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Some Properties for a Kind of the Heston Stochastic Volatility Model with Jump

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Abstract. Stochastic volatility models play an important role in finance modeling. In this work, we study the existence, uniqueness, continuity and some estimates of the solution to a kind of the Heston stochastic volatility model with jump.

Keywords: Heston stochastic volatility model with jump; Existence; Uniqueness; Continuity

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1. Introduction

In the classical finance model, a basic assumption is that the volatility is constant. However, several models proposed in recent years, such as the model found in [1], have allowed the volatility to be nonconstant or a stochastic variable [1,3]. Over the past decades, several stochastic volatility models have been developed to overcome the shortcomings of the Black–Scholes model [4-10], such as a missing smile or skew of the volatility. Among these models, the Heston stochastic volatility model [11] plays an important role, since it reproduces market smiles and skews and can be calibrated rapidly using semi-analytical formulas.

In this paper we study a jump-type Heston model (also called a stochastic volatility with jumps model, SVJ model)

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sqrt{v_1(t)}S(t)\mathrm{d}W_1(t) + \gamma S(t-)\mathrm{d}\tilde{N}_1(t), \tag{1.1}$$

$S(0) = s_0$ .The variance process $\{v(t), t \geq 0\}$ driven by another jump-diffusion progress satisfy

$$\mathrm{d}v(t) = \kappa(\theta - v(t))\mathrm{d}t + \sigma_1\sqrt{v(t)}\mathrm{d}W_2(t) + \sigma_2\sqrt{v(t-)}\mathrm{d}\tilde{N}_2(t), \tag{1.2}$$

$$\mathrm{d}W_1(t)\cdot\mathrm{d}W_2(t) = \mathrm{d}N_1(t)\cdot\mathrm{d}N_2(t) = \mathrm{d}W_1(t)\cdot\mathrm{d}N_2(t) = \mathrm{d}N_1(t)\cdot\mathrm{d}W_2(t) = 0,$$

where $v(0) = v_0$, v_0 and s_0 are given positive values, the non negative constants r, γ, θ and κ represent the interest rate, the volatility of jump diffusion term, the long variance, and the rate at which v reverts to θ. σ_1 and σ_2 have an impact on the volatility of variance process. The integrals with respect to the Wiener process $\{B(t), t \geq 0\}$ and compensate Poisson progress $\{\tilde{N}(t), t \geq 0\}$ are described as the Ito integral.

The main goal of this work is to investigate the existence, uniqueness and continuity of solutions to the dynamic model (1.1)-(1.2). The existence and uniqueness are analyzed in Section 2. Section 3 studies the continuity of the solution to the dynamic model (1.1)-(1.2).

2. The existence and uniqueness

In this section, we prove the existence and uniqueness of solution for the mixed fractional CEV model by extending the idea of [11-14].

Theorem 2.1. The volatility equation of the fractional mixed CEV model has a unique positive solution $v(t)$ where $t \in [0,T)$ and $T = \inf\{t>0 \mid v(t)=0\}$.

Proof: First, we confirm the solution existence for the volatility equation (1.2). We first define $Y_t^0 = v_0$ and $Y_t^{(k)} = Y_t^{(k)}(\omega)$ inductively as follows

$$Y_t^{(k+1)} = v_0 + \int_0^t \kappa(\theta - Y_s^{(k)})\mathrm{d}s + \int_0^t \sigma_1\sqrt{Y_s^{(k)}}\mathrm{d}W_2(s) + \int_0^t \sigma_1\sqrt{Y_{s-}^{(k)}}\mathrm{d}\tilde{N}_2(s)\,. \tag{2.1}$$

Therefore

$$\begin{aligned}&E[\left|Y_t^{(k+1)} - Y_t^{(k)}\right|^2]\\ &= E[\left|\kappa\int_0^t Y_s^{(k)} - Y_s^{(k-1)}\mathrm{d}s + \sigma_1\int_0^t \sqrt{Y_s^{(k)}} - \sqrt{Y_s^{(k-1)}}\mathrm{d}W_2(s) + \sigma_2\int_0^t \sqrt{Y_{s-}^{(k)}} - \sqrt{Y_{s-}^{(k-1)}}\mathrm{d}\tilde{N}_2(s)\right|^2].\end{aligned}$$

We know that $(a+b+c)^n \le 2^{n-1}(|a|^n + |b|^n + |c|^n)$, so

$$\begin{aligned}&E[\left|Y_t^{(k+1)} - Y_t^{(k)}\right|^2]\\ &\le 3\kappa^2 E[\left|\int_0^t (Y_s^{(k)} - Y_s^{(k-1)})\mathrm{d}s\right|^2] + 3\sigma_1^2 E[\left|\int_0^t \sqrt{Y_s^{(k)}} - \sqrt{Y_s^{(k-1)}}\mathrm{d}W_2(s)\right|^2]\\ &\quad + 3\sigma_2^2 E[\left|\int_0^t \sqrt{Y_s^{(k)}} - \sqrt{Y_s^{(k-1)}}\mathrm{d}\tilde{N}_2(s)\right|^2].\end{aligned} \tag{2.2}$$

Using the Holder inequality, one derives

$$E[\left|\int_0^t (Y_t^{(k)} - Y_t^{(k-1)})\mathrm{d}s\right|^2] \le t\int_0^t E[|Y_s^{(k)} - Y_s^{(k-1)}|^2]\mathrm{d}s \le T\int_0^t E[|Y_s^{(k)} - Y_s^{(k-1)}|^2]\mathrm{d}s\,. \tag{2.3}$$

Next we consider $E[\left|\int_0^t \sqrt{Y_s^{(k)}} - \sqrt{Y_s^{(k-1)}}\mathrm{d}W_2(s)\right|^2]$. Using the Ito isometry[12], we obtain

$$E[\left|\int_0^t \sqrt{Y_s^{(k)}} - \sqrt{Y_s^{(k-1)}}\mathrm{d}W_2(s)\right|^2] \le \int_0^t E[|\sqrt{Y_s^{(k)}} - \sqrt{Y_s^{(k-1)}}|^2]\mathrm{d}s \le 4\int_0^t E[|Y_s^{(k)} - Y_s^{(k-1)}|^2]\mathrm{d}s, \tag{2.4}$$

$$E[\left|\int_0^t \sqrt{Y_{s-}^{(k)}} - \sqrt{Y_{s-}^{(k-1)}}\mathrm{d}\tilde{N}_2(s)\right|^2] \le \lambda\int_0^t E[|\sqrt{Y_s^{(k)}} - \sqrt{Y_s^{(k-1)}}|^2]\mathrm{d}s \le 4\lambda\int_0^t E[|Y_s^{(k)} - Y_s^{(k-1)}|^2]\mathrm{d}s.$$

Here we used the fact that $\left|\sqrt{a} - \sqrt{b}\right| \le 2\left|a-b\right|$ for any $a>0$, $b>0$. Putting together (2.2), (2.3), and (2.4), we have

$$E[\left|Y_t^{(k+1)} - Y_t^{(k)}\right|^2] \le M_1\int_0^t E[|Y_s^{(k)} - Y_s^{(k-1)}|^2]\mathrm{d}s\,, \tag{2.5}$$

where $M_1 = 3\kappa^2 T + 12\sigma_1^2 + 12\lambda\sigma_2^2$.

Next, we pay attention to $h_1 = E[\left|Y_t^{(1)} - v_0\right|^2]$. Taking $k=0$ in (2.1), one obtains

$$h_1 = E[\left|\int_0^t \kappa(\theta - v_0)\mathrm{d}s + \int_0^t \sigma_1\sqrt{v_0}\mathrm{d}W_2(s) + \int_0^t \sigma_2\sqrt{v_0}\mathrm{d}\tilde{N}_2(s)\right|^2]$$
$$= E[\left|\kappa\theta t - \kappa v_0 t + \sigma_1\sqrt{v_0}W_2(t) + \sigma_2\sqrt{v_0}\tilde{N}_2(t)\right|^2].$$

Now we use $(a+b+c)^n \le 2^{n-1}(|a|^n + |b|^n + |c|^n)$ and Holder inequality to obtain

$$h_1 \le 3E[\left|\kappa\theta t - \kappa v_0 t\right|^2] + 3E[\left|\sigma_1\sqrt{v_0}W_2(t)\right|^2] + 3E[\left|\sigma_2\sqrt{v_0}\tilde{N}_2(t)\right|^2]$$
$$\le 3\kappa^2 E[\left|\theta - v_0\right|^2]t^2 + \frac{3}{2}(\sigma_1^2 + \sigma_2^2)E[|v_0|] + \frac{3}{2}\sigma_1^2 E[|W_2(t)|^2] + \frac{3}{2}\sigma_2^2 E[|\tilde{N}_2(t)|^2]$$
$$\le 3\kappa^2 E[\left|\theta - v_0\right|^2]t^2 + \frac{3}{2}(\sigma_1^2 + \sigma_2^2)E[|v_0|] + \frac{3}{2}\sigma_1^2 t + \frac{3}{2}\lambda\sigma_2^2 t.$$

Recall that $E[|W_2(t)|^2] = t$, $E[|\tilde{N}_2(t)|^2] = \lambda t$,

$$h_1 \le M_2 + M_3 t + M_4 t^2,$$

where $M_2 = \frac{3}{2}(\sigma_1^2 + \sigma_2^2)E[|v_0|]$, $M_3 = \frac{3}{2}\sigma_1^2 + \frac{3}{2}\lambda\sigma_2^2$, $M_4 = 3\kappa^2 E[\left|\theta - v_0\right|^2]$. Similarly, with the induction on k we obtain

$$h_{k+1} = E[\left|Y_t^{(k+1)} - Y_t^{(k)}\right|^2] \le \frac{M_5^k t^k}{(k+1)!}.$$

for some suitable constant M_5 only depends on M_i, $i = 1,2,3,4$. Thus, the existence follows from the Doob martingale inequality and Fatou lemma.

We now show that the solution of (1.2) is unique. Suppose $Y(t,\omega)$ and $Z(t,\omega)$ satisfy (1.2), $Y(0,\omega) = Y$ and $Z(0,\omega) = Z$. Therefore,

$$E[|Y(t,\omega) - Z(t,\omega)|^2]$$
$$= E[|Y - Z + \int_0^t \kappa(Y(s,\omega) - Z(s,\omega))\mathrm{d}s$$
$$+ \sigma_1\int_0^t \sqrt{Y(s,\omega)} - \sqrt{Z(s,\omega)}\mathrm{d}W_2(s) + \sigma_2\int_0^t \sqrt{Y(s-,\omega)} - Z(s-,\omega)\mathrm{d}\tilde{N}_2(s)|^2].$$

We may use Young's inequality to obtain

$$E[|Y(t,\omega) - Z(t,\omega)|^2]$$
$$\le 4E[|Y - Z|^2] + 4E[\left(\kappa\int_0^t Y(s,\omega)) - Z(s,\omega)\mathrm{d}s\right)^2] \tag{2.6}$$
$$+ 4\sigma_1^2 E[\left(\int_0^t \sqrt{Y(s,\omega)} - \sqrt{Z(s,\omega)}\mathrm{d}W_2(s)\right)^2] + 4\sigma_2^2 E[\left(\int_0^t \sqrt{Y(s-,\omega)} - \sqrt{Z(s-,\omega)}\mathrm{d}\tilde{N}_2(s)\right)^2].$$

Following the similar proof of (2.3), (2.4) and (2.5), we obtian

$$E[\left(\int_0^t Y(s,\omega) - Z(s,\omega)\mathrm{d}s\right)^2] \le t\int_0^t E[|Y(s,\omega) - Z(s,\omega)|^2]\mathrm{d}s \le T\int_0^t E[|Y(s,\omega) - Z(s,\omega)|^2]\mathrm{d}s, \tag{2.7}$$

and

$$E[\left(\int_0^t \sqrt{Y(s,\omega)} - \sqrt{Z(s,\omega)}\mathrm{d}W_2(s)\right)^2] \le 4\int_0^t E[|Y(s,\omega) - Z(s,\omega)|^2]\mathrm{d}s, \tag{2.8}$$

$$E[\left(\int_0^t \sqrt{Y(s-,\omega)} - \sqrt{Z(s-,\omega)}\mathrm{d}\tilde{N}_2(s)\right)^2] \le 4\lambda\int_0^t E[|Y(s,\omega) - Z(s,\omega)|^2]\mathrm{d}s. \tag{2.9}$$

Substituting (2.7) and (2.8) into (2.6) and letting $M_6 = 4T\kappa + 16T\sigma_1^2 + 16\lambda T\sigma_2^2$, we have

$$E[|Y(t,\omega)-Z(t,\omega)|^2]\le 4E[|Y-Z|^2]+M_6\int_0^t E[|Y(s,\omega)-Z(s,\omega)|^2]\mathrm{d}s\ .$$

Using Gronwall inequality, we have

$$E[|Y(t,\omega)-Z(t,\omega)|^2]\le 4E[|Y-Z|^2]\exp\{M_5 t\}\ .$$

The uniqueness of solution can be proved using

$$Y=Y(0,\omega)=Z(0,\omega)=Z\ .$$

Consequently, the theorem is proved.□

Next, we will derive L_p estimate for the solution of the volatility equation.

Lemma 2.1. Assume that Assumption 1 holds, and let $T>0$ be fixed. Then for any positive constant $M_7=M_7(v_0,T,p,\kappa,\theta,\sigma_1,\sigma_2)$, we have

$$E[\sup_{t\in[0,T]}\left|v(t)\right|^p]\le M_7\ . \tag{2.10}$$

Proof: Since for any $t\in[0,T]$,

$$v(t)=v_0+\int_0^t\kappa(\theta-v(s))\mathrm{d}s+\sigma_1\int_0^t\sqrt{v(s)}\mathrm{d}W_2(s)+\sigma_2\int_0^t\sqrt{v(s-)}\mathrm{d}\tilde{N}_2(s)\ .$$

Using Young' s inequality, we have for any $p\ge 2$ that

$$\left|v(t)\right|^p\le 4^{p-1}(|v_0|^p+A_1+A_2+A_3)\ , \tag{2.11}$$

where $A_1=\left|\int_0^t\kappa(\theta-v(s))\mathrm{d}s\right|^p$, $A_2=\left|\int_0^t\sqrt{v(s)}\mathrm{d}W_2(s)\right|^p$, $A_3=\left|\int_0^t\sqrt{v(s-)}\mathrm{d}\tilde{N}_2(s)\right|^p$. Now, we compute $E[A_1]$, $E[A_2]$ and $E[A_3]$. Using Holder inequality, we obtain

$$E[A_1]\le E[\left|\kappa\theta T+\kappa\int_0^t v(s)\mathrm{d}s\right|^p]\le 2^{p-1}\kappa^p\theta^pT^p+\kappa^pT\int_0^t E[|v(s)|^p]\mathrm{d}s\ . \tag{2.12}$$

By B-D-G's inequality[11] and Holder inequality, we obtain

$$E[A_2]=E[\left|\int_0^t\sqrt{v(s)}\mathrm{d}W_2(s)\right|^p]\le\left|\int_0^t E[|v(s)|]\mathrm{d}s\right|^{p/2}\le T\int_0^t E[|v(s)|^{p/2}]\mathrm{d}s\ .$$

Note that $x\le 1+x^2$ for any $x\ge 0$,

$$E[A_2]\le T+T\int_0^t E[|v(s)|^p]\mathrm{d}s\ . \tag{2.13}$$

Following the similar proof with (2.13),

$$E[A_3]\le\lambda T+\lambda T\int_0^t E[|v(s)|^p]\mathrm{d}s\ . \tag{2.14}$$

Substituting (2.12) and (2.13) into (2.11), and letting

$$M_8=4^{p-1}E[|v_0|^p]+8^{p-1}\kappa^p\theta^pT^p+4^{p-1}T(1+\lambda)\ ,\ M_9=4^{p-1}T(1+\lambda+\kappa^2)\ ,$$

we obtain

$$\left|v(t)\right|^p\le M_8+M_9\int_0^t E[|v(s)|^p)]\mathrm{d}s\ . \tag{2.15}$$

Hence the Gronwall inequality implies that

$$\sup_{t\in[0,T]}E[\left|v(t)\right|^p]\le M_8\exp\{M_9T\}=M_{10},\ p\ge 2\ . \tag{2.16}$$

Second, we prove that (2.10) still holds for any $1\le p<2$. Using Cauchy inequality, we obtain

$$E[\left|v(t)\right|^p]\le E[\left|v(t)\right|^{2p}]^{\frac{1}{2}}\le\left[\sup_{t\in[0,T]}E[\left|v(t)\right|^{2p}]\right]^{\frac{1}{2}}.$$

Considering $2p \ge 2$, and using (2.16), we obtain

$$E[|v(t)|^p] \le \sqrt{M_{10}} .$$

Because $t \in [0,T]$ is arbitrary, (2.10) is proved when $1 \le p < 2$.

Finally, if $0 < p < 1$, note that

$$|v(t)|^p = |v(t)|^p I_{\{|v(t)| \ge 1\}} + |v(t)|^p I_{\{|v(t)|<1\}} \le |v(t)|^{p+1} I_{\{|v(t)|\ge 1\}} + |v(t)|^p I_{\{|v(t)|<1\}} .$$

Further we have

$$|v(t)|^p \le |v(t)|^{p+1} I_{\{|v(t)|\ge 1\}} + 1 \le |v(t)|^{p+1} + 1 .$$

Hence it follows from the case $0 < p < 1$,

$$\sup_{t\in[0,T]} E[|v(t)|^p] \le M_{10} + 1 .$$

This completes the proof of the lemma□

Following the proof of Theorem 2.1 and Lemma2.1, we can prove the following lemma for stock price equation.

Lemma 2.2. Stock price equation of CEV model has a unique solution. In the case that $\beta(\cdot)$ and $\sigma(\cdot)$ satisfies Assumption 1, then

$$\sup_{t\in[0,T]} E[|S(t)|^p] \le M_{11}(v_0, s_0, T, p, r, \gamma, \kappa, \sigma_1, \sigma_2) . \qquad (2.17)$$

3. The model

In this section we discuss the continuity to stock price equation of CEV model.

Theorem 3.1. Stock price process of Black-Scholes model $\{S(t), t \ge 0\}$ is continuous.

Proof: Note that for any $0 \le s < t \le T$,

$$S(t) - S(s) = \int_s^t \mu S(s)\mathrm{d}s + \int_s^t \sqrt{v(s)} S(s)\mathrm{d}W_1(s) + \int_s^t \gamma S(s)\mathrm{d}\tilde{N}_1(s) .$$

Using $(a+b)^4 \le 2^3(a^4 + b^4)$, we obtain

$$|S(t) - S(s)|^2 \le 3A_4 + 3A_5 + 3A_6 , \qquad (3.1)$$

where $A_4 = \left|\int_s^t rS(s)\mathrm{d}s\right|^2$, $A_5 = \left|\int_s^t \sqrt{v(s)} S(s)^\alpha \mathrm{d}w(s)\right|^2$, $A_6 = \left|\int_s^t \sqrt{v(s)} S(s)^\alpha \mathrm{d}w(s)\right|^2$. It follows Cauchy inequality,

$$E[A_4] \le r^4 E[\left|\int_s^t S(s)\mathrm{d}s\right|^2] \le (t-s)\int_s^t E[|S(s)|^2]\mathrm{d}s . \qquad (3.2)$$

(2.17) and (3.2) imply that

$$E[A_4] \le r^2 M_{12} |t-s|^2 . \qquad (3.3)$$

Now we pay attention to $E[A_5]$. Using B-D-G inequality[11] and Holder inequality we obtain

$$E[A_5] \le \int_s^t E[|v(s)| \cdot |S(s)|^2]\mathrm{d}s] \le \int_s^t \sqrt{E[|v(s)|^2]E[|S(s)|^4]}\mathrm{d}s] .$$

It follows by (2.10) and (2.17) that

$$E[A_5] \le \sqrt{M_7 M_{11}} |t-s|, \; E[A_6] \le \lambda\sqrt{M_7 M_{11}} |t-s| . \qquad (3.4)$$

Substituting (3.3) and (3.4) into (3.1), we yeild

$$E[|S(t)-S(s)|^4] \le 3(1+\lambda)\sqrt{M_7 M_{11}}\,|t-s| + 3r^2 M_{12}\,|t-s|^2 . \tag{3.5}$$

Therefore, the theorem is proved. □

4. Conclusion

Stochastic volatility models play an important role in finance modeling. In this paper, we proposed a new version of the Heston model with long-range dependence by considering the properties of mixed fractional Brownian motion and showed this model has a unique solution. Moreover, we proved the continuity and some estimates of the solution.

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