Homomorphisms and $L$-Fuzzy Ideals in Universal Algebras

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Abstract. In this paper, we study the homomorphic images (respectively pre-images) of $L$-fuzzy ideals, $L$-fuzzy prime ideals, maximal $L$-fuzzy ideals and $L$-fuzzy semi-prime ideals of universal algebras.

Keywords: Homomorphisms; $L$-fuzzy ideals; $L$-fuzzy prime ideals; maximal $L$-fuzzy ideals; $L$-fuzzy semi-prime ideals.

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1. Introduction
The theory of fuzzy sets introduced by Zadeh [22] has evoked tremendous interest among researchers working in different branches of mathematics. Rosenfield in his pioneering paper [15] introduced the notions of fuzzy subgroups of a group. Since then, many researches have been studying fuzzy subalgebras of several algebraic structures (see [7, 12, 13, 14, 19]). As suggested by Gougen [9], the unit interval $[0,1]$ is not sufficient to take the truth values of general fuzzy statements. U. M. Swamy and D. V. Raju [17, 18] studied the general theory of algebraic fuzzy systems by introducing the notion of a fuzzy $\mathcal{U} \subseteq X$ which is a subset of a set $X$ corresponding to a given class $\mathcal{U}$ of subsets of $X$ having truth values in a complete lattice satisfying the infinite meet distributive law.

Ideals in universal algebras have been studied in a series of papers [4, 5, 6, 11, 20] as a generalization of those familiar structures: normal subgroups (in groups), normal subloops (in loops), ideals (in rings), submodules (in modules), subspaces (in vector spaces) and filters (in implication algebras or Heyting algebras).

In [1], we have introduced the concept $L$-fuzzy ideals in universal algebras and we gave a necessary and sufficient condition for a variety of algebras to be an ideal determined. In [2], we study $L$-fuzzy prime ideals and maximal $L$-fuzzy ideals of universal algebras and gave an internal characterization for $L$-fuzzy prime ideals analogous to the characterization of Swamy and Swamy [16] in the case of rings. In [3], we continued our study and we define $L$-fuzzy semi-prime ideals and the radical of $L$-fuzzy ideals in universal algebras in the frame work of $L$-fuzzy ideals given in [1]. In the present paper, we study the image and pre-image of $L$-fuzzy ideals of universal
algebras under a homomorphism. We make a theoretical study on their properties and give several characterizing theorems.

2. Preliminaries
This section contains some definitions and results which will be used in the paper. We refer to the readers [8, 10], for the standard concepts in universal algebras. Throughout this paper $\mathcal{A} \in \mathcal{K}$, where $\mathcal{K}$ is a class of algebras of a fixed type $\Omega$ and assume that there is an equationally definable constant in all algebras of $\mathcal{K}$ denoted by $0$. For a positive integer $n$, we write $\bar{a} \in \mathcal{A}$ to denote the $n$-tuple $\langle a_1, a_2, \ldots, a_n \rangle \in A^n$.

**Definition 2.1.** [11] A term $P(\bar{x}, \bar{y})$ is said to be an ideal term in $\mathcal{A}$ if and only if $P(\bar{x}, 0) = 0$.

**Definition 2.2.** [11] A nonempty subset $I$ of $\mathcal{A}$ is called an ideal of $\mathcal{A}$ if and only if $P(\bar{a}, 0) \in I$ for all $\bar{a} \in A^n, \bar{b} \in I^n$ and any ideal term $P(\bar{x}, \bar{y})$ in $\mathcal{A}$.

We denote the class of all ideals of $\mathcal{A}$, by $I(\mathcal{A})$.

**Definition 2.3.** [11, 20] A term $t(\bar{x}, \bar{y}, \bar{z})$ is said to be a commutator term in $\mathcal{A}$ if and only if it is an ideal term in $\mathcal{A}$ and an ideal term in $J(\mathcal{A})$.

**Definition 2.4.** [11] In an ideal determined variety, the commutator $[I, J]$ of ideals $I$ and $J$ is the zero congruence class of the commutator congruence $[I^\delta, J^\delta]$.

It is characterized in [11] as follows:

**Theorem 2.5.** [11, 20] In an ideal determined variety, $[I, J] = \{ t(\bar{a}, \bar{b}, \bar{c}) : \bar{a} \in A^n, \bar{b} \in I^n, \bar{c} \in J^n, \text{where } t(\bar{x}, \bar{y}, \bar{z}) \text{ is a commutator term in } \mathcal{A} \}$

For subsets $H, G$ of $\mathcal{A}, [H, G]$ denotes the product $\langle (H), (G) \rangle$. In particular, for $a, b \in A, [(a), (b)]$ is denoted by $[a, b]$.

**Definition 2.6.** [20] A proper ideal $P$ of $\mathcal{A}$ is called prime if and only if for all $I, J \in J(\mathcal{A})$:

$[I, J] \subseteq P \Rightarrow \text{either } I \subseteq P \text{ or } J \subseteq P$

**Theorem 2.7.** [20] A proper ideal $P$ of $\mathcal{A}$ is prime if and only if:

$[a, b] \subseteq P \Rightarrow \text{either } a \in P \text{ or } b \in P$

for all $a, b \in A$.

**Definition 2.8.** [20] An ideal $Q$ of $\mathcal{A}$ is called semiprime if and only if for all $I \in J(\mathcal{A})$:

$[I, I] \subseteq Q \Rightarrow I \subseteq Q$

**Definition 2.9.** [20] The prime radical of an ideal $I$ of $\mathcal{A}$, denoted by $\sqrt{I}$ is the intersection of all prime ideals of $\mathcal{A}$ containing $I$.

Throughout this paper $L = (L, \wedge, \vee, 0, 1)$ is a complete Brouwerian lattice; i.e., $L$ is a complete lattice satisfying the infinite meet distributive law. By an $L$-fuzzy subset of $A$,
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we mean a mapping $\mu: A \to L$. For each $\alpha \in L$, the $\alpha$ – level set of $\mu$ denoted by $\mu_\alpha$ is a subset of $A$ given by:

$$\mu_\alpha = \{x \in A: \alpha \leq \mu(x)\}$$

For fuzzy subsets $\mu$ and $\nu$ of $A$, we write $\mu \leq \nu$ to mean $\mu(x) \leq \nu(x)$ in the ordering of $L$.

**Definition 2.10.** [21] For each $x \in A$ and $0 \neq \alpha \in L$, the fuzzy subset $x_\alpha$ of $A$ given by:

$$x_\alpha(z) = \begin{cases} \alpha & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$$

is called the fuzzy point of $A$. In this case $x$ is called the support of $x_\alpha$ and $\alpha$ its value.

For a fuzzy subset $\mu$ of $A$ and a fuzzy point $x_\alpha$ of $A$, we write $x_\alpha \in \mu$ whenever $\mu(x) \geq \alpha$.

**Definition 2.11.** [1] An $L$–fuzzy subset $\mu$ of $A$ is said to be an $L$–fuzzy ideal of $A$ (or shortly a fuzzy ideal of $A$) if and only if the following conditions are satisfied:

1. $\mu(0) = 1$, and
2. If $P(\tilde{x}, \tilde{y})$ is an ideal term in $\tilde{y}$ and $\tilde{a} \in A^n, \tilde{b} \in A^m$, then

$$\mu(P(\tilde{a}, \tilde{b})) \geq \mu^m(\tilde{b})$$

We denote by $\mathcal{FI}(A)$, the class of all fuzzy ideals of $A$.

**Definition 2.12.** [2] The commutator of fuzzy ideals $\mu$ and $\sigma$ of $A$ denoted by $[\mu, \sigma]$ is a fuzzy subset of $A$ defined by:

$$[\mu, \sigma](x) = \lor \{\alpha \land \beta: \alpha, \beta \in L, x \in [\mu_\alpha, \sigma_\beta]\}$$

= $\lor \{\lambda \in L: x \in [\mu_\lambda, \sigma_\lambda]\}$

for all $x \in A$.

**Theorem 2.13.** [2] For each $x \in A$, and fuzzy ideals $\mu$ and $\sigma$ of $A$:

$$[\mu, \sigma](x) = \lor \{\mu^m(\tilde{b}) \land \sigma^k(\tilde{c}): x = t(\tilde{a}, \tilde{b}, \tilde{c}), \text{ where } \tilde{a} \in A^n, \tilde{b} \in A^m, \tilde{c} \in A^k, \text{ and } t(\tilde{x}, \tilde{y}, \tilde{z}) \text{ is a commutator term in } \tilde{y}, \tilde{z}\}$$

**Definition 2.14.** [2] A non-constant fuzzy ideal $\mu$ of $A$ is called a fuzzy prime ideal if and only if:

$$[\nu, \sigma] \leq \mu \Rightarrow \text{either } \nu \leq \mu \text{ or } \sigma \leq \mu$$

for all $\nu, \sigma \in \mathcal{FI}(A)$.

**Theorem 2.15.** [2] A non-constant fuzzy ideal $\mu$ is a fuzzy prime ideal if and only if

$\text{Im}(\mu) = \{1, \alpha\}$, where $\alpha$ is a prime element in $L$ and the set

$$\mu_* = \{x \in A: \mu(x) = 1\}$$

is a prime ideal of $A$.

**Definition 2.16.** [3] A fuzzy ideal $\mu$ of $A$ is called fuzzy semi-prime if:

$$[\theta, \theta] \leq \mu \Rightarrow \theta \leq \mu$$

for all $\theta \in \mathcal{FI}(A)$.  

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According to [20], the prime radical of an ideal \( I \) of \( A \), denoted by \( \sqrt{I} \), is the intersection of all prime ideals of \( A \) containing \( I \). Here we define the prime radical of fuzzy ideals using their level ideals.

**Definition 2.17.** [3] For a fuzzy ideal \( \mu \) of \( A \), its prime radical of \( \mu \) denoted by \( \sqrt{\mu} \) is defined as a fuzzy subset of \( A \) such that, for each \( x \in A \):

\[
\sqrt{\mu}(x) = \alpha \quad \text{if and only if} \quad x \in \sqrt{\mu_\alpha} \quad \text{and} \quad x \notin \sqrt{\mu_\beta} \quad \text{for all} \quad \beta > \alpha \quad \text{in} \quad L.
\]

**Lemma 2.18.** [3] Let \( \mu \) be a fuzzy ideal of \( A \) and \( x \in A \). Then

\[
\sqrt{\mu}(x) = \bigvee \{ \alpha \in L : x \in \sqrt{\mu_\alpha} \}
\]

### 3. Homomorphisms and fuzzy ideals

Let \( A \) and \( B \) be algebras of the same type \( \Omega \). A mapping \( h : A \to B \) is called a homomorphism from \( A \) to \( B \) if:

\[
h(f^A(a_1, a_2, \ldots, a_n)) = f^B(h(a_1), h(a_2), \ldots, h(a_n))
\]

for each \( n \)-ary operation \( f \) in \( \Omega \) and each sequence \( a_1, a_2, \ldots, a_n \) from \( A \). It is observed that if \( p \) is an \( n \)-ary term of type \( \Omega \), then

\[
h(f^A(a_1, a_2, \ldots, a_n)) = p^B(h(a_1), h(a_2), \ldots, h(a_n))
\]

for all \( a_1, a_2, \ldots, a_n \in A \).

**Theorem 3.1.** Let \( h : A \to B \) be a homomorphism. Then we have the following:

1. If \( \sigma \) is a fuzzy ideal of \( B \), then \( h^{-1}(\sigma) \) is a fuzzy ideal of \( A \).
2. If \( \mu \) is a fuzzy ideal of \( A \) and \( h \) is surjective, then \( h(\mu) \) is a fuzzy ideal of \( B \).

**Proof:** Let \( h : A \to B \) be a homomorphism.

1. Suppose that \( \sigma \) is a fuzzy ideal of \( B \) and let \( a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in A \). Then \( h(a_1), h(a_2), \ldots, h(a_n), h(b_1), h(b_2), \ldots, h(b_m) \in B \). If \( p(x, y) \) is an \( n + m \) ideal term in \( \tilde{\sigma} \), then we get:

\[
\sigma(p^B(h(a_1), h(a_2), \ldots, h(a_n), h(b_1), h(b_2), \ldots, h(b_m))) \geq \sigma(h(b_1)) \land \ldots \land \sigma(h(b_m))
\]

Now consider the following:

\[
h^{-1}(\sigma)(p^A(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)) = \sigma(h(p^A(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)))
\]

\[
= \sigma(p^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m)))
\]

\[
= \sigma(h(b_1)) \land \ldots \land \sigma(h(b_m))
\]

\[
= h^{-1}(\sigma)(b_1) \land \ldots \land h^{-1}(\sigma)(b_m)
\]

Therefore \( h^{-1}(\sigma) \) is a fuzzy ideal of \( A \).

2. Suppose that \( h \) is surjective and let \( \mu \) be a fuzzy ideal of \( A \). If \( u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m \in B \), then there exist \( a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in A \) such that \( h(a_i) = u_i \) and \( h(b_j) = v_j \) for all \( i, j \). If \( p(x, y) \) is an \( n + m \) ideal term in \( \tilde{\sigma} \), then we get:

\[
h(p^A(a_1, \ldots, a_n, b_1, \ldots, b_m)) = p^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m))
\]

\[
= p^B(u_1, \ldots, u_n, v_1, \ldots, v_m)
\]

So that \( p^A(a_1, \ldots, a_n, b_1, \ldots, b_m) \in h^{-1}(p^B(u_1, \ldots, u_n, v_1, \ldots, v_m)) \). Now consider the following:

\[
h(\mu)(p^B((u_1, \ldots, u_n, v_1, \ldots, v_m)) = \bigvee \{ \mu(a) : a \in h^{-1}(p^B((u_1, \ldots, u_n, v_1, \ldots, v_m)) \}
\]
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\[ \geq \mu(p^A(a_1, \ldots, a_n, b_1, \ldots, b_m)) \geq \mu(b_1) \land \ldots \land \mu(b_m) \]

Since \( b_j \) is arbitrary in \( h^{-1}(v_j) \) for all \( j = 1, 2, \ldots, m \), it follows that

\[ h(\mu)(p^B((u_1, \ldots, u_n, v_1, \ldots, v_m)) \geq \bigvee_{b_j \in h^{-1}(v_j)} \mu(b_1) \land \ldots \land \mu(b_m) \]

\[ = h(\mu)(v_j) \land \ldots \land h(\mu)(v_m) \]

Therefore \( h(\mu) \) is a fuzzy ideal of \( B \).

**Theorem 3.2.** Let \( h: A \to B \) be a homomorphism, \( \mu \) and \( \nu \) be fuzzy ideals of \( A \). Then

\[ h(\mu \lor \nu) = h(\mu) \lor h(\nu) \]

**Proof:** We show that \( h(\mu \lor \nu) \) is the smallest fuzzy ideal of \( B \) containing both \( h(\mu) \) and \( h(\nu) \). By Theorem 3.1, \( h(\mu \lor \nu) \) is a fuzzy ideal of \( B \). Now let \( y \in B \). If \( h^{-1}(y) = \emptyset \), then \( h(\mu)(y) = 0 \leq h(\mu \lor \nu)(y) \). Also if \( h^{-1}(y) \neq \emptyset \), then consider the following:

\[ h(\mu)(y) = \lor \{\mu(x) : x \in h^{-1}(y)\} \leq \lor \{\mu \lor \nu(x) : x \in h^{-1}(y)\} = h(\mu \lor \nu)(y) \]

So that \( h(\mu) \leq h(\mu \lor \nu) \). Similarly, we can verify that \( h(\nu) \leq h(\mu \lor \nu) \). Now for any fuzzy ideal \( \eta \) of \( B \):

\[ h(\mu) \leq \eta, h(\nu) \leq \eta \Rightarrow h^{-1}(h(\mu)) \leq h^{-1}(\eta), h^{-1}(h(\nu)) \leq h^{-1}(\eta) \]

\[ \Rightarrow \mu \leq h^{-1}(\eta), \nu \leq h^{-1}(\eta) \]

\[ \Rightarrow h(\mu \lor \nu) \leq h(h^{-1}(\eta)) \leq \eta \]

Therefore \( h(\mu \lor \nu) \) is the smallest fuzzy ideal of \( B \) containing both \( h(\mu) \) and \( h(\nu) \). So that, \( h(\mu \lor \nu) = h(\mu) \lor h(\nu) \).

**Theorem 3.3.** Let \( h: A \to B \) be a homomorphism, and \( \mu \) and \( \nu \) be fuzzy ideals of \( A \). Then

\[ h(\mu \land \nu) \leq h(\mu) \land h(\nu) \]

Moreover, if either \( \mu \) or \( \nu \) is \( h \) -invariant, then the equality holds.

**Proof:** Let \( y \) be any element in \( B \). If \( h^{-1}(y) = \emptyset \), then \( h(\mu)(y) = 0 = h(\nu)(y) = h(\mu \land \nu)(y) \). Let \( h^{-1}(y) \neq \emptyset \). Then consider the following:

\[ h(\mu \land \nu)(y) = \lor \{\mu \land \nu(x) : x \in h^{-1}(y)\} \]

\[ = \lor \{\mu(a) \land \nu(b) : a, b \in h^{-1}(y)\} \]

\[ = h(\mu)(y) \land h(\nu)(y) \]

Therefore \( h(\mu \land \nu) \leq h(\mu) \land h(\nu) \). Moreover, assume without loss of generality that \( \mu \) is \( h \) -invariant. Then \( \mu(a) = \mu(b) \), whenever \( h(a) = h(b) \). Now for each \( y \in B \), with \( h^{-1}(y) \neq \emptyset \), consider the following:

\[ h(\mu)(y) \land h(\nu)(y) = \lor \{\mu(a) : a \in h^{-1}(y)\} \land \lor \{\nu(b) : b \in h^{-1}(y)\} \]

\[ = \lor \{\mu(a) \land \nu(b) : a, b \in h^{-1}(y)\} \]

\[ = h(\mu)(y) \land h(\nu)(y) \]

Therefore \( h(\mu \land \nu) \leq h(\mu) \land h(\nu) \). Moreover, assume without loss of generality that \( \mu \) is \( h \) -invariant. Then \( \mu(a) = \mu(b) \), whenever \( h(a) = h(b) \). Now for each \( y \in B \), with \( h^{-1}(y) \neq \emptyset \), consider the following:

\[ h(\mu)(y) \land h(\nu)(y) \]
Theorem 3.4. Let \( h: A \to B \) be a homomorphism, and and \( \sigma \) and \( \theta \) be fuzzy ideals of \( B \). Then
\[
\overline{h^{-1}(\sigma)} \vee \overline{h^{-1}(\theta)} \leq \overline{h^{-1}(\sigma \vee \theta)}
\]
Moreover, the equality holds whenever \( h \) is surjective.

Proof: For each \( x \in A \), consider:
\[
\overline{h^{-1}(\sigma)}(x) = \sigma(h(x)) \\
\leq (\sigma \vee \theta)(h(x)) \\
= \overline{h^{-1}(\sigma \vee \theta)}(x)
\]
So that \( \overline{h^{-1}(\sigma)} \leq \overline{h^{-1}(\sigma \vee \theta)} \). Similarly it can be verified that \( \overline{h^{-1}(\theta)} \leq \overline{h^{-1}(\sigma \vee \theta)} \). Further, let us assume that \( h \) is surjective. To prove the equality, it is enough if we show that \( \overline{h^{-1}(\sigma \vee \theta)} \) is the smallest fuzzy ideal of \( A \) containing both \( \sigma \) and \( \theta \). From Theorem 3.1 we have that \( \overline{h^{-1}(\sigma \vee \theta)} \) is a fuzzy ideal of \( A \). From the above inequality also we have \( \overline{h^{-1}(\sigma)} \leq \overline{h^{-1}(\sigma \vee \theta)} \) and \( \overline{h^{-1}(\theta)} \leq \overline{h^{-1}(\sigma \vee \theta)} \). Now let \( \mu \) be any other fuzzy ideal of \( A \) such that \( \overline{h^{-1}(\sigma)} \leq \mu \) and \( \overline{h^{-1}(\theta)} \leq \mu \). Since \( h \) is surjective, it follows that \( \sigma \leq h(\mu) \) and \( \theta \leq h(\mu) \). So that, \( \sigma \vee \theta \leq h(\mu) \), which gives \( h^{-1}(\sigma \vee \theta) \leq \overline{h^{-1}(h(\mu))} \).

Our aim is to show that \( \overline{h^{-1}(\sigma \vee \theta)} \leq \mu \). Suppose not. Then there exists \( a \in A \) such that \( h^{-1}(\sigma \vee \theta)(a) > \mu(a) \). If we put \( z = h(a) \), then we get \( (\sigma \vee \theta)(z) > h(\mu)(z) \), which is a contradiction. Therefore \( \overline{h^{-1}(\sigma \vee \theta)} \leq \mu \) and hence the equality holds.

Theorem 3.5. Let \( h: A \to B \) be a homomorphism, and \( \sigma \) and \( \theta \) be fuzzy ideals of \( B \). Then
\[
\overline{h^{-1}(\sigma \land \theta)} = \overline{h^{-1}(\sigma)} \land \overline{h^{-1}(\theta)}
\]

Proof: For each \( a \in A \), consider the following:
\[
\overline{h^{-1}(\sigma \land \theta)}(a) = (\sigma \land \theta)(h(a)) \\
= \sigma(h(a)) \land \theta(h(a)) \\
= \overline{h^{-1}(\sigma \land \theta)}(a) \\
= (\overline{h^{-1}(\sigma)} \land \overline{h^{-1}(\theta)})(a)
\]
Therefore \( \overline{h^{-1}(\sigma \land \theta)} = \overline{h^{-1}(\sigma)} \land \overline{h^{-1}(\theta)} \).

Theorem 3.6. Let \( h: A \to B \) be a surjective homomorphism. For any \( h \)-invariant fuzzy subset \( \mu \) of \( A \), we have:
\[
h(\langle \mu \rangle) = \langle h(\mu) \rangle
\]

Proof: For any \( y \in B \), consider:
\[
h(\langle \mu \rangle)(y) = \vee \{ (\mu(x) \mid x \in h^{-1}(y)) \}
\]
\[
= \vee \{ \alpha \in L : x \in \langle \mu_\alpha \rangle \mid x \in h^{-1}(y) \}
\]
\[
= \vee \{ \alpha \in L : x \in \langle \mu_\alpha \rangle \land dh(x) = y \}
\]
\[
= \vee \{ \alpha \in L : y \in h(\langle \mu_\alpha \rangle) \}
\]
on the other hand
\[
\langle h(\mu) \rangle(y) = \vee \{ \alpha \in L : y \in \langle h(\mu)_\alpha \rangle \}
\]
Now it is enough to show that
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Let $z \in h(\langle \mu_\alpha \rangle)$. Then $z = h(x)$ for some $x \in \langle \mu_\alpha \rangle$. There exists $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_m \in \mu_\alpha$ and an ideal term $p(\tilde{x}, \tilde{y})$ in $\tilde{y}$ such that $x = p^A(a_1, \ldots, a_n \in A, b_1, \ldots, b_m)$. So,

$$z = h(x)$$

$$= h(p^A(a_1, \ldots, a_n, b_1, \ldots, b_m))$$

$$= p^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m))$$

For each $j = 1, 2, \ldots, m$ we have

$$h(\mu)(h(b_j)) = \bigvee \{\mu(x) : x \in h^{-1}(b_j)\}$$

Since $\mu$ is $h$-invariant and each $b_j \in \mu_\alpha$, we get

$$h(\mu)(h(b_j)) = \mu(b_j) \geq \alpha$$

for all $j = 1, 2, \ldots, m$; that is $h(b_j) \in h(\langle \mu_\alpha \rangle)$ for all $j$ and $z = p^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m))$. This means $z \in h(\langle \mu_\alpha \rangle)$. So that

$$h(\langle \mu_\alpha \rangle) \subseteq h(\langle \mu_\alpha \rangle)$$

To prove the other inclusion, let $z \in h(\langle \mu_\alpha \rangle)$. Then $z = p^B(\tilde{u}, \tilde{v})$ for some $u_1, \ldots, u_n \in B, v_1, \ldots, v_m \in h(\langle \mu_\alpha \rangle)$ and some ideal term $p(\tilde{x}, \tilde{y})$ in $\tilde{y}$. Since $h$ is surjective, there exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$ such that $h(a_i) = u_i$ and $h(b_j) = v_j$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. As each $v_j \in h(\langle \mu_\alpha \rangle)$, we have $h(\mu)(h(b_j)) \geq \alpha$. Since $\mu$ is $h$-invariant we get $\mu(b_j) \geq \alpha$; that is, $b_j \in \mu_\alpha$ for all $j$. Put $x = p^A(a_1, \ldots, a_n, b_1, \ldots, b_m)$. Then $x \in \langle \mu_\alpha \rangle$.

Moreover

$$h(x) = h(p^A(a_1, \ldots, a_n, b_1, \ldots, b_m))$$

$$= p^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m))$$

$$= p^B(u_1, \ldots, u_n, v_1, \ldots, v_m)$$

$$= z$$

That is, $z = h(x)$, where $x \in \langle \mu_\alpha \rangle$, which gives $z \in h(\langle \mu_\alpha \rangle)$. Thus $h(\langle \mu_\alpha \rangle) \subseteq h(\langle \mu_\alpha \rangle)$. Hence $h(\langle \mu_\alpha \rangle) = h(\langle \mu_\alpha \rangle)$ and this completes the proof.

4. Homomorphisms and fuzzy prime ideals

**Theorem 4.1.** Let $h : A \to B$ be a surjective homomorphism.

1. If $\mu$ and $\sigma$ are fuzzy ideals of $A$, then

$$h(\langle \mu, \sigma \rangle) = [h(\mu), h(\sigma)]$$

2. If $\sigma$ and $\theta$ are fuzzy ideals of $B$, then

$$[h^{-1}(\sigma), h^{-1}(\theta)] \leq h^{-1}(\langle \sigma, \theta \rangle)$$

**Proof:** (1) Let $y \in B$. Since $h$ is assumed to be surjective, the set $h^{-1}(y)$ is always nonempty. By definition we have:

$$h(\langle \mu, \eta \rangle)(y) = \bigvee \{\mu, \eta\}(x) : x \in h^{-1}(y)\}$$

$$= \bigvee \{\mu^m(\tilde{u}) \land \eta^k(\tilde{c}) : x = t^A(\tilde{u}, \tilde{b}, \tilde{c}) : x \in h^{-1}(y)\}$$

$$= \bigvee \{\mu^m(\tilde{b}) \land \eta^k(\tilde{c}) : y = h(t^A(\tilde{u}, \tilde{b}, \tilde{c}))\}$$

which gives

$$h(\langle \mu, \eta \rangle)(y) \geq \mu^m(\tilde{b}) \land \eta^k(\tilde{c})$$

(4.1)
for any $b_1, \ldots, b_m, c_1, \ldots, c_k \in A$. Now let $y = t^B(\tilde{u}, \tilde{v}, \tilde{w})$ be any expression of $y$ using commutator terms, where $u_1, \ldots, u_n, v_1, \ldots, v_m, w_1, \ldots, w_k \in B$. Since $h$ is surjective there exist $a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_k \in A$ such that $h(a_i) = u_i$, $h(b_j) = v_j$ and $h(c_r) = w_r$ for all $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, k$. Equivalently, each $a_i \in h^{-1}(u_i)$, $b_j \in h^{-1}(v_j)$ and $c_r \in h^{-1}(w_r)$. Now consider:

$$h(t^A(\tilde{a}, \tilde{b}, \tilde{c})) = t^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m), h(c_1), \ldots, h(c_k))$$

$$= t^B(u_1, \ldots, u_n, v_1, \ldots, v_m, w_1, \ldots, w_k)$$

$$= y$$

So, by eq. (4.1) we get

$$h([\mu, \eta])(y) \geq \mu^m(\tilde{v}) \wedge \eta^k(\tilde{w})$$

Since each $b_j$ (respectively $c_r$) is arbitrary in $h^{-1}(v_j)$ (respectively in $h^{-1}(w_r)$), it follows that

$$h([\mu, \eta])(y) \geq \bigvee_{b \in h^{-1}(v)} \mu(b) \wedge \bigvee_{c \in h^{-1}(w)} \eta(c)$$

This gives

$$h([\mu, \eta])(y) \geq \bigvee \{ h(\mu)^m(\tilde{v}) \wedge h(\eta)^k(\tilde{w}) : y = t^B(\tilde{u}, \tilde{v}, \tilde{w}) \}$$

Therefore $[\mu, h(\eta)] \leq h([\mu, \eta])$. To prove the other inequality, consider

$$[h(\mu), h(\eta)](y) = \bigvee \{ h(\mu)^m(\tilde{v}) \wedge h(\eta)^k(\tilde{w}) : y = t^B(\tilde{u}, \tilde{v}, \tilde{w}) \}$$

So that

$$[h(\mu), h(\eta)](y) \geq h(\mu)^m(\tilde{v}) \wedge h(\eta)^k(\tilde{w})$$

for all $v_1, \ldots, v_m, w_1, \ldots, w_k \in B$, with $y = t^B(\tilde{u}, \tilde{v}, \tilde{w})$, for some commutator term $t(\tilde{x}, \tilde{y}, \tilde{z})$ in $\tilde{y}, \tilde{z}$ and some $u_1, \ldots, u_n \in B$. Now let $y = t^A(\tilde{a}, \tilde{b}, \tilde{c})$ for some $a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_k \in A$ and commutator term $t(\tilde{x}, \tilde{y}, \tilde{z})$ in $\tilde{y}, \tilde{z}$. That is,

$$y = t^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m), h(c_1), \ldots, h(c_k))$$

By eq. (4.2) we get

$$[h(\mu), h(\eta)](y) \geq h(\mu)^m(\tilde{v}) \wedge \bigvee \{ h(\eta)^k(\tilde{w}) : y = t^A(\tilde{a}, \tilde{b}, \tilde{c}) \}$$

using the fact that $h(\mu)(h(a)) \geq \mu(a)$ for all $a \in A$, we get the following:

$$[h(\mu), h(\eta)](y) \geq h(\mu)^m(\tilde{v}) \wedge \bigvee \{ h(\eta)^k(\tilde{w}) : y = t^A(\tilde{a}, \tilde{b}, \tilde{c}) \}$$

Since these $b_j$’s and $c_j$’s are arbitrary, it follows that

$$[h(\mu), h(\eta)](y) \geq \bigvee \{ \mu^m(\tilde{v}) \wedge \eta^k(\tilde{w}) : y = h(t^A(\tilde{a}, \tilde{b}, \tilde{c})) \}$$

which gives $h([\mu, \eta]) \leq [h(\mu), h(\eta)]$ and therefore the equality holds.

(2) Let $x \in A$ be any element. Then
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$h^{-1}([\sigma, \theta])(x) = [h^{-1}(\sigma), h^{-1}(\theta)](x)$

$= \vee \{\mu^m(\tilde{b}) \land \sigma^k(\tilde{c}) : x = t(\tilde{a}, \tilde{b}, \tilde{c}) \text{ where } (\tilde{x}, \tilde{y}, \tilde{z}) \text{ is a commutator term} \}$

Now let $x = t^A(\tilde{a}, \tilde{b}, \tilde{c})$; for some commutator term $t(\tilde{x}, \tilde{y}, \tilde{z})$ in $\tilde{y}, \tilde{z}$ and $a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_k \in A$. Then

$h(\tilde{x}) = h(t^A(\tilde{a}, \tilde{b}, \tilde{c}))$

$= t^B(h(a_1), \ldots, h(a_n), h(b_1), \ldots, h(b_m), h(c_1), \ldots, h(c_k))$

Consider the following:

$h^{-1}([\sigma, \theta])(x) = [\sigma, \theta](h(x))$

$= \vee \{\sigma^m(\tilde{v}) \land \theta^k(\tilde{w}) : h(x) = t^B(\tilde{u}, \tilde{v}, \tilde{w}) \}$

$\geq \sigma(h(b_1)) \land \ldots \land \sigma(h(b_m)) \land \theta(h(c_1)) \land \ldots \land \theta(h(c_k))$

$= h^{-1}(\sigma)(b_1) \land \ldots \land h^{-1}(\sigma)(b_m) \land h^{-1}(\theta)(c_1) \land \ldots \land h^{-1}(\theta)(c_k)$

$= (h^{-1}(\sigma))^m(\tilde{b}) \land (h^{-1}(\theta))^k(\tilde{c})$

Since each $a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_k$ are arbitrary, we get

$h^{-1}([\sigma, \theta])(x) \geq \vee \{(h^{-1}(\sigma))^m(\tilde{b}) \land (h^{-1}(\theta))^k(\tilde{c}) : x = t^A(\tilde{a}, \tilde{b}, \tilde{c}) \}$

$= [h^{-1}(\sigma), h^{-1}(\theta)](x)$

Therefore $[h^{-1}(\sigma), h^{-1}(\theta)] \leq h^{-1}([\sigma, \theta])$.

**Theorem 4.2.** If $h : A \to B$ is an onto homomorphism and $\mu$ is an $h$-invariant fuzzy prime ideal of $A$, then $h(\mu)$ is a fuzzy prime ideal of $B$.

**Proof:** Suppose that $\mu$ is an $h$-invariant fuzzy prime ideal of $A$. It follows from Theorem 3.1 that $h(\mu)$ is a fuzzy ideal of $B$. Let $\sigma$ and $\theta$ be fuzzy ideals of $B$ such that

$[\sigma, \theta] \leq h(\mu)$

Then

$h^{-1}([\sigma, \theta]) \leq h^{-1}(h(\mu))$

Since $\mu$ is given to be an $h$-invariant, we have $h^{-1}(h(\mu)) = \mu$. So that,

$h^{-1}([\sigma, \theta]) \leq \mu$

Also, by (2) of Theorem 4.1, we have

$[h^{-1}(\sigma), h^{-1}(\theta)] \leq h^{-1}([\sigma, \theta])$

which gives

$[h^{-1}(\sigma), h^{-1}(\theta)] \leq \mu$

Since $\mu$ is fuzzy prime, either $h^{-1}(\sigma) \leq \mu$ or $h^{-1}(\theta) \leq \mu$, which implies either $h(h^{-1}(\sigma)) \leq h(\mu)$ or $h(h^{-1}(\theta)) \leq h(\mu)$; that is, either $\sigma \leq h(\mu)$ or $\theta \leq h(\mu)$. This means $h(\mu)$ is fuzzy prime.

**Theorem 4.3.** If $h$ is a homomorphism from $A$ onto $B$ and $\sigma$ is a fuzzy prime ideal of $B$, then $h^{-1}(\sigma)$ is a fuzzy prime ideal of $A$.

**Proof:** Suppose that $\theta$ is a fuzzy prime ideal of $B$. By Theorem 3.1 $h^{-1}(\theta)$ is a fuzzy ideal of $A$. Let $\mu$ and $\eta$ be fuzzy ideals of $A$ such that

$[\mu, \eta] \leq h^{-1}(\theta)$

Then

$h([\mu, \eta]) \leq h(h^{-1}(\theta))$

Since $h$ is surjective, $h(h^{-1}(\theta)) = \theta$ and by (1) of Theorem 4.1, we have $h([\mu, \eta]) = [h(\mu), h(\eta)]$. So that
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[\(h(\mu), h(\eta)\)] \(\leq\) \(\theta\)

Since \(\theta\) is fuzzy prime, either \(h(\mu) \leq \theta\) or \(h(\eta) \leq \theta\). This provides that either \(\mu \leq h^{-1}(\theta)\) or \(\eta \leq h^{-1}(\theta)\). Therefore \(h^{-1}(\theta)\) is fuzzy prime.

**Theorem 4.4.** If \(h: A \rightarrow B\) is an onto homomorphism, then the mapping \(\mu \mapsto h(\mu)\) defines a one-to-one correspondence between the set of all \(h\)-invariant fuzzy prime ideals of \(A\) and the set of all fuzzy prime ideals of \(B\).

**Proof:** The above two theorems confirm that \(\mu \mapsto h(\mu)\) is an onto map from the set of all \(h\)-invariant fuzzy prime ideals of \(A\) to the set of all fuzzy prime ideals of \(B\). It remains to show that it one-one. Let \(\mu_1\) and \(\mu_2\) be an \(h\)-invariant fuzzy prime ideals of \(A\) such that \(h(\mu_1) = h(\mu_2)\). Let \(x \in A\). Then \(h(x) \in B\) and \(h(\mu_1)(h(x)) = h(\mu_2)(h(x))\). Since \(\mu_1\) is \(h\)-invariant we have \(\mu_1(x) = \mu_1(a)\) for all \(a \in h^{-1}(x)\). So,

\[
\mu_1(x) = \vee \{\mu_1(a) : a \in h^{-1}(x)\}
\]

\[
= h(\mu_1)(h(x))
\]

\[
= h(\mu_2)(h(x))
\]

\[
= \vee \{\mu_2(b) : b \in h^{-1}(x)\}
\]

\[
= \mu_2(x)
\]

Thus \(\mu_1 = \mu_2\) and hence the map \(\mu \mapsto h(\mu)\) is a one-to-one correspondence.

**Theorem 4.5.** If \(h: A \rightarrow B\) is an onto homomorphism and \(\mu\) is an \(h\)-invariant maximal fuzzy ideal of \(A\), then \(h(\mu)\) is a maximal fuzzy ideal of \(B\).

**Proof:** Suppose that \(\mu\) is an \(h\)-invariant maximal fuzzy ideal of \(A\). Let \(\sigma\) be a proper fuzzy ideal of \(B\) such that

\[
h(\mu) \leq \sigma
\]

Then

\[
h^{-1}(h(\mu)) \leq h^{-1}(\sigma)
\]

Since \(\mu\) is \(h\)-invariant, we have \(\mu = h^{-1}(h(\mu))\). So that

\[
\mu \leq h^{-1}(\sigma)
\]

By Theorem 4.1, \(h^{-1}(\sigma)\) is a fuzzy ideal of \(A\). Moreover, since \(\sigma\) is proper, there exists \(y \in B\) such that \(\sigma(y) < 1\); that is, \(\sigma(y) = h(h^{-1}(\sigma))(y) < 1\), which gives \(h^{-1}(\sigma)(x) < 1\) for all \(x \in h^{-1}(y)\). This means, \(h^{-1}(\sigma)\) is a proper fuzzy ideal of \(A\) such that \(\mu \leq h^{-1}(\sigma)\). Since \(\mu\) is maximal, we get that \(\mu = h^{-1}(\sigma)\), which implies \(h(\mu) = h(h^{-1}(\sigma)) = \sigma\). Therefore \(h(\mu)\) is a maximal fuzzy ideal in \(A\).

**Theorem 4.6.** If \(h\) is a homomorphism from \(A\) onto \(B\) and \(\sigma\) is a maximal fuzzy ideal of \(B\), then \(h^{-1}(\sigma)\) is a maximal fuzzy ideal of \(A\).

**Theorem 4.7.** If \(h: A \rightarrow B\) is an onto homomorphism, then the mapping \(\mu \mapsto h(\mu)\) defines a one-to-one correspondence between the set of all \(h\)-invariant maximal fuzzy ideals of \(A\) and the set of all maximal fuzzy ideals of \(B\).

**Theorem 4.8.** Let \(h: A \rightarrow B\) be an onto homomorphism. If \(\mu\) is an \(h\)-invariant fuzzy ideal of \(A\), then

\[
h(\sqrt{\mu}) = \sqrt{h(\mu)}
\]
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**Theorem 4.9.** Let $h: A \to B$ is an homomorphism. If $\sigma$ is a fuzzy ideal of $B$, then

$$h^{-1}(\sqrt{\mu}) = \sqrt{h^{-1}(\mu)}$$

**Theorem 4.10.** If $h: A \to B$ is an onto homomorphism and $\mu$ is an $h$-invariant fuzzy semi-prime ideal of $A$, then $h(\mu)$ is a fuzzy semi-prime ideal of $B$.

**Theorem 4.11.** If $h$ is a homomorphism from $A$ to $B$ and $\sigma$ is a fuzzy semi-prime ideal of $B$, then $h^{-1}(\sigma)$ is a fuzzy semi-prime ideal of $A$.

**Theorem 4.12.** If $h: A \to B$ is an onto homomorphism, then the mapping $\mu \mapsto h(\mu)$ defines a one-to-one correspondence between the set of all $h$-invariant fuzzy semi-prime ideals of $A$ and the set of all fuzzy semi-prime ideals of $B$.

**REFERENCES**