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Application of Non Polynomial Spline to Solve the Third Order Boundary Value Problems

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Abstract. In this paper we used quartic non-polynomial splines to improve a new numerical method for computing approximations to the solution of third order boundary value problems, and shown the new method gives approximations, which are better than those produced by other collocation, finite-difference, and spline methods. Convergence analysis of the method is discussed through standard procedures. A numerical example is given here to illustrate the efficiency and applicability of the novel method.

Keywords: Non-polynomial Spline method, boundary value problems, convergence

AMS Mathematics Subject Classification (2010): 41A15

1. Introduction

Consider the following third order boundary value problems

$$u^{(3)}(x) + p(x)u(x) = r(x), \; x \in [a, b] \tag{1}$$

With the boundary conditions:

$$u(a) = \gamma, u'(a) = \varepsilon \;, u'(b) = \sigma \tag{2}$$

where γ , ε and σ are finite real constants. The functions $p(x)$ and $r(x)$ are continuous on the interval $[a, b]$ The analytical solution of the problem (1-2) cannot be obtained for arbitrary choices of $p(x)$ and $r(x)$. The numerical analysis literature contains a few numbers of other methods developed to explain an approximate solution of this problem. The third-order problems solved by using quartic spline Noor et al. [1]. By using quintic splines for the numerical solution of third-order boundary-value problems Khan et al. [2]. Forth-degree B-spline functions for numerical solution of third-order boundary-value problems with Caglar et al. [4] and other [5] numerical solutions of third-order system of boundary value problems. The aim of this paper is to construct a non-polynomial spline function to construct a new spline method based on a that has a trigonometric part, and a polynomial part to develop numerical methods in order to obtaining smooth approximations to find the solution of the problem (1) with the boundary conditions (2). Recently, new methods based on a non-polynomial spline function that have three parts the trigonometric part, exponential part and a polynomial part are used to develop numerical methods for obtaining numerical approximation for boundary-value problems, [7, 8]. A postponement of exertion of Islam et al. [9]. In this paper we propose also for the numerical solution of third order boundary value problem (1) the quartic non-polynomial spline method with boundary conditions. in this paper define a non-polynomial spline method is systematized into six sections. In section number 2 gives a brief derivation for method. While in Section 3 the method is made in a matrix form. which explain in Section 4 Convergence analysis of the method. While in section number 5,demonstrate the efficiency of my method I presented the numerical examples and their comparison with the existing methods . In section 6, we explain the conclusion of the numerical results. of the proposed method.

2. The method description

To derive non-polynomial spline approximation for equation (1) with boundary conditions in equations (2), the interval $[a,b]$ was distributed into n like subintervals :

$x_i = a + ih$, $i = 0,1,\dots,n$ where $a = x_0$, $b = x_{n+1}$ and $h = \frac{b-a}{n}$

To the exact solution assume the approximation $u(x)$ was considered to be , which was obtained by using the Quartic non-polynomial spline $S_i(x)$, passing by the points (x_i,u_i) and (x_{i+1},u_{i+1}) Then $S_i(x)$ was required to satisfy the conditions at (x_i,x_{i+1}) the boundary conditions in equations (1),and also the continuity condition of first derivative at the point (x_i,u_i) For every partition $[x_i,x_{i+1}]$, the non-polynomial spline $S_i(x)$ can be written in the following

$$S_i(x) = a_i \cos\omega(x - x_i) + b_i e^{-\omega(x-x_i)} + c_i(x - x_i)^2 + d_i(x - x_i) + e_i \quad (3)$$

Here a_i, b_i, c_i, d_i and e_i are arbitrary constant values and ω denotes the free parameter. The non-polynomial function $S_i(x)$. Chosen from class $C^2[a,b]$, interpolates $u(x)$, at the common knots x_i, $i = 0,1,\dots,n$ rely on the parameter and then converted to an ordinary quartic spline $S_i(x)$ over $[a,b]$ when $\omega \to 0$.

Let

$$\begin{cases} S_i(x_i) = u_i\,, S_i(x_{i+1}) = u_{i+1}\,, S^{(1)}(x_i) = V_i\,, \\ S^{(1)}(x_{i+1}) = V_{i+1}\,, S^{(3)}(x_i) = Z_i\,, S^{(3)}(x_{i+1}) = Z_{i+1} \end{cases}$$

By using the conditions of continuity on first and second derivatives at the points (x_i,u_i) and through simple algebraic manipulation, the constants in equation (3) can be obtained in the form,

$$a_i = \frac{h^3\left(Z_{i+1} + Z_i e^{-\theta}\right)}{\theta^2 \sin(\theta)}, \qquad b_i = \frac{-h^3 Z_i}{\theta^3}$$

$$c_i = \frac{h^3\left(\sin(\theta)\left(Z_i e^{-\theta} - Z_i + V_i + \theta Z_i\right) + \theta^3(1 - u_i)\right) + \cos(\theta)\left(e^{-\theta} - Z_{i+1}\right)\right)}{\theta^3 h^2 \sin(\theta)}$$

$$d_i = V_i - \frac{kh^3 Z_i}{\theta^3}, \qquad e_i = u_i + \frac{(-Z_{i+1} + Z_i e^{-\theta})}{\theta^3 \sin(\theta)} + \frac{h^3 Z_i}{\theta^3}$$

And at the points (x_i, u_i) we obtained the consistency relation as following :

$$V_i + V_{i-1} = \frac{2}{h}(u_i - u_{i-1}) + \frac{h^3\left(e^{-\theta}(2\cos(\theta) - \theta\sin(\theta) + \sin(\theta) - 2) + \sin(\theta)(2\theta+1)\right)Z_{i-1}}{\theta^3 \sin(\theta)} + h^3\frac{(\theta\sin(\theta) - \cos(\theta) - 2)}{\theta^3 \sin(\theta)} Z_i \qquad (4)$$

$$V_i - V_{i-1} = \frac{2}{h}(u_{i+1} - u_{i-1}) + \frac{1}{2}h^3\left(\frac{\sin(\theta)(\theta^2+\theta)}{\theta^3\sin(\theta)} + \frac{2(1-\cos(\theta))}{\theta^3\sin(\theta)}\right)Z_{i+1} + \frac{1}{2}h^3\left(\frac{e^{-\theta}(2\cos(\theta) + \sin(\theta) + \theta^2\cos(\theta) + \theta^2\sin(\theta)) + \sin(\theta)(\theta+1)}{\theta^3\sin(\theta)}\right)Z_{i-1} + \frac{1}{2}h^3\left(\frac{e^{-\theta}(2\cos(\theta) + \theta\sin(\theta) + 2\sin(\theta) - \theta^2 - 2) - \cos(\theta)(\theta^2 - 1) + (2 + \theta^3\sin(\theta))}{\theta^3\sin(\theta)}\right)Z_i \qquad (5)$$

Adding equations (4) and (5) we get

$$V_i = \frac{1}{2h}(u_{i+1} - u_{i-1}) + h^2\left(\frac{h^3(2 - \theta^2 - 2\cos(\theta))}{4h\theta^3\sin(\theta)}\right)Z_{i+1} + h^2\left(\frac{\left(\cos(\theta)(\sin(\theta)+1) - \sin(\theta)(\theta^2 + 2e^{-\theta}\right) + \theta^3 e^{-\theta}}{4\theta^3\sin(\theta)}\right)Z_i + h^2\left(\frac{\left(e^{-\theta}\sin(\theta)(4\theta + 2 + \theta^2 + 2) + \cos(\theta)(2 - \theta^2) + 2\right)}{4\theta^3\sin(\theta)}\right)Z_{i-1} \qquad (6)$$

by replace i with $i-1$ in equation (6) we get the following

$$V_{i-1} = \frac{1}{2h}(u_{i+1} - u_{i-1}) + h^2\left(\frac{(2-\theta^2-2\cos(\theta))}{4\theta^3\sin(\theta)}\right)Z_i + h^2\left(\frac{\left(\cos(\theta)(\sin(\theta)+1) - \sin(\theta)(\theta^2+2e^{-\theta}\right)+\theta^3 e^{-\theta}}{4\theta^3\sin(\theta)}\right)Z_{i-1} + h^2\left(\frac{\left(e^{-\theta}\sin(\theta)(4\theta+2+\theta^2+2) + \cos(\theta)(2-\theta^2)+2\right)}{4\theta^3\sin(\theta)}\right)Z_{i-2} \qquad (7)$$

Substituting equation (6) and (7) into equation (5) we get the following equation

$$- u_{i-2} + 3\, u_{i-1} - 3u_i + u_{i+1} = h^2 \left(\frac{e^{-\theta}(1+6\cos(\theta)+6\sin(\theta)+4\cos(\theta)-3\sin(\theta))+\sin(\theta)(4\theta-6)+3\,\theta^2\cos(\theta)}{4\theta^3 \sin(\theta)} \right) (Z_{i-2} + Z_{i+1}) + h^2 \left(\frac{(e^{-\theta}(4\,\cos(\theta)-4\,\sin\theta-\theta)+\sin(\theta))+(4\theta-\theta^2+4\,\theta\sin\theta+2)-2\cos(\theta)}{4\theta^3 \sin(\theta)} \right) (Z_i + Z_{i-1}) \qquad (8)$$

For simplicity, equation (8) can be re-written in the following form:

$$-u_{i-2} + 3\, u_{i-1} - 3u_i + u_{i+1} = h^3 \lfloor \xi\, (Z_{i-2} + Z_{i+1}) + \lambda\, (Z_i + Z_{i-1}) \rfloor \qquad (9)$$

where

$$\xi = \frac{e^{-\theta}(1+6\cos(\theta)+6\sin(\theta)+4\cos(\theta)-3\sin(\theta))+\sin(\theta)(4\theta-6)+3\,\theta^2\cos(\theta)}{4\theta^3 \sin(\theta)}$$

$$\lambda = \frac{e^{-\theta}(4\,\cos(\theta) - 4\,\sin\theta - \theta) + \sin(\theta)) + (4\theta - \theta^2 + 4\,\theta\sin\theta + 2 - 2\cos(\theta)}{4\theta^3 \sin(\theta)}$$

Equation. (9) gives $n - 2$ equations in the n unknowns $u_i, i = 0,1, \dots n$. We must added two equations to the direct computation of $u_i, i = 0,1, \dots, n$. These 2 equations are developed by Taylor series and by the method of undetermined coefficients,

$$3\, u_0 - 4\, u_1 + u_1 = -2\, hV_0 + h^3 \left(\frac{1}{4} Z_0 + \frac{1}{3} Z_1 + \frac{1}{12} Z_2 \right) \qquad i = 1 \qquad (10)$$

$$3\, u_{n-2} - 8\, u_{n-1} - 5u_n = h^3 \left(-\frac{1}{6} Z_{n-2} - Z_{n-1} - \frac{5}{2} Z_n \right) \qquad i = n \qquad (11)$$

The local truncation errors $t_i, i = 1,2,3, \dots n.$,. associated with equation (9 - 11) can be obtained as follows:

$$t_i = \begin{cases} \frac{1}{10} u_i^{(5)} + O(h^6)\,, \quad i = 1. \\ h^3(1 - 2\,\xi - 2\lambda)u_i^{(3)} + h^4 \left(\frac{1}{2} + \xi + \lambda \right) u_i^{(4)} + h^5 \left(\frac{1}{8} - \frac{3}{5}\xi + \frac{1}{4}\lambda \right) u_i^{(5)} \\ +h^6 \left(\frac{-1}{12} + \frac{7}{6}\,\xi + \frac{1}{6}\lambda \right) u_i^{(6)} + h^7 \left(\frac{1}{40} - \frac{17}{24}\xi - \frac{1}{24}\lambda \right) u_i^{(7)} + O(h^8)\,, \quad i = 2,3, \dots, n-1\,. \\ \frac{-h^5}{60} u_i^{(5)} + O(h^6), \quad i = n. \end{cases}$$

3. Non polynomial Spline function

Spline solution of (1) with the boundary condition (2) is based on linear equations given by (9-11). Let $W = w_i$, , $U = u_i$, $Q = q_i$, $T = t_i$,$E = U - t_i$ be n-dimensional column vectors. Where W , U, T and E are Exact, approximate, truncation error and error n-column vectors respectively. Then I can write the standard matrix equations in the form

$$AW = Q + T, \qquad AU = W, AE = T \tag{12}$$

We also have

$$A = A_0 + hRP, \qquad P = \mathrm{diag}(p_i) \tag{13}$$

where A_0 and R are four-band matrices. The four-band matrix A_0 has the form

$$A_0 = \begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 0 \\ 3 & -3 & 1 & 0 & 0 & 0 \\ -1 & 3 & -3 & 1 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & 3 & -3 & 1 \\ 0 & 0 & 0 & 3 & -8 & 5 \end{bmatrix} \tag{14}$$

and the four-band matrix G has the form

$$G = \begin{bmatrix} \frac{1}{3} & \frac{1}{12} & 0 & 0 & 0 & 0 \\ \lambda & \lambda & \xi & 0 & 0 & 0 \\ \xi & \lambda & \lambda \quad 0 & \xi \quad \ddots & \ddots \quad \ddots & 0 \quad 0 \\ 0 & 0 & & \lambda & & \xi \\ 0 & 0 & \xi & \frac{-1}{8} & \lambda & \frac{-15}{6} \\ 0 & 0 & 0 & & -1 & \end{bmatrix} \tag{15}$$

For the vector Q, we have

$$q_i = \begin{cases} -3\gamma - 2h\varepsilon + h^3\left(\frac{1}{4}(r_0 - p_0\gamma) + \frac{r_0}{3} + \frac{r_2}{12}\right), & i = 1. \\ \gamma h^3\big(\xi(r_0 - p_0\gamma + r_3) + \lambda(r_1 + r_2)\big), & i = 2 \\ h^3\big(\xi(r_{i-2} + r_{i+1}) + \lambda(r_{i-2} + r_{i+1})\big), & i = 3,4,\ldots,n-1. \\ -2h\sigma + h^3\left(\frac{r_{n-1}}{6} + \frac{15r_{n-1}}{6}\right), & i = n. \end{cases} \tag{16}$$

4. Convergence analysis of the method

Our main purpose here is to derive a bound ||E||. We now turn back to the error equation in (12) and rewrite it in the following form

$E = A^{-1}T = [A_0 + h^3RP]^{-1}T = \left[I + h^3{A_0}^{-1}P\right]^{-1}{A_0}^{-1}T$

$$\|E\|_\infty \le \frac{\left\|{A_0}^{-1}\right\|_\infty \|T\|_\infty}{h^3\left\|{A_0}^{-1}\right\|_\infty \|R\|_\infty \|P\|_\infty} \tag{17}$$

provided that $h^3\left\|{A_0}^{-1}\right\|_\infty \|R\|_\infty \|P\|_\infty$. From equation (6) we have

From [4] we get $\|T\|_\infty = h^3\left(\frac{1}{8} - \frac{3}{5}\,\xi - \frac{1}{4}\lambda\right)A_0{}^{-1}, A_0{}^{-1} = \max_{a\le x\le b}\left|u^5(x)\right|$ that

$$\left\|D_0{}^{-1}\right\|_\infty \le \frac{\Omega}{h^3} \tag{18}$$

where Ω is constant independent h . Then from equation. (17) we have

$$\|E\|_\infty \le \frac{\Omega A_0{}^{-1} h^2\left(\frac{1}{8}-\frac{3}{5}\,\xi-\frac{1}{4}\lambda\right)h}{1-\Omega\|P\|_\infty\,\|R\|_\infty} \le O(h^2) \tag{19}$$

5. Numerical examples

Now to illustrating the comparative performance of our method we consider 2 numerical examples (12) over other existing methods. All calculations are implemented by Maple 13.

Example 1.

Consider the boundary value problems:

$u^{(3)}(x) + u = (x-4)\sin(x) + (1-x)\cos(x) \qquad x \in [0,1]$

$u(0) = 0, u'(0) = 0\ , u'(1) = \sin(1)$

The analytic solution is $u(x) = x(x-1)\sin(x)$

The numerical solution in term of maximum absolute error are given in Table(1).

Example 2.

Consider the boundary value problems:

$u^{(3)}(x) - xu = (x^3 - 2x^3 - 5x - 3)e^x \qquad x \in [1,2]$

$u(0) = 0, u'(1) = 0\ , u'(2) = -e$

The analytic solution is $u(x) = x(1-x)e^x$

The numerical solution in term of maximum absolute error are given in Table (2)

Table 1: The maximum absolute errors for Example 1

H	Our method	Abd El-Salam et al [6]
0.0625	1.7412×10^{-10}	2.3819×10^{-8}
0.03125	2.8531×10^{-11}	1.1184×10^{-9}
0.015625	1.7915×10^{-11}	6.3020×10^{-9}
0.0078125	2.1483×10^{-13}	3.7640×10^{-11}

Table 2: Maximum absolute errors for Example 2

H	Our method	Abdullah et al [3]
0.0625	6.311×10^{-7}	5.7055×10^{-5}
0.03125	8.5729×10^{-8}	6.8944×10^{-6}
0.015625	7.6141×10^{-9}	8.4476×10^{-7}
0.0078125	2.6174×10^{-9}	1.0442×10^{-7}

6. Conclusion

In this paper we used the Quartic non polynomial spline function to develop a numerical method to solving third order boundary value problem. We shows that the developed method maintains a very remarkable and high accuracy that makes it so encouraging to deal with the solutions of singularly perturbed boundary value problems.

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