the fuzzy relation $m : \Omega \to [0, \infty]$ be defined as

$$m(S) = x \ \text{ iff } x \le |S| \quad \text{that is} \quad m(S) = \{x \,/\, x \in [0, |S|]\}, \ \forall S \in \Omega$$

and the membership function $\mu_m : m \to [0,1]$ be defined as

$$\mu_m(S, x) = \frac{x}{|S|}, \ x \le |S|, \ \text{for } S \ne \phi \quad \text{and} \quad \mu_m(\phi, 0) = 1$$

m is a fuzzy measure and also m is null- additive.

Example 3.3. Let $X = \{a, b\}$, and $\Omega = P(X)$.

Let the fuzzy relation $m : \Omega \to [0,\infty]$ be defined as

$$m(S) = \begin{cases} 1 & \text{if } S = X \\ 0 & \text{if } S \neq X \end{cases} \quad \forall S \in \Omega$$

and the membership function $\mu_m : m \to [0,1]$ be defined as

$$\mu_m(S,x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x = 0 \end{cases} \quad \mathit{for}\ S \neq \phi \quad \text{and} \quad \mu_m(\phi, 0) = 1$$

m is a fuzzy measure but it is not null-additive.

Theorem 3.4. If for any nonempty set $A \in \Omega$, $\left(\sup\limits_{m(A)=x} x \right) \neq 0$, then m is null-additive.

Proof: If there exists some set $B \in \Omega$, such that $\left(\sup\limits_{m(B)=y} y \right) = 0$, then $B = \phi$.

We get, for any $A \in \Omega$, $\sup\limits_{m(A\cup B)=y} y = \sup\limits_{m(A)=x} x$. Hence m is null-additive.

Theorem 3.5. If m is a null-additive fuzzy measure, and $A \in \Omega$. Then, we have $\lim\limits_{n} \left(\sup\limits_{m(A\cup B_n)=y} y \right) = \sup\limits_{m(A)=x} x$, for any decreasing set sequence $\{B_n\} \subset \Omega$ for which $\lim\limits_{n} \left(\sup\limits_{m(B_n)=x} x \right) = 0$ and there exists at least one positive integer n' such that $\left(\sup\limits_{m(A\cup B_{n'})=y} y \right) < \infty$ as $\left(\sup\limits_{m(A)=x} x \right) < \infty$.

Proof: It is enough to prove this theorem for $\left(\sup\limits_{m(A)=x} x \right) < \infty$.

Suppose that $B = \bigcap\limits_{n=1}^{\infty} B_n$, we get $\sup\limits_{m(B)=y} y = \lim\limits_{n} \left(\sup\limits_{m(B_n)=x} x \right) = 0$.

Since $A \cup B_n \searrow A \cup B$, $\lim\limits_{n} \left(\sup\limits_{m(A\cup B_n)=y} y \right) = \sup\limits_{m(A\cup B)=x} x = \sup\limits_{m(A)=x} x$, by the continuity and the null-additivity.

Definition 3.6. Subadditive fuzzy measure

A fuzzy measure m is said to be subadditive if

$$\sup_{m(A\cup B)=x} x \leq \left(\sup_{m(A)=y} y \right) + \left(\sup_{m(B)=z} z \right) \text{ whenever } A, B \in \Omega \text{ and } A \cup B \in \Omega .$$

Definition 3.7. Superadditive fuzzy measure

A fuzzy measure m is said to be superadditive if

$\sup_{m(A\cup B)=x} x \geq \left(\sup_{m(A)=y} y\right) + \left(\sup_{m(B)=z} z\right)$ whenever $A, B \in \Omega$ and $A \cup B \in \Omega$ and $A \cap B = \phi$.

Example 3.8. The fuzzy measure m in the example 3.2 is subadditive and also superadditive.

Example 3.9. Consider the example given in [1].
Let $X = \{a, b, c\}$, and $\Omega = P(X) = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$,
where
$S_1 = \phi,\ S_2 = \{a\},\ S_3 = \{b\},\ S_4 = \{c\},\ S_5 = \{a, b\}, S_6 = \{a, c\}, S_7 = \{b, c\}, S_8 = X$
Let the fuzzy relation $m : \Omega \to [0, \infty]$ be defined as

$$m(S) = \left\{ x \ / \ 0 \leq x \leq \left(\frac{|S|}{3}\right)^2 + 0.2 \right\} \quad , \quad \forall S \in \Omega \text{ and } S \neq \phi \quad \text{and} \qquad m(\phi) = 0$$

and the membership function $\mu_m : m \to [0,1]$ be defined as

$$\mu_m(S, x) = \begin{cases} \dfrac{x}{\left(\dfrac{|S|}{3}\right)^2} & if \quad 0 \leq x \leq \left(\dfrac{|S|}{3}\right)^2 \\[2ex] \dfrac{\left(\dfrac{|S|}{3}\right)^2 + 0.2 - x}{0.2} & if \quad \left(\dfrac{|S|}{3}\right)^2 \leq x \leq \left(\dfrac{|S|}{3}\right)^2 + 0.2 \end{cases}$$

$for\ S \neq \phi$ and $\mu_m(\phi, 0) = 1$
The fuzzy measure m is not subadditive but it is superadditive.

Definition 3.10. Autocontinuous fuzzy measure
A fuzzy measure m is said to be autocontinuous from above (or from below) if

$\lim_n\left(\sup_{m(A\cup B_n)=y} y\right) = \sup_{m(A)=x} x \quad \left(\text{or } \lim_n\left(\sup_{m(A-B_n)=y} y\right) = \sup_{m(A)=x} x\right)$ whenever $A, B_n \in \Omega$,

n = 1, 2, 3,.... and $\lim_n\left(\sup_{m(B_n)=x} x\right) = 0$. m is autocontinuous if and only if it is both autocontinuous from above and autocontinuous from below.

Example 3.11. Consider the example given in [1].
Let $X = \{a_1, a_2, a_3\}$, and $\Omega = P(X) = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$, where
$S_1 = \phi, S_2 = \{a_1\}, S_3 = \{a_2\}, S_4 = \{a_3\}, S_5 = \{a_1, a_2\},\ S_6 = \{a_1, a_3\},$
$S_7 = \{a_2,\ a_3\}, \quad S_8 = X$
Let the fuzzy relation $m : \Omega \to [0, \infty]$ be defined as

$$m(S)=\begin{cases}1 \ \ if \ \ x_0\in S\\ 0 \ \ if \ \ x_0\notin S\ ,\end{cases} \qquad \forall S\in\Omega \quad \text{and} \quad x_0 \ \text{ is a fixed point in } \ X$$

and the membership function $\mu_m : m \to [0,1]$ be defined as

$$\mu_m(S,x)=1-\frac{1}{|S|+x} \quad , for \ \ S\neq\phi \quad \text{and} \quad \mu_m(\phi,0)=1$$

The fuzzy measure m is auto continuous.

Theorem 3.12. If a fuzzy measure m is autocontinuous from above, or autocontinuous from below, then it is null-additive.

Proof: Let $A,B\in\Omega$, and $\left(\sup\limits_{m(B)=x} x\right)=0$. Take $B_n=B,\ n=1,2,3,\ldots$.

Then $\lim\limits_n\left(\sup\limits_{m(B_n)=x} x\right)=\left(\sup\limits_{m(B)=x} x\right)=0$.

Suppose that m is autocontinuous from above, we have

$$\sup_{m(A\cup B)=x} x=\lim_n\left(\sup_{m(A\cup B_n)=y} y\right)=\sup_{m(A)=x} x\ .$$

Hence m is null-additive.

Suppose that m is autocontinuous from below,

$$\sup_{m(A\cup B)=x} x=\lim_n\left(\sup_{m((A\cup B)-B_n)=y} y\right)=\sup_{m(A)=x} x\,.$$

Hence m is null-additive.

Theorem 3.13. A fuzzy measure m is autocontinuous if and only if

$$\lim_n\left(\sup_{m(A\Delta B_n)=y} y\right)=\sup_{m(A)=x} x \qquad \text{whenever } A\in\Omega,\{B_n\}\subset\Omega\ ,\ \lim_n\left(\sup_{m(B_n)=x} x\right)=0\,.$$

Proof: Assume that m is autocontinuous.

For any $A\in\Omega,\{B_n\}\subset\Omega$ with $\lim\limits_n\left(\sup\limits_{m(B_n)=x} x\right)=0$,

$A-B_n\subset A\Delta B_n\subset A\cup B_n$. By the monotonicity of m,

$$\sup_{m(A-B_n)=y} y\ \le\ \sup_{m(A\Delta B_n)=y} y\le\ \sup_{m(A\cup B_n)=y} y\,.$$

Hence, we get $\lim\limits_n\left(\sup\limits_{m(A\Delta B_n)=y} y\right)=\sup\limits_{m(A)=x} x$, by autocontinuity.

Conversely, assume that, $\lim\limits_n\left(\sup\limits_{m(A\Delta B_n)=y} y\right)=\sup\limits_{m(A)=x} x$.

For any $A\in\Omega,\{B_n\}\subset\Omega$ with $\lim\limits_n\left(\sup\limits_{m(B_n)=x} x\right)=0$, we have

$B_n-A\in\Omega$ and $\sup\limits_{m(B_n-A)=y} y\ \le\ \sup\limits_{m(B_n)=y} y$.

Therefore, $\lim_{n}\left(\sup_{m(B_n-A)=y} y\right)=0$.

$$\Rightarrow \lim_{n}\left(\sup_{m(A\cup B_n)=y} y\right)=\lim_{n}\left(\sup_{m(A\Delta(B_n-A))=y} y\right)=\sup_{m(A)=x} x .$$

Hence, m is autocontinuous from above.

Also, we have $B_n \cap A \subset B_n$ and $\sup_{m(B_n\cap A)=y} y \le \sup_{m(B_n)=y} y$. Therefore, $\lim_{n}\left(\sup_{m(B_n\cap A)=y} y\right)=0$.

$$\Rightarrow \lim_{n}\left(\sup_{m(A-B_n)=y} y\right)=\lim_{n}\left(\sup_{m(A\Delta(B_n\cap A))=y} y\right)=\sup_{m(A)=x} x .$$

Hence m is autocontinuous from below. So, m is autocontinuous.

Definition 3.14. Uniformly autocontinuous fuzzy measure

A fuzzy measure m is said to be uniformly autocontinuous from above (or uniformly autocontinuous from below) if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\left(\text{or } \left(\sup_{m(A)=x} x\right)-\varepsilon \le \sup_{m(A-B)=y} y \le \left(\sup_{m(A)=x} x\right)+\varepsilon\right)$$

whenever $A, B \in \Omega$, and $\left|\sup_{m(B)=y} y\right| \le \delta$.

m is uniformly autocontinuous if and only if it is both uniformly autocontinuous from above and uniformly autocontinuous from below.

Theorem 3.15. If m is a fuzzy measure, then the following statements are equivalent:

(i) m is uniformly autocontinuous

(ii) m is uniformly autocontinuous from above

(iii) m is uniformly autocontinuous from below

(iv) for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\left(\sup_{m(A)=x} x\right)-\varepsilon \le \sup_{m(A\Delta B)=y} y \le \left(\sup_{m(A)=x} x\right)+\varepsilon$$

whenever $A, B \in \Omega$, and $\sup_{m(B)=y} y \le \delta$.

Proof :

(i) $\Rightarrow$ (ii): Clearly by definition.

(ii) $\Rightarrow$ (iii): Since $\sup_{m(A\cap B)=y} y \le \sup_{m(B)=y} y \le \delta$,

$$\left(\sup_{m(A)=x} x\right)=\sup_{m((A-B)\cup(A\cap B))=x} x \le \left(\sup_{m(A-B)=y} y\right)+\varepsilon .$$

(iii) $\Rightarrow$ (iv): Since $\sup_{m(A\cap B)=y} y \le \sup_{m(B)=y} y \le \delta$,

$$\left(\sup_{m(A\Delta B)=y} y\right)=\sup_{m((A\cup B)-(A\cap B))=y} y \ge \left(\sup_{m(A\cup B)=y} y\right)-\varepsilon \ge \left(\sup_{m(A)=x} x\right)-\varepsilon .$$

On the other hand, since

$$\sup_{m(B-A)=y} y \le \sup_{m(B)=y} y \le \delta, \ \sup_{m(A)=x} x \ge \sup_{m((A-B))=y} y = \sup_{m((A\Delta B)-(B-A))=y} y \ge \left(\sup_{m(A\Delta B)=y} y \right) - \varepsilon \ .$$

Therefore $\left(\sup_{m(A)=x} x \right) - \varepsilon \le \sup_{m(A\Delta B)=y} y \le \left(\sup_{m(A)=x} x \right) + \varepsilon$.

(iv) $\Rightarrow$ (i): Clearly by assumption.

4. Finite fuzzy measure

Definition 4.1. Finite fuzzy measure

A fuzzy measure m is said to be finite if $\left(\sup_{m(A)=x} x \right) < \infty$ for any $A \in \Omega$

Definition 4.2. σ -Finite fuzzy measure

A fuzzy measure m is said to be σ - finite if

$$\left(\sup_{m(A_n)=x} x \right) < \infty \ \text{ for any } n = 1, 2, 3, ... \text{ and } X = \bigcup_{n=1}^{\infty} A_n$$

Example 4.3. The fuzzy measure in the example 3.2 is finite fuzzy measure and also σ - finite fuzzy measure.

Theorem 4.4. If m is a finite fuzzy measure, then we have

$\lim_n \left(\sup_{m(A_n)=x} x \right) = \sup_{m\left(\lim_n A_n \right)=x} x$, for any sequence $\{A_n\}$ for which $\lim_n A_n$ exists.

Proof: Let $\{A_n\}$ be a sequence of sets in Ω with $\lim_n A_n$ exists and

$$A = \lim_n A_n = \limsup_n A_n = \liminf_n A_n .$$

Case (i): If $\{A_n\}$ is a monotone sequence of sets in Ω with $\lim_n A_n$ exists, then we get $\lim_n \left(\sup_{m(A_n)=x} x \right) = \sup_{m\left(\lim_n A_n \right)=x} x$ by continuity condition of fuzzy measure.

Case (ii): If $\{A_n\}$ is not a monotone sequence of sets in Ω with $\lim_n A_n$ exists, then we have

$$\begin{aligned} \sup_{m(A)=x} x &= \sup_{m\left(\limsup_n A_n \right)=x} x = \limsup_n \left(\sup_{m\left(\bigcup_{k=n}^{\infty} A_k \right)=x} x \right) \\ &\ge \limsup_n \left(\sup_{m(A_n)=x} x \right) \\ &\ge \liminf_n \left(\sup_{m(A_n)=x} x \right) \end{aligned}$$

$$\geq \liminf_n \left(\sup_{m\left(\bigcap_{k=n}^{\infty} A_k\right)=x} x \right)$$

$$= \sup_{m\left(\liminf_n A_n\right)=x} x \quad = \sup_{m(A)=x} x$$

Hence $\lim_n \left(\sup_{m(A_n)=x} x \right) = \sup_{m\left(\lim_n A_n\right)=x} x$.

Definition 4.5. Exhaustive fuzzy measure

A fuzzy measure m is called exhaustive if $\lim_n \left(\sup_{m(A_n)=x} x \right) = 0$ for any disjoint sequence $\{A_n\}$.

Example 4.6. The fuzzy measure in the example 3.2 is also exhaustive fuzzy measure.

Theorem 4.7. If m is a finite upper semi continuous fuzzy measure, then it is exhaustive.
Proof: Let $\{A_n\}$ be a disjoint sequence of sets in Ω

and $B_n = \bigcup_{k=n}^{\infty} A_k$. Then $\{B_n\}$ is a decreasing sequence and

$\lim_n B_n = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \limsup_n A_n = \phi$.

By the finiteness and the continuity from above of m, we have

$$\lim_n \left(\sup_{m(B_n)=x} x \right) = \sup_{m\left(\lim_n B_n\right)=x} x = \sup_{m(\phi)=x} x = 0$$

Since $B_n = \bigcup_{k=n}^{\infty} A_k$,

$$0 \leq \sup_{m(A_n)=x} x \leq \sup_{m(B_n)=y} y \Rightarrow \lim_n \left(\sup_{m(A_n)=x} x \right) \leq \lim_n \left(\sup_{m(B_n)=y} y \right) = 0 \Rightarrow \lim_n \left(\sup_{m(A_n)=x} x \right) = 0 .$$

Hence m is exhaustive.

5. Conclusion

We have defined some structural characteristics such as null-additivity, autocontinuity and uniform autocontinuity and some types of fuzzy measure like finite, and σ - Finite fuzzy measure. We have also established the relationship between the characteristics. Further investigations can be made to define more characteristics and to study the relationships between them.

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