Counting Distinct Fuzzy Subgroups of Symmetric Group \( S_5 \) by a New Equivalence Relation

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Abstract. One of the significant aspects of fuzzy group theory is a classification of the fuzzy subgroups of finite groups under a suitable equivalence relation. In this paper, we determine the number of distinct fuzzy subgroups of finite symmetric group \( S_5 \) by the new equivalence relation introduced by Tărnăuceanu. In this case, the corresponding equivalence classes of fuzzy subgroups of a group \( G \) are closely connected to the automorphism group and the chains of subgroups of \( G \).

Keywords: Equivalence Relation, Chains of Subgroups, Automorphism group

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1. Introduction

Classifying the fuzzy subgroups of a finite group has undergone rapid development, in recent years. The fuzzy subgroups of a group \( G \) can be classified up to some natural equivalence relations on the set consisting of all fuzzy subsets of \( G \). Many papers have treated the classification of the fuzzy subgroups for particular cases of finite groups with respect to suitable equivalence relation. Murali and Makamba studied equivalence classes of fuzzy subgroups of a given group under a suitable equivalence relation. Ogiugo and EniOluwafe have also the number of fuzzy subgroups of \( S_5 \) under the natural equivalence relation. We have computed the number of fuzzy subgroups of \( S_5 \) by this new equivalence relation [1]. This new equivalence relation generalizes the natural equivalence relation defined on the lattice of fuzzy subgroups. The paper, we present some preliminary definitions and necessary results on fuzzy subgroups and recall the new equivalence relation. Other types of fuzzy algebraic structures are available in [7-9].

2. Preliminaries

A fuzzy subset of a set \( X \) is a function \( \mu : X \to [0,1] \). Let \( G \) be a group with a multiplicative binary operation and identity \( e \), and let \( \mu : G \to [0,1] \) be a fuzzy subset of \( G \). Then \( \mu \) is said
to be a fuzzy subgroup of $G$ if

\begin{align*}
(1) & \quad \mu(xy) \geq \min\{\mu(x), \mu(y)\}, \\
(2) & \quad \mu(x^{-1}) \geq \mu(x) \text{ for all } x, y \in G
\end{align*}

The following elementary facts about fuzzy subgroups follow easily from the axioms: 

$$
\mu(x) = \mu(x^{-1}) \quad \text{and} \quad \mu(x) \leq \mu(e), \text{ for all } x \in G.
$$

Also, $\mu$ satisfies conditions (1) and (2) of Definition if and only if 

$$
\min\{\mu(x), \mu(y)\} \leq \mu(xy^{-1}), \text{ for all } x, y \in G.
$$

The set \{\mu(x) \mid x \in G\} is called the image of $\mu$ and is denoted by $\mu(G)$. For each $\alpha \in \mu(G)$, the set 

$$
\mu_\alpha := \{x \in G \mid \mu(x) \geq \alpha\}
$$

is called a level subset of $\mu$. It follows that $\mu$ is a fuzzy subgroup of $G$ if and only if its level subsets are subgroups of $G$. These subsets are useful in the characterization of fuzzy subgroups. (see [6])

Suppose $G$ is a finite group; then the number of subgroups of $G$ is finite whereas the number of level subgroups of a fuzzy subgroup $A$ appears to be infinite. But, since every level subgroup is indeed a subgroup of $G$, not all these level subgroups are distinct. In this paper, we count the classified distinct fuzzy subgroups of $S_5$.

Without any equivalence relation on fuzzy subgroups of group $G$, the number of fuzzy subgroups is infinite, even for the trivial group \{e\}. So we define an equivalence relation on the set of all fuzzy subgroups of a given group. We say that $\mu$ is equivalent to $\nu$, written as $\mu \sim \nu$, if we have

$$
\mu(x) > \mu(y) \iff \nu(x) > \nu(y), \text{ for all } x, y \in G
$$

and

$$
\mu(x) = 0 \iff \nu(x) = 0, \text{ for all } x \in G.
$$

Note that the condition $\mu(x) = 0$ holds if and only if $\nu(x) = 0$ simply says that the supports of $\mu$ and $\nu$ are equal and two fuzzy subgroups $\mu, \nu$ of $G$ are said to be distinct if $\mu \neq \nu$.

3. Methodology

Let $G$ be a finite group. Then it is well-defined the following action of $Aut(G)$ on $FL(G)$

$$
\rho : FL(G) \times Aut(G) \to FL(G)
$$

$$
\rho(\mu, f) = \mu \circ f, \text{ for all } (\mu, f) \in FL(G) \times Aut(G)
$$

Let us denote by $\sim_\rho$ the equivalence relation on $FL(G)$ induced by $\rho$, namely $\mu \sim_\rho \nu$ if and only if there exists $f \in Aut(G)$ such that $\nu = \mu \circ f$.

In this paper, it is called a new equivalence relation, [5]. There are other different versions of fuzzy equivalence relations in Literature.

The problem of classifying the fuzzy subgroup of finite group $G$ by using a new equivalence relation $\approx$ on the lattice of all fuzzy subgroups of $G$, its definition has a consistent group theoretical foundation, by involving the knowledge of the automorphism group associated to $G$. The approach is motivated by the realization that in a theoretical study of fuzzy groups, fuzzy subgroups are distinguished by their level subgroups and not by their images in $[0,1]$. Various enumeration techniques that are used in the counting of distinct fuzzy subgroups of a finite group. These counting techniques are derived from the interpretation of the definition of fuzzy equivalence relations used. The equivalence classes are called the orbits of the action, the orbit of a chain $C \in \mathcal{L}$ is $\{f(C) \mid f \in Aut(G)\}$, while the set of all chains in $\mathcal{L}$ that are fixed by an automorphism $f$ of $G$ is
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$$\text{Fix}_C(f) = \{ C \in C \mid f(C) = C \}$$

Now, the number $N$ is obtained by applying the Burnside’s lemma:

$$N = \frac{1}{|\text{Aut}(G)|} \sum_{f \in \text{Aut}(G)} |\text{Fix}_C(f)|$$

The Burnside’s lemma plays an important role in the explicit formula to compute the number of distinct fuzzy subgroups $N$ of a finite group $G$ with respect to a certain equivalence relation on the lattice of fuzzy subgroups, induced by an action of the automorphism group $\text{Aut}(G)$ associated to $G$ (see[5]).

4. Main results
Let $C \in \text{Fix}_C(f)$, where $C : H_1 \subset H_2 \subset \cdots \subset H_m = S_n$. Then $f(C) = C$, that is $f(H_i) = H_i$, for all $i = 1 \leq i \leq m$.

Then every automorphism of $S_n$ is of the form $f_{\sigma}$ with $\sigma \in S_n$. In fact, for $n \neq 2, 6$ the symmetric group is complete group. It is well-known that $S_5$ has 120 elements. The arrangement of elements according to their cycle structure reveals the conjugacy classes in $S_5$.

**Proposition 4.1.** Two elements of $S_n$ are conjugate if and only if they have the same cycle structure. It is well-known that cycle types determine the conjugacy classes in $S_n$.

**Proposition 4.2.** Two elements of $S_n$ are conjugate if and only if they have the same cycle structure.

It is well-known from classical group theory, an automorphism of a group $G$ permutes the conjugacy classes in $G$, and the inner automorphisms preserve each conjugacy class.

**Theorem 4.1.** [1] Let $a, b$ of $S_n$ be conjugate, then the set of the number of chains of subgroups of $S_n$ for $n \neq 2, 6$ fixed by the automorphism $f$ is equal.

The set of chains of subgroups of $S_5$ fixed by the automorphism $f$ can be represented by the cycle structure of $S_5$. The inner automorphisms of $S_5$ preserve each conjugacy class in $S_5$.

**Theorem 4.2.** [3] The number of chains of subgroups of $S_5$ that ends in $S_5$ is 3784.

We shall compute the number of all distinct fuzzy subgroups of $S_5$ using the counting technique (#) from the definition of the new equivalence relation induced by the action of automorphism groups.

$$f_{\tau} : S_5 \rightarrow S_5, \quad f_{\tau}(\sigma) = \tau^{-1}\sigma \tau$$

Then every automorphism of $S_5$ is of the form $f_{\sigma}$ with $\sigma \in S_5$

$$\text{Aut}(S_5) = \{ f_1, f_2, f_3, f_6, f_6, f_7, \cdots, f_{118}, f_{119}, f_{120} \}$$

$$f_1 = f_e = 3784$$
The number of the set of all chains in $S_5$ that are fixed by an automorphism $f$ of $S_5$ is 26880

$$h(S_5) = \frac{1}{|\text{Aut}(S_5)|} \sum_{f \in \text{Aut}(S_5)} |\text{Fix}_E(f)|$$

Theorem 4.3. The number of all distinct fuzzy subgroups with respect to $\approx$ of the symmetric group $S_5$ is 224

5. Conclusion
In this paper, we have determined the number of distinct fuzzy subgroups for symmetric group $S_5$ by the new equivalence relation which generalizes the natural equivalence relation used in [2].

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