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The Generalized Quarternion *p*-Group of Order 2^n: Discovering the Fuzzy Subgroups

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Abstract. In this paper, the classification of finite *p*-groups is extended to the cartesian product of the generalized quarternion group of order 2^n with a cyclic group of order 2 which also belongs to the class of the famous nilpotent groups .

Keywords: Finite *p*-groups, nilpotent group, fuzzy subgroups, dihedral group, quaternion group, inclusion-exclusion principle, maximal subgroups

AMS Mathematics Subject Classification (2010): 20D15, 20E28, 20F18, 20N25, 20K27

1. Introduction

The following properties for the fuzzy subgroups of G were known.

1. The level sets of a fuzzy subset of a finite set form a chain.
2. λ is a fuzzy subgroup of G iff its level sets are subgroups of G'
3. The relation $:$ is an equivalence relation on fuzzy subgroups of G,where

for fuzzy subgroups μ,ν of G , $\mu:\nu$ iff $\forall x,y\in G,(\mu(x)>\mu(y) iff \nu(x)>\nu(y))$

Some related algebraic structures are discussed in [7-10].

2. Priliminaries

Suppose that $(G,\cdot,e)$ is a group with identity e. Let $S(G)$ denote the collection of all fuzzy subsets of G. An element $\lambda\in S(G)$ is said to be a fuzzy subgroup of G if the following two conditions are sat.

1. $\lambda(ab)\geq\in\{\lambda(a),\lambda(b)\},\ \forall\ a,b\in G$;
2. $\lambda(a^{-1}\geq\lambda(a)$ for any $a\in G$.

And, since $(a^{-1})^{-1}=a$, we have that $\lambda(a^{-1})=\lambda(a)$, for any $a\in G$.

Also, by this notation and definition, $\lambda(e)=\sup\lambda(G)$. (Marius [3] and [4]).

Now, concerning the subgroups, the set $FL(G)$ possessing all fuzzy subgroups of G forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of G.

In what follows, the method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of G. Let $h(G)$ denote the number of chains of subgroups of G which ends in G. The method of computing $h(G)$ is based on the application of the Inclusion-Exclusion Principle. If A is the set of chains in G of type $C_1 \subset C_2 \subset \cdots \subset C_r = G$, and A' represents the set of chains of A' which are contained in M_r, $r = 1, \ldots, t$.

Then we have:

$$|A| = 1 + |A'| = \left|\bigcup_{r=1}^{t} A_r\right|$$

$$= 1 + \sum_{r=1}^{t} |A_r| - \sum_{1 \le r_1 \le r_2 \le t} |A_{r_1} \cap A_{r_2}| + \cdots + (-1)^{t-1}\left|\bigcap_{r=1}^{t} A_r\right|$$

Observe that, for every $1 \le w \le t$ and $1 \le r_1 < r_2 < \cdots < r_w \le t$, the set $\bigcap_{i=1}^{w} A_{r_i}$ consists of all chains of A' which are included in $\bigcap_{i=1}^{w} M_{r_i}$. We have that

$$\left|\bigcap_{i=1}^{w} A_{r_i}\right| = 2h\left(\bigcap_{i=1}^{w} M_{r_i}\right)^{-1}$$

$$\therefore\ |A| = 1 + \sum_{r=1}^{t}(2h(M_r) - 1) - \sum_{1 \le r_1 < r_2 \le t}(2h(M_{r_1} \cap M_{r_2}) - 1)$$

$$+ \cdots + (-1)^{t-1}\left(2h\left(\bigcap_{r=1}^{t} M_r\right) - 1\right)$$

$$= 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1 \le r_1 < r_2 \le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right) + C$$

And

$$C = 1 + \sum_{r=1}^{t}(-1) - \sum_{1 \le r_1 < r_2 \le t}(-1) + \cdots + (-1)^{t-1}(-1)$$

$$= (1-1)^t = 0$$

we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1 \le r_1 < r_2 \le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right) \quad (3)$$

(3) was used to obtain the explicit formulas of $h(D_{2n})$ for some positive integers n.

Theorem 1. The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is:

1. $h(\mathbb{Z}_{p^n}) = 2^n$
2. $h(D_{2^n}) = 2^{2n-1}$
3. $h(\varphi_{2^n}) = 2^{2n-2}$
4. $h(S_{2^n}) = 3.2^{2n-3}$
5. $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2+(n-1)p]$

Following our paper[1](Also see [2] and [5]) the following equation(#) based on the usual Inclusive-Exclusive technique is appliied :

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\le r_1<r_2\le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right) (\#)$$

In [4], (1) was used to obtain the explicit formulas of $h(D_{2n})$ for some positive integers n.

Theorem 2. [3] The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is:

(i) $h(\mathbb{Z}_{p^n}) = 2^n$

(ii) $h(D_{2^n}) = 2^{2n-1}$

(iii) $h(\varphi_{2^n}) = 2^{2n-2}$

(iv) $h(S_{2^n}) = 3.2^{2n-3}$

(v) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2+(n-1)p]$

3. The Cartesian product of the quaternion group of order 2n and a cyclic group of order 2

We begin from the simplest form of it, putting $n = 3$.
By the application of equation (1), we have:

$$\frac{1}{2}h(G) = 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + 4h(Q_{2^3}) - 12h(\mathbb{Z}_{2^2}) - 2h(\mathbb{Z}_2 \times \mathbb{Z}_2) + 8h(\mathbb{Z}_2) = 88$$

$$\therefore\ h(Q_{2^3} \times C_2) = 2\times 88 = 176.$$

4. **Computation of the Fuzzy Subgroups of $Q_{2^4} \times C_2$**

Let $G = Q_{2^4} \times C_2$. Then, by equation (c)

$\frac{1}{2}h(G) = h(Z_2 \times Z_{2^3}) + 2h(Q_{2^3} \times C_2) + 4h(Q_{2^4}) - 8h(Q_{2^3}) - 2h(Z_2 \times Z_{2^2}) - 4h(Z_{2^3}) + 8h(Z_{2^2}) = 496$

$\therefore \ h(Q_{2^4} \times C_2) = 2 \times 496 = 992.$

By observing the structure of the nilpotent group $Q_{2^n} \times C_2$, in general, the group possesses seven maximal subgroups. And so, by using equation (c), putting $G = Q_{2^n} \times C_2$, we have

$$\frac{1}{2}h(G) = 2^n + 2^{2n-1} + 2h(Q_{2^{n-1}} \times C_2)$$

$$\therefore \ h(Q_{2^n} \times C_2) = n.2^{2n} - 2^{n+1}, \ n \geq 3.$$

Theorem 3. Let $G = Q_{2^n} \times C_2$, the nilpotent group obtained by taking the cartesian product of the generalised quaternion group of order 2^n, and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of G is :

$$h(G) = n \cdot 2^{2n} - 2^{n+1}, \ \text{for} \ n > 3$$

Proof: One maximal subgroup of this group is isomorphic to $Z_2 \times Z_{2^{n-1}}$, two of the maximal subgroups are isomorphic to $Q_{2^{n-1}} \times C_2$, while 2^2 of the maximal subgroups are isomorphic to Q_{2^n}. And so, by using equation (3.2.1), putting $G = Q_{2^n} \times C_2$, we have

$\frac{1}{2}h(G) = h(Z_2 \times Z_{2^{n-1}}) + 2h(Q_{2^{n-1}} \times C_2) + 4h(Q_{2^n}) - 6h(Z_{2^{n-1}}) - 3h(Z_2 \times Z_{2^{n-2}}) - 12h(Q_{2^{n-1}}) + 28h(Z_{2^{n-2}}) + 2h(Z_{2^{n-1}}) + h(Z_2 \times Z_{2^{n-2}}) + 4h(Q_{2^{n-1}}) - 20h(Z_{2^{n-2}})$

Putting in the values of the surfacing structures from we have that :

$h(Q_{2^n} \times C_2) = n.2^{2n} - 2^{n+1}, \ n \geq 3.$ Applying induction on n, by setting F(n) = $h(Q_{2^n} \times C_2)$, ascertaining the truth of

F(k) = $2^{k+1} + 2^{2k} + 4h(Q_{2^{k-1}} \times C_2) = k.2^{2k} - 2^{k+1}, k \geq 3,$

we show that F(k+1) is also true.

Hence, F(k+1) = $2^{k+2} + 2^{2(k+1)} + 4h(Q_{2^k} \times C_2) = 2^2[(k+1)2^{2k} - 2^k].$

Theorem 4. (see [2] and [6]) Suppose that $G = D_{2^n} \times C_4$. Then, the number of distinct fuzzy subgroups of G is given by :

$$2^{2(n-2)}(64n + 173) + 3\sum_{j=1}^{n-3} 2^{(n-1+j)}(2n + 1 - 2j)$$

6. Conclusion

Finally, the product of the generalised quarternion p-group of order 2^n and a cyclic group of order 2 have been successfully classified and the number of distinct fuzzy subgroups were directly computed using comprehensive analysis and the application of GAP(Group Algorithms and Programming, Version 4.8.7; https:www.gap-system.org)

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