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# Homomorphisms and *L*-Fuzzy Ideals in Universal Algebras

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***Abstract.*** In this paper, we study the homomorphic images (respectively pre-images) of $L$-fuzzy ideals, $L$ −fuzzy prime ideals, maximal $L$ −fuzzy ideals and $L$-fuzzy semi-prime ideals of universal algebras.

***Keywords:*** Homomorphisms; $L$ −fuzzy ideals; $L$ −fuzzy prime ideals; maximal $L$ −fuzzy ideals; $L$ −fuzzy semi-prime ideals.

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## 1. Introduction

The theory of fuzzy sets introduced by Zadeh [22] has evoked tremendous interest among researchers working in different branches of mathematics. Rosenfield in his pioneering paper [15] introduced the notions of fuzzy subgroups of a group. Since then, many researches have been studying fuzzy subalgebras of several algebraic structures (see [7, 12, 13, 14, 19]). As suggested by Gougen [9], the unit interval [0,1] is not sufficient to take the truth values of general fuzzy statements. U. M. Swamy and D. V. Raju [17, 18] studied the general theory of algebraic fuzzy systems by introducing the notion of a fuzzy $\mathfrak{L}-$ subset of a set $X$ corresponding to a given class $\mathfrak{L}$ of subsets of $X$ having truth values in a complete lattice satisfying the infinite meet distributive law.

Ideals in universal algebras have been studied in a series of papers [4, 5, 6, 11, 20] as a generalization of those familiar structures: normal subgroups (in groups), normal subloops (in loops), ideals (in rings), submodules (in modules), subspaces (in vector spaces) and filters (in implication algebras or Heyting algebras).

In [1], we have introduced the concept $L$ −fuzzy ideals in universal algebras and we gave a necessary and sufficient condition for a variety of algebras to be an ideal determined. In [2], we study $L$ −fuzzy prime idelas and maximal $L$ −fuzzy ideals of universal algebras and gave an internal characterization for $L$ − fuzzy prime idelas analogous to the characterization of Swamy and Swamy [16] in the case of rings. In [3], we continued our study and we define $L$ −fuzzy semi-prime ideals and the radical of $L$ −fuzzy ideals in universal algebras in the frame work of $L$ −fuzzy ideals given in [1]. In the present paper, we study the image and pre-image of $L$-fuzzy ideals of of universal algebras under a homomorphism. We make a theoretical study on their properties and give several characterizing theorems.

**2. Preliminaries**

This section contains some definitions and results which will be used in the paper. We refer to the readers [8, 10], for the standard concepts in universal algebras. Throughout this paper $A \in \mathcal{K}$, where $\mathcal{K}$ is a class of algebras of a fixed type $\Omega$ and assume that there is an equationaly definable constant in all algebras of $\mathcal{K}$ denoted by $0$. For a positive integer $n$, we write $\overleftarrow{a}$ to denote the $n$ −tuple $\langle a_1, a_2, \dots, a_n\rangle \in A^n$.

**Definition 2.1.** *[11] A term $P(\overleftarrow{x}, \overleftarrow{y})$ is said to be an ideal term in $\overleftarrow{y}$ if and only if* $P(\overleftarrow{x}, \overleftarrow{0}) = 0$.

**Definition 2.2.** *[11] A nonempty subset $I$ of $A$ is called an ideal of $A$ if and only if $P(\overleftarrow{a}, \overleftarrow{b}) \in I$ for all $\overleftarrow{a} \in A^n, \overleftarrow{b} \in I^m$ and any ideal term $P(\overleftarrow{x}, \overleftarrow{y})$ in $\overleftarrow{y}$.*

We denote the class of all ideals of $A$, by $\mathcal{J}(A)$.

**Definition 2.3.** *[11, 20] A term $t(\overleftarrow{x}, \overleftarrow{y}, \overleftarrow{z})$ is said to be a commutator term in $\overleftarrow{y}, \overleftarrow{z}$ if and only if it is an ideal term in $\overleftarrow{y}$ and an ideal term in $\overleftarrow{z}$.*

**Definition 2.4.** *[11] In an ideal determined variety, the commutator $[I, J]$ of ideals I and $J$ is the zero congruence class of the commutator congruence $[I^\delta, J^\delta]$.*

It is characterized in [11] as follows:

**Theorem 2.5.** *[11, 20] In an ideal determined variety,*

$$[I, J] = \{t(\overleftarrow{a}, \overleftarrow{\imath}, \overleftarrow{\jmath}) : \overleftarrow{a} \in A^n, \overleftarrow{\imath} \in I^m and \overleftarrow{\jmath} \in J^k where t(\overleftarrow{x}, \overleftarrow{y}, \overleftarrow{z}) \text{ is a commutator term in } \overleftarrow{y}, \overleftarrow{z}\}$$

For subsets $H, G$ of $A$, $[H, G]$ denotes the product $[\langle H\rangle, \langle G\rangle]$. In particular, for $a, b \in A$, $[\langle a\rangle, \langle b\rangle]$ is denoted by $[a, b]$.

**Definition 2.6.** *[20] A proper ideal $P$ of $A$ is called prime if and only if for all $I, J \in \mathcal{J}(A)$:*

$$[I, J] \subseteq P \Rightarrow either I \subseteq P or J \subseteq P$$

**Theorem 2.7.** *[20] A proper ideal $P$ of $A$ is prime if and only if:*

$$[a, b] \subseteq P \Rightarrow either a \in P or b \in P$$

for all $a, b \in A$.

**Definition 2.8.** *[20] An ideal $Q$ of $A$ is called semiprime if and only if for all $I \in \mathcal{J}(A)$:*

$$[I, I] \subseteq Q \Rightarrow I \subseteq Q$$

**Definition 2.9.** *[20] The prime radical of an ideal $I$ of $A$, denoted by $\sqrt{I}$ is the intersection of all prime ideals of $A$ containing $I$.*

Throughout this paper $L = (L, \wedge, \vee, 0, 1)$ is a complete Brouwerian lattice; i.e., $L$ is a complete lattice satisfying the infinite meet distributive law. By an $L$ −fuzzy subset of $A$, we mean a mapping $\mu: A \to L$. For each $\alpha \in L$, the $\alpha$ −level set of $\mu$ denoted by $\mu_\alpha$ is a subset of $A$ given by:

$$\mu_\alpha = \{x \in A: \alpha \leq \mu(x)\}$$

For fuzzy subsets $\mu$ and $\nu$ of $A$, we write $\mu \leq \nu$ to mean $\mu(x) \leq \nu(x)$ in the ordering of $L$.

**Definition 2.10.** *[21] For each $x \in A$ and $0 \neq \alpha$ in $L$, the fuzzy subset $x_\alpha$ of $A$ given by:*

$$x_\alpha(z) = \begin{cases} \alpha & if z = x \\ 0 & otherwise \end{cases}$$

is called the fuzzy point of $A$. In this case $x$ is called the support of $x_\alpha$ and $\alpha$ its value.
For a fuzzy subset $\mu$ of $A$ and a fuzzy point $x_\alpha$ of $A$, we write $x_\alpha \in \mu$ whenever $\mu(x) \geq \alpha$.

**Definition 2.11.** *[1] An $L$ −fuzzy subset $\mu$ of $A$ is said to be an $L$ −fuzzy ideal of $A$ (or shortly a fuzzy ideal of $A$) if and only if the following conditions are satisfied:*

1. $\mu(0) = 1$, and
2. If $P(\overleftarrow{x}, \overleftarrow{y})$ is an ideal term in $\overleftarrow{y}$ and $\overleftarrow{a} \in A^n, \overleftarrow{b} \in A^m$, then

$$\mu(P(\overleftarrow{a}, \overleftarrow{b})) \geq \mu^m(\overleftarrow{b})$$

We denote by $\mathcal{FI}(A)$, the class of all fuzzy ideals of $A$.

**Definition 2.12.** *[2] The commutator of fuzzy ideals $\mu$ and $\sigma$ of $A$ denoted by $[\mu, \sigma]$ is a fuzzy subset of $A$ defined by:*

$$[\mu, \sigma](x) = \vee \{\alpha \wedge \beta: \alpha, \beta \in L, x \in [\mu_\alpha, \sigma_\beta]\}$$

$$= \vee \{\lambda \in L: x \in [\mu_\lambda, \sigma_\lambda]\}$$

for all $x \in A$.

**Theorem 2.13.** *[2] For each $x \in A$, and fuzzy ideals $\mu$ and $\sigma$ of $A$:*

$$[\mu, \sigma](x) = \vee \{\mu^m(\overleftarrow{b}) \wedge \sigma^k(\overleftarrow{c}): x = t(\overleftarrow{a}, \overleftarrow{b}, \overleftarrow{c}), where\ \overleftarrow{a} \in A^n, \overleftarrow{b} \in A^m, \overleftarrow{c} \in A^k,$$
$$and t(\overleftarrow{x}, \overleftarrow{y}, \overleftarrow{z}) is\ a\ commutator\ term\ in\ \overleftarrow{y}, \overleftarrow{z}\}$$

**Definition 2.14.** *[2] A non-constant fuzzy ideal $\mu$ of $A$ is called a fuzzy prime ideal if and only if:*

$$[\nu, \sigma] \leq \mu \Rightarrow either\ \nu \leq \mu\ or\ \sigma \leq \mu$$

for all $\nu, \sigma \in \mathcal{FI}(A)$.

**Theorem 2.15.** *[2] A non-constant fuzzy ideal $\mu$ is a fuzzy prime ideal if and only if $Img(\mu) = \{1, \alpha\}$, where $\alpha$ is a prime element in $L$ and the set*

$$\mu_* = \{x \in A: \mu(x) = 1\}$$

*is a prime ideal of $A$.*

**Definition 2.16.** *[3] A fuzzy ideal $\mu$ of $A$ is called fuzzy semi-prime if:*

$$[\theta, \theta] \leq \mu \Rightarrow \theta \leq \mu$$

for all $\theta \in \mathcal{FI}(A)$.

According to [20], the prime radical of an ideal $I$ of $A$, denoted by $\sqrt{I}$ is the intersection of all prime ideals of $A$ containing $I$. Here we define the prime radical of fuzzy ideals using their level ideals.

**Definition 2.17.** *[3] For a fuzzy ideal $\mu$ of $A$, its prime radical of $\mu$ denoted by $\sqrt{\mu}$ is defined as a fuzzy subset of $A$ such that, for each $x \in A$:*

$$\sqrt{\mu}(x) = \alpha \text{ if and only if } x \in \sqrt{\mu_\alpha} \text{ and } x \notin \sqrt{\mu_\beta} \text{ for all } \beta > \alpha \text{ in } L.$$

**Lemma 2.18.** *[3] Let $\mu$ be a fuzzy ideal of $A$ and $x \in A$. Then*

$$\sqrt{\mu}(x) = \vee \{\alpha \in L : x \in \sqrt{\mu_\alpha}\}$$

**3. Homomorphisms and fuzzy ideals**

Let $A$ and $B$ be algebras of the same type $\Omega$. A mapping $h: A \to B$ is called a homomorphism from $A$ to $B$ if:

$$h(f^A(a_1, a_2, \ldots, a_n)) = f^B(h(a_1), h(a_2), \ldots, h(a_n))$$

for each $n-$ary operation $f$ in $\Omega$ and each sequence $a_1, a_2, \ldots, a_n$ from $A$. It is observed that if $p$ is an $n-$ary term of type $\Omega$, then

$$h(f^A(a_1, a_2, \ldots, a_n)) = p^B(h(a_1), h(a_2), \ldots, h(a_n))$$

for all $a_1, a_2, \ldots, a_n \in A$.

**Theorem 3.1.** *Let $h: A \to B$ be a homomorphism. Then we have the following:*

1. If $\sigma$ is a fuzzy ideal of $B$, then $h^{-1}(\sigma)$ is a fuzzy ideal of $A$
2. If $\mu$ is a fuzzy ideal of $A$ and $h$ is surjective, then $h(\mu)$ is a fuzzy ideal of $B$.

**Proof:** Let $h: A \to B$ be a homomorphism.

1) Suppose that $\sigma$ is a fuzzy ideal of $B$ and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in A$. Then $h(a_1), h(a_2), \ldots, h(a_n), h(b_1), h(b_2), \ldots, h(b_m) \in B$. If $p(\vec{x}, \vec{y})$ is an $n+m$ ideal term in $\vec{y}$, then we get:

$$\sigma(p^B(h(a_1), h(a_2), \ldots, h(a_n), h(b_1), h(b_2), \ldots, h(b_m))) \geq \sigma(h(b_1)) \wedge \ldots \wedge \sigma(h(b_m))$$

Now consider the following:

$$\begin{aligned} h^{-1}(\sigma)(p^A(a_1, \ldots, a_n, b_1, \ldots, b_m)) &= \sigma(h(p^A(a_1, \ldots, a_n, b_1, \ldots, b_m))) \\ &= \sigma(p^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m))) \\ &= \sigma(h(b_1)) \wedge \ldots \wedge \sigma(h(b_m)) \\ &= h^{-1}(\sigma)(b_1) \wedge \ldots \wedge h^{-1}(\sigma)(b_m) \end{aligned}$$

Therefore $h^{-1}(\sigma)$ is a fuzzy ideal of $A$.

2) Suppose that $h$ is surjective and let $\mu$ be a fuzzy ideal of $A$. If $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m \in B$, then there exist $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in A$ such that $h(a_i) = u_i$ and $h(b_j) = v_j$ for all $i, j$. If $p(\vec{x}, \vec{y})$ is an $n+m$ ideal term in $\vec{y}$, then we get:

$$\begin{aligned} h(p^A(a_1, \ldots, a_n, b_1, \ldots, b_m)) &= p^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m)) \\ &= p^B(u_1, \ldots, u_n, v_1, \ldots, v_m) \end{aligned}$$

So that $p^A(a_1, \ldots, a_n, b_1, \ldots, b_m) \in h^{-1}(p^B(u_1, \ldots, u_n, v_1, \ldots, v_m))$. Now consider the following:

$$h(\mu)(p^B((u_1, \ldots, u_n, v_1, \ldots, v_m)) = \vee \{\mu(a) : a \in h^{-1}(p^B((u_1, \ldots, u_n, v_1, \ldots, v_m))\}$$

$$\geq \mu(p^A(a_1,\ldots,a_n,b_1,\ldots,b_m))$$
$$\geq \mu(b_1)\wedge\ldots\wedge\mu(b_m)$$

Since $b_j$ is arbitrary in $h^{-1}(v_j)$ for all $j=1,2,\ldots,m$, it follows that $h(\mu)(p^B((u_1,\ldots,u_n,v_1,\ldots,v_m)) \geq \bigvee_{b_1\in h^{-1}(v_1)} \mu(b_1)\wedge\ldots\wedge \bigvee_{b_m\in h^{-1}(v_m)} \mu(b_m)$

$$= h(\mu)(v_1)\wedge\ldots\wedge h(\mu)(v_m)$$

Therefore $h(\mu)$ is a fuzzy ideal of $B$.

**Theorem 3.2.** *Let $h: A\to B$ be a homomorphism, $\mu$ and $\nu$ be fuzzy ideals of $A$. Then*

$$h(\mu\vee\nu)=h(\mu)\vee h(\nu)$$

**Proof:** We show that $h(\mu\vee\nu)$ is the smallest fuzzy ideal of $B$ containing both $h(\mu)$ and $h(\nu)$. By Theorem 3.1, $h(\mu\vee\nu)$ is a fuzzy ideal of $B$. Now let $y\in B$. If $h^{-1}(y)=\emptyset$, then $h(\mu)(y)=0\leq h(\mu\vee\nu)(y)$. Also if $h^{-1}(y)\neq\emptyset$, then consider the following:

$$h(\mu)(y) = \vee\{\mu(x): x\in h^{-1}(y)\}$$
$$\leq \vee\{(\mu\vee\nu)(x): x\in h^{-1}(y)\}$$
$$= h(\mu\vee\nu)(y)$$

So that $h(\mu)\leq h(\mu\vee\nu)$. Similarly, we can verify that $h(\nu)\leq h(\mu\vee\nu)$. Now for any fuzzy ideal $\eta$ of $B$:

$$h(\mu)\leq\eta, h(\nu)\leq\eta \Rightarrow h^{-1}(h(\mu))\leq h^{-1}(\eta), h^{-1}(h(\nu))\leq h^{-1}(\eta)$$
$$\Rightarrow \mu\leq h^{-1}(\eta), \nu\leq h^{-1}(\eta)$$
$$\Rightarrow \mu\vee\nu\leq h^{-1}(\eta)$$
$$\Rightarrow h(\mu\vee\nu)\leq h(h^{-1}(\eta))\leq\eta$$

Therefore $h(\mu\vee\nu)$ is the smallest fuzzy ideal of $B$ containing both $h(\mu)$ and $h(\nu)$. So that, $h(\mu\vee\nu)=h(\mu)\vee h(\nu)$.

**Theorem 3.3.** *Let $h: A\to B$ be a homomorphism, and $\mu$ and $\nu$ be fuzzy ideals of $A$. Then*

$$h(\mu\wedge\nu)\leq h(\mu)\wedge h(\nu)$$

Moreover, if either $\mu$ or $\nu$ is $h-$invariant, then the equality holds.

**Proof:** Let $y$ be any element in $B$. If $h^{-1}(y)=\emptyset$, then $h(\mu)(y)=0=h(\nu)(y)=h(\mu\wedge\nu)(y)$. Let $h^{-1}(y)\neq\emptyset$. Then consider the following:

$$h(\mu\wedge\nu)(y) = \vee\{(\mu\wedge\nu)(x): x\in h^{-1}(y)\}$$
$$= \vee\{\mu(x)\wedge\nu(x): x\in h^{-1}(y)\}$$
$$\leq \vee\{\mu(a)\wedge\nu(b): a,b\in h^{-1}(y)\}$$
$$= \vee\{\mu(a): a\in h^{-1}(y)\}\wedge\vee\{\nu(b): b\in h^{-1}(y)\}$$
$$= h(\mu)(y)\wedge h(\nu)(y)$$

Therefore $h(\mu\wedge\nu)\leq h(\mu)\wedge h(\nu)$. Moreover, assume without loss of generality that $\mu$ is $h-$invariant. Then $\mu(a)=\mu(b)$, whenever $h(a)=h(b)$. Now for each $y\in B$, with $h^{-1}(y)\neq\emptyset$, consider the following:

$$h(\mu)(y)\wedge h(\nu)(y) = \vee\{\mu(a): a\in h^{-1}(y)\}\wedge\vee\{\nu(b): b\in h^{-1}(y)\}$$
$$= \vee\{\mu(a)\wedge\nu(b): a,b\in h^{-1}(y)\}$$
$$= \vee\{\mu(x)\wedge\nu(x): x\in h^{-1}(y)\}$$
$$= \vee\{(\mu\wedge\nu)(x): x\in h^{-1}(y)\}$$

$= h(\mu \wedge \nu)(y)$

Therefore $h(\mu \wedge \nu) = h(\mu) \wedge h(\nu)$.

**Theorem 3.4.** *Let $h: A \rightarrow B$ be a homomorphism, and and $\sigma$ and $\theta$ be fuzzy ideals of $B$. Then*

$$h^{-1}(\sigma) \vee h^{-1}(\theta) \leq h^{-1}(\sigma \vee \theta)$$

Moreover, the equality holds whenever $h$ is surjective.

**Proof:** For each $x \in A$, consider:

$$\begin{aligned} h^{-1}(\sigma)(x) &= \sigma(h(x)) \\ &\leq (\sigma \vee \theta)(h(x)) \\ &= h^{-1}(\sigma \vee \theta)(x) \end{aligned}$$

So that $h^{-1}(\sigma) \leq h^{-1}(\sigma \vee \theta)$. Similarly it can be verified that $h^{-1}(\theta) \leq h^{-1}(\sigma \vee \theta)$. Therefore $h^{-1}(\sigma) \vee h^{-1}(\theta) \leq h^{-1}(\sigma \vee \theta)$. Further, let we assume that $h$ is surjective. To prove the equality, it is enough if we show that $h^{-1}(\sigma \vee \theta)$ is the smallest fuzzy ideal of $A$ containing both $\sigma$ and $\theta$. From Theorem 3.1 we have that $h^{-1}(\sigma \vee \theta)$ is a fuzzy ideal of $A$. From the above inequality also we have $h^{-1}(\sigma) \leq h^{-1}(\sigma \vee \theta)$ and $h^{-1}(\theta) \leq h^{-1}(\sigma \vee \theta)$. Now let $\mu$ be any other fuzzy ideal of $A$ such that $h^{-1}(\sigma) \leq \mu$ and $h^{-1}(\theta) \leq \mu$. Then $h(h^{-1}(\sigma)) \leq h(\mu)$ and $h(h^{-1}(\theta)) \leq h(\mu)$. Since $h$ is surjective, it follows that $\sigma \leq h(\mu)$ and $\theta \leq h(\mu)$. So that, $\sigma \vee \theta \leq h(\mu)$, which gives $h^{-1}(\sigma \vee \theta) \leq h^{-1}(h(\mu))$. Our aim is to show that $h^{-1}(\sigma \vee \theta) \leq \mu$. Suppose not. Then there exists $a \in A$ such that $h^{-1}(\sigma \vee \theta)(a) > \mu(a)$. If we put $z = h(a)$, then we get $(\sigma \vee \theta)(z) > h(\mu)(z)$, which is a contradiction. Therefore $h^{-1}(\sigma \vee \theta) \leq \mu$ and hence the equality holds.

**Theorem 3.5.** *Let $h: A \rightarrow B$ be a homomorphism, and $\sigma$ and $\theta$ be fuzzy ideals of $B$. Then*

$$h^{-1}(\sigma \wedge \theta) = h^{-1}(\sigma) \wedge h^{-1}(\theta)$$

**Proof:** For each $a \in A$, consider the following:

$$\begin{aligned} h^{-1}(\sigma \wedge \theta)(a) &= (\sigma \wedge \theta)(h(a)) \\ &= \sigma(h(a)) \wedge \theta(h(a)) \\ &= h^{-1}(\sigma)(a) \wedge h^{-1}(\theta)(a) \\ &= (h^{-1}(\sigma) \wedge h^{-1}(\theta))(a) \end{aligned}$$

Therefore $h^{-1}(\sigma \wedge \theta) = h^{-1}(\sigma) \wedge h^{-1}(\theta)$.

**Theorem 3.6.** *Let $h: A \rightarrow B$ be a surjective homomorphism. For any $h$-invariant fuzzy subset $\mu$ of $A$, we have:*

$$h(\langle\mu\rangle) = \langle h(\mu)\rangle$$

**Proof:** For any $y \in B$, consider:

$$\begin{aligned} h(\langle\mu\rangle)(y) &= \vee \{\langle\mu\rangle(x) : x \in h^{-1}(y)\} \\ &= \vee \{\vee \{\alpha \in L : x \in \langle\mu_\alpha\rangle\} : x \in h^{-1}(y)\} \\ &= \vee \{\alpha \in L : x \in \langle\mu_\alpha\rangle and h(x) = y\} \\ &= \vee \{\alpha \in L : y \in h(\langle\mu_\alpha\rangle)\} \end{aligned}$$

on the other hand

$$\langle h(\mu)\rangle(y) = \vee \{\alpha \in L : y \in \langle h(\mu)_\alpha\rangle\}$$

Now it is enough to show that

$$h(\langle\mu_\alpha\rangle) = \langle h(\mu)_\alpha\rangle$$

Let $z \in h(\langle\mu_\alpha\rangle)$. Then $z = h(x)$ for some $x \in \langle\mu_\alpha\rangle$. There exists $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_m \in \mu_\alpha$ and an ideal term $p(\overleftarrow{x}, \overleftarrow{y})$ in $\overleftarrow{y}$ such that $x = p^A(a_1, \ldots, a_n \in A$, $b_1, \ldots, b_m)$. So,

$$\begin{aligned} z &= h(x) \\ &= h(p^A(a_1, \ldots, a_n, b_1, \ldots, b_m)) \\ &= p^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m)) \end{aligned}$$

For each $j = 1,2, \ldots, m$ we have

$$h(\mu)(h(b_j)) = \vee\{\mu(x): x \in h^{-1}(b_j)\}$$

Since $\mu$ is $h$-invariant and each $b_j \in \mu_\alpha$, we get

$$h(\mu)(h(b_j)) = \mu(b_j) \geq \alpha$$

for all $j = 1,2, \ldots, m$; that is $h(b_j) \in h(\mu)_\alpha$ for all $j$ and $z = p^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m))$. This means $z \in \langle h(\mu)_\alpha\rangle$. So that

$$h(\langle\mu_\alpha\rangle) \subseteq \langle h(\mu)_\alpha\rangle$$

To prove the other inclusion, let $z \in \langle h(\mu)_\alpha\rangle$. Then $z = p^B(\overleftarrow{u}, \overleftarrow{v})$ for some $u_1, \ldots, u_n \in B$, $v_1, \ldots, v_m \in h(\mu)_\alpha$ and some ideal term $p(\overleftarrow{x}, \overleftarrow{y})$ in $\overleftarrow{y}$. Since $h$ is surjective, there exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$ such that $h(a_i) = u_i$ and $h(b_j) = v_j$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. As each $v_j \in h(\mu)_\alpha$, we have $h(\mu)(h(b_j)) \geq \alpha$. Since $\mu$ is $h$-invariant we get $\mu(b_j) \geq \alpha$; that is, $b_j \in \mu_\alpha$ for all $j$. Put $x = p^A(a_1, \ldots a_n, b_1, \ldots b_m)$. Then $x \in \langle\mu_\alpha\rangle$. Moreover

$$\begin{aligned} h(x) &= h(p^A(a_1, \ldots, a_n, b_1, \ldots, b_m)) \\ &= p^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m))) \\ &= p^B(u_1, \ldots, u_n, v_1, \ldots, v_m) \\ &= z \end{aligned}$$

That is, $z = h(x)$, where $x \in \langle\mu_\alpha\rangle$, which gives $z \in h(\langle\mu_\alpha\rangle)$. Thus $\langle h(\mu)_\alpha\rangle \subseteq h(\langle\mu_\alpha\rangle)$. Hence $h(\langle\mu_\alpha\rangle) = \langle h(\mu)_\alpha\rangle$ and this completes the proof.

## 4. Homomorphisms and fuzzy prime ideals

**Theorem 4.1.** *Let $h: A \to B$ be a surjective homomorphism.*

1. If $\mu$ and $\sigma$ are fuzzy ideals of $A$, then
$$h([\mu, \sigma]) = [h(\mu), h(\sigma)]$$

2. If $\sigma$ and $\theta$ are fuzzy ideals of $B$, then
$$[h^{-1}(\sigma), h^{-1}(\theta)] \leq h^{-1}([\sigma, \theta])$$

**Proof:** (1) Let $y \in B$. Since $h$ is assumed to be surjective, the set $h^{-1}(y)$ is always nonempty. By definition we have:

$$\begin{aligned} h([\mu, \eta])(y) &= \vee\{[\mu, \eta](x): x \in h^{-1}(y)\} \\ &= \vee\{\vee\{\mu^m(\overleftarrow{b}) \wedge \eta^k(\overleftarrow{c}): x = t^A(\overleftarrow{a}, \overleftarrow{b}, \overleftarrow{c})\}: x \in h^{-1}(y)\} \\ &= \vee\{\mu^m(\overleftarrow{b}) \wedge \eta^k(\overleftarrow{c}): y = h(t^A(\overleftarrow{a}, \overleftarrow{b}, \overleftarrow{c}))\} \end{aligned}$$

which gives

$$h([\mu, \eta])(y) \geq \mu^m(\overleftarrow{b}) \wedge \eta^k(\overleftarrow{c}) \tag{4.1}$$

for any $b_1, \ldots, b_m, c_1, \ldots, c_k \in A$, with $y = h(t^A(\overleftarrow{a}, \overleftarrow{b}, \overleftarrow{c}))$ for some commutator term $t(\overleftarrow{x}, \overleftarrow{y}, \overleftarrow{z})$ in $\overleftarrow{y}, \overleftarrow{z}$ and some $a_1, \ldots, a_n \in A$. Now let $y = t^B(\overleftarrow{u}, \overleftarrow{v}, \overleftarrow{w})$ be any expression of $y$ using commutator terms, where $u_1, \ldots, u_n, v_1, \ldots, v_m, w_1, \ldots, w_k \in B$. Since $h$ is surjective there exist $a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_k \in A$ such that $h(a_i) = u_i$, $h(b_j) = v_j$ and $h(c_r) = w_r$ for all $i = 1,2, \ldots, n$, $j = 1,2, \ldots, m$ and $r = 1,2, \ldots, k$. Equivalently, each $a_i \in h^{-1}(u_i)$, $b_j \in h^{-1}(v_j)$ and $c_r \in h^{-1}(w_r)$. Now consider:

$$\begin{aligned} h(t^A(\overleftarrow{a}, \overleftarrow{b}, \overleftarrow{c})) &= t^B(h(a_1), \ldots, h(a_a), h(b_1), \ldots, h(b_m), h(c_1), \ldots, h(c_k)) \\ &= t^B(u_1, \ldots, u_n, v_1, \ldots, v_m, w_1, \ldots, w_k) \\ &= t^B(\overleftarrow{u}, \overleftarrow{v}, \overleftarrow{w}) \\ &= y \end{aligned}$$

So, by eq. (4.1) we get

$$h([\mu, \eta])(y) \geq \mu^m(\overleftarrow{b}) \wedge \eta^k(\overleftarrow{c})$$

Since each $b_j$ (respectively $c_r$) is arbitrary in $h^{-1}(v_j)$ (respectively in $h^{-1}(w_r)$), it follows that

$$\begin{aligned} h([\mu, \eta])(y) &\geq \mu^m(\overleftarrow{b}) \wedge \eta^k(\overleftarrow{c}) \\ &= \mu(b_1) \wedge \ldots \wedge \mu(b_m) \wedge \eta(c_1) \wedge \ldots \wedge \eta(c_k) \\ &\geq \bigvee_{b_1 \in h^{-1}(v_1)} \mu(b_1) \wedge \ldots \wedge \bigvee_{b_m \in h^{-1}(v_m)} \mu(b_m) \wedge \bigvee_{c_1 \in h^{-1}(w_1)} \eta(c_1) \wedge \ldots \wedge \bigvee_{c_k \in h^{-1}(w_k)} \eta(c_k) \\ &= h(\mu)(v_1) \wedge \ldots \wedge h(\mu)(v_m) \wedge h(\eta)(w_1) \wedge \ldots \wedge h(\eta)(w_k) \\ &= h(\mu)^m(\overleftarrow{v}) \wedge h(\eta)^k(\overleftarrow{w}) \end{aligned}$$

This gives

$$\begin{aligned} h([\mu, \eta])(y) &\geq \bigvee \{h(\mu)^m(\overleftarrow{v}) \wedge h(\eta)^k(\overleftarrow{w}) : y = t^B(\overleftarrow{u}, \overleftarrow{v}, \overleftarrow{w})\} \\ &= [h(\mu), h(\eta)](y) \end{aligned}$$

Therefore $[h(\mu), h(\eta)] \leq h([\mu, \eta])$. To prove the other inequality, consider

$$[h(\mu), h(\eta)](y) = \bigvee \{h(\mu)^m(\overleftarrow{v}) \wedge h(\eta)^k(\overleftarrow{w}) : y = t^B(\overleftarrow{u}, \overleftarrow{v}, \overleftarrow{w})\}$$

So that

$$[h(\mu), h(\eta)](y) \geq h(\mu)^m(\overleftarrow{v}) \wedge h(\eta)^k(\overleftarrow{w}) \tag{4.2}$$

for all $v_1, \ldots, v_m, w_1, \ldots, w_k \in B$, with $y = t^B(\overleftarrow{u}, \overleftarrow{v}, \overleftarrow{w})\}$, for some commutator term $t(\overleftarrow{x}, \overleftarrow{y}, \overleftarrow{z})$ in $\overleftarrow{y}, \overleftarrow{z}$ and some $u_1, \ldots, u_n \in B$. Now let $y = h(t^A(\overleftarrow{a}, \overleftarrow{b}, \overleftarrow{c}))$ for some $a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_k \in A$ and commutator term $t(\overleftarrow{x}, \overleftarrow{y}, \overleftarrow{z})$ in $\overleftarrow{y}, \overleftarrow{z}$. That is,

$$y = t^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m)), h(c_1), \ldots, h(c_k))$$

By eq. (4.2) we get

$$[h(\mu), h(\eta)](y) \geq h(\mu)(h(b_1)) \wedge \ldots \wedge h(\mu)(h(b_m)) \wedge h(\eta)(h(c_1)) \wedge \ldots \wedge h(\eta)(h(c_k))$$

using the fact that $h(\mu)(h(a)) \geq \mu(a)$ for all $a \in A$, we get the following:

$$\begin{aligned} [h(\mu), h(\eta)](y) &\geq h(\mu)(h(b_1)) \wedge \ldots \wedge h(\mu)(h(b_m)) \wedge h(\eta)(h(c_1)) \wedge \ldots \wedge h(\eta)(h(c_k)) \\ &\geq \mu(b_1) \wedge \ldots \wedge \mu(b_m) \wedge \eta(c_1) \wedge \ldots \wedge \eta(c_k) \\ &= \mu^m(\overleftarrow{b}) \wedge \eta^k(\overleftarrow{c}) \end{aligned}$$

Since these $b_i$'s and $c_j$'s are arbitrary, it follows that

$$\begin{aligned} [h(\mu), h(\eta)](y) &\geq \bigvee \{\mu^m(\overleftarrow{b}) \wedge \eta^k(\overleftarrow{c}) : y = h(t^A(\overleftarrow{a}, \overleftarrow{b}, \overleftarrow{c}))\} \\ &= h([\mu, \eta])(y) \end{aligned}$$

which gives $h([\mu, \eta]) \leq [h(\mu), h(\eta)]$ and therefore the equality holds.

(2) Let $x \in A$ be any element. Then

$$h^{-1}([\sigma,\theta])(x) = [h^{-1}(\sigma), h^{-1}(\theta)](x)$$
$$= \vee \{\mu^m(\overleftarrow{b}) \wedge \sigma^k(\overleftarrow{c}) : x = t(\overleftarrow{a}, \overleftarrow{b}, \overleftarrow{c})\ where t(\overleftarrow{x}, \overleftarrow{y}, \overleftarrow{z}) is a commutator term in \overleftarrow{y}, \overleftarrow{z}\}$$

Now let $x = t^A(\overleftarrow{a}, \overleftarrow{b}, \overleftarrow{c})$ ; for some commutator term $t(\overleftarrow{x}, \overleftarrow{y}, \overleftarrow{z})$ in $\overleftarrow{y}, \overleftarrow{z}$ and $a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_k \in A$. Then

$$h(x) = h(t^A(\overleftarrow{a}, \overleftarrow{b}, \overleftarrow{c}))$$
$$= t^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m), h(c_1), \ldots, h(c_k))$$

Consider the following:

$$h^{-1}([\sigma,\theta])(x) = [\sigma,\theta](h(x))$$
$$= \vee \{\sigma^m(\overleftarrow{v}) \wedge \theta^k(\overleftarrow{w}) : h(x) = t^B(\overleftarrow{u}, \overleftarrow{v}, \overleftarrow{w})\}$$
$$\geq \sigma(h(b_1)) \wedge \ldots \wedge \sigma(h(b_m)) \wedge \theta(h(c_1)) \wedge \ldots \wedge \theta(h(c_k)))$$
$$= h^{-1}(\sigma)(b_1) \wedge \ldots \wedge h^{-1}(\sigma)(b_m) \wedge h^{-1}(\theta)(c_1) \wedge \ldots \wedge h^{-1}(\theta)(c_k)$$
$$= (h^{-1}(\sigma))^m(b) \wedge (h^{-1}(\theta))^k(\overleftarrow{c})$$

Since each $a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_k$ are arbitrary, we get

$$h^{-1}([\sigma,\theta])(x) \geq \vee \{(h^{-1}(\sigma))^m(\overleftarrow{b}) \wedge (h^{-1}(\theta))^k(\overleftarrow{c}) : x = t^A(\overleftarrow{a}, \overleftarrow{b}, \overleftarrow{c})\}$$
$$= [h^{-1}(\sigma), h^{-1}(\theta)](x)$$

Therefore $[h^{-1}(\sigma), h^{-1}(\theta)] \leq h^{-1}([\sigma,\theta])$.

**Theorem 4.2.** *If $h: A \to B$ is an onto homomorphism and $\mu$ is an $h$-invariant fuzzy prime ideal of $A$, then $h(\mu)$ is a fuzzy prime ideal of $B$.*

**Proof:** Suppose that $\mu$ is an $h$-invariant fuzzy prime ideal of $A$. It follows from Theorem 3.1 that $h(\mu)$ is a fuzzy ideal of $B$. Let $\sigma$ and $\theta$ be fuzzy ideals of $B$ such that

$$[\sigma,\theta] \leq h(\mu)$$

Then

$$h^{-1}([\sigma,\theta]) \leq h^{-1}(h(\mu))$$

Since $\mu$ is given to be an $h$-invariant, we have $h^{-1}(h(\mu)) = \mu$. So that,

$$h^{-1}([\sigma,\theta]) \leq \mu$$

Also, by (2) of Theorem 4.1, we have

$$[h^{-1}(\sigma), h^{-1}(\theta)] \leq h^{-1}([\sigma,\theta])$$

which gives

$$[h^{-1}(\sigma), h^{-1}(\theta)] \leq \mu$$

Since $\mu$ is fuzzy prime, either $h^{-1}(\sigma) \leq \mu$ or $h^{-1}(\theta) \leq \mu$ , which implies either $h(h^{-1}(\sigma)) \leq h(\mu)$ or $h(h^{-1}(\theta)) \leq h(\mu)$; that is, either $\sigma \leq h(\mu)$ or $\theta \leq h(\mu)$. This means $h(\mu)$ is fuzzy prime.

**Theorem 4.3.** *If $h$ is a homomorphism from $A$ onto $B$ and $\sigma$ is a fuzzy prime ideal of $B$, then $h^{-1}(\sigma)$ is a fuzzy prime ideal of $A$.*

**Proof:** Suppose that $\theta$ is a fuzzy prime ideal of $B$. By Theorem 3.1 $h^{-1}(\theta)$ is a fuzzy ideal of $A$. Let $\mu$ and $\eta$ be fuzzy ideals of $A$ such that

$$[\mu,\eta] \leq h^{-1}(\theta)$$

Then

$$h([\mu,\eta]) \leq h(h^{-1}(\theta))$$

Since $h$ is surjective, $h(h^{-1}(\theta)) = \theta$ and by (1) of Theorem 4.1, we have $h([\mu,\eta]) = [h(\mu), h(\eta)]$. So that

$$[h(\mu), h(\eta)] \leq \theta$$

Since $\theta$ is fuzzy prime, either $h(\mu) \leq \theta$ or $h(\eta) \leq \theta$. This provides that either $\mu \leq h^{-1}(\theta)$ or $\eta \leq h^{-1}(\theta)$. Therefore $h^{-1}(\theta)$ is fuzzy prime.

**Theorem 4.4.** *If $h: A \to B$ is an onto homomorphism, then the mapping $\mu \mapsto h(\mu)$ defines a one-to-one correspondence between the set of all $h$-invariant fuzzy prime ideals of $A$ and the set of all fuzzy prime ideals of $B$.*

**Proof:** The above two theorems confirm that $\mu \mapsto h(\mu)$ is an onto map from the set of all $h$-invariant fuzzy prime ideals of $A$ to the set of all fuzzy prime ideals of $B$. It remains to show that it one-one. Let $\mu_1$ and $mu_2$ be an $h$-invariant fuzzy prime ideals of $A$ such that $h(\mu_1) = h(\mu_2)$. Let $x \in A$. Then $h(x) \in B$ and $h(\mu_1)(h(x)) = h(\mu_2)(h(x))$. Since $\mu_1$ is $h$-invariant we have $\mu_1(x) = \mu_1(a)$ for all $a \in h^{-1}(x)$. So,

$$\mu_1(x) = \vee \{\mu_1(a) : a \in h^{-1}(x)\}$$

$$= h(\mu_1)(h(x))$$

$$= h(\mu_2)(h(x))$$

$$= \vee \{\mu_2(b) : b \in h^{-1}(x)\}$$

$$= \mu_2(x)$$

Thus $\mu_1 = \mu_2$ and hence the map $\mu \mapsto h(\mu)$ is a one-to-one correspondence.

**Theorem 4.5.** *If $h: A \to B$ is an onto homomorphism and $\mu$ is an $h$-invariant maximal fuzzy ideal of $A$, then $h(\mu)$ is a maximal fuzzy ideal of $B$.*

**Proof:** Suppose that $\mu$ is an $h$-invariant maximal fuzzy ideal of $A$. Let $\sigma$ be a proper fuzzy ideal of $B$ such that

$$h(\mu) \leq \sigma$$

Then

$$h^{-1}(h(\mu)) \leq h^{-1}(\sigma)$$

Since $\mu$ is $h$-invariant, we have $\mu = h^{-1}(h(\mu))$. So that

$$\mu \leq h^{-1}(\sigma)$$

By Theorem 4.1, $h^{-1}(\sigma)$ is a fuzzy ideal of $A$. Moreover, since $\sigma$ is proper, there exists $y \in B$ such that$\sigma(y) < 1$; that is, $\sigma(y) = h(h^{-1}(\sigma))(y) < 1$, which gives $h^{-1}(\sigma)(x) < 1$ for all $x \in h^{-1}(y)$. This means, $h^{-1}(\sigma)$ is a proper fuzzy ideal of $A$ such that $\mu \leq h^{-1}(\sigma)$. Since $\mu$ is maximal, we get that $\mu = h^{-1}(\sigma)$, which implies $h(\mu) = h(h^{-1}(\sigma)) = \sigma$. Therefore $h(\mu)$ is a maximal fuzzy ideal in $A$.

**Theorem 4.6.** *If $h$ is a homomorphism from $A$ onto $B$ and $\sigma$ is a maximal fuzzy ideal of $B$, then $h^{-1}(\sigma)$ is a maximal fuzzy ideal of $A$.*

**Theorem 4.7.** *If $h: A \to B$ is an onto homomorphism, then the mapping $\mu \mapsto h(\mu)$ defines aone-to-one correspondence between the set of all $h$-invariant maximal fuzzy ideals of $A$ and the set of all maximal fuzzy ideals of $B$.*

**Theorem 4.8.** *Let $h: A \to B$ is an onto homomorphism. If $\mu$ is an $h$-invariant fuzzy ideal of $A$, then*

$$h(\sqrt{\mu}) = \sqrt{h(\mu)}$$

**Theorem 4.9.** *Let $h: A \to B$ is an homomorphism. If $\sigma$ is a fuzzy ideal of $B$, then*

$$h^{-1}(\sqrt{\mu}) = \sqrt{h^{-1}(\mu)}$$

**Theorem 4.10.** *If $h: A \to B$ is an onto homomorphism and $\mu$ is an $h$-invariant fuzzy semi-prime ideal of $A$, then $h(\mu)$ is a fuzzy semi-prime ideal of $B$.*

**Theorem 4.11.** *If $h$ is a homomorphism from $A$ to $B$ and $\sigma$ is a fuzzy semi-prime ideal of $B$, then $h^{-1}(\sigma)$ is a fuzzy semi-prime ideal of $A$.*

**Theorem 4.12.** *If $h: A \to B$ is an onto homomorphism, then the mapping $\mu \mapsto h(\mu)$ defines a one-to-one correspondence between the set of all $h$-invariant fuzzy semi-prime ideals of $A$ and the set of all fuzzy semi-prime ideals of $B$.*

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