Brief note
The General Classification of the Modular Group of the Structure $M_p^n \times C_p$

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Abstract. In this paper, the classification of finite $p$-groups is extended to the group of the modular structure $M_p^n \times C_p$, and the number of distinct subgroups were computed, making an entire classification of the given structure possible for any given prime $p$

Keywords: Finite $p$-Groups, Nilpotent Group, Fuzzy subgroups, Dihedral Group, Inclusion-Exclusion Principle, Maximal subgroups. Explicit formulae, non-cyclic subgroup, prime.

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1. Introduction
Let $h(G)$ be the number of chains of subgroups of $G$ which ends in $G$. This actually represents the distinct number of the fuzzy subgroups of $G$ (see [5]). Suppose that $M_1, M_2, ..., M_t$ are the maximal subgroups of $G$ The method of computing $h(G)$ is based on the application of the Inclusion-Exclusion Principle. This had been extensively discussed in our article [1]. Following our paper [1] (Also see [3] and [4]) the following equation (x) based on the usual Inclusive-Exclusive technique is applied :

$$h(G) = 2 \left( \sum_{r=1}^{t} h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h \left( \bigcap_{r=1}^{t} M_r \right) \right)$$

The fuzzy subgroup for the nilpotent group of the form: $M_p^n \times C_p$.

We approach this particular concept from two distinct perspectives namely; when $p = 2$ and in the subsequent case, $p > 2$. And, of course, $p$ is a prime.

The nilpotent 2-group of the form $M_{2^n} \times C_2$.
This case was already settled in one of our papers.
The derivation of $h(M_p^n \times C_p)$ for $p > 2$

We begin with the case $p = 3$ and $n = 3$ By theorem (γ), there exists 13 distinct maximal subgroups for $M_3^3 \times C_3$.

By (x), we have:

$$-27h(\ZZ_3^2) - 12h(\ZZ_3 \times \ZZ_3) + 27h(\ZZ_3) \therefore h(M_3^3 \times C_3) = 420 + 2h(\ZZ_3 \times \ZZ_3) = 736.$$

2. Determination of $h(M_p^n \times C_2)$

Following a careful analysis and subsequent operations on the maximal subgroups, we have in general, an estimate given by:

Theorem 1. Let $G = M_p^n \times C_2$, the modular nilpotent group formed by taking the cartesian product of the modular $p$-group of order $p^n$ and a cyclic group of order $p$, where $p$ is a prime.

Then, the number of distinct fuzzy subgroups of $G$ for $n > 4$ is given by:

$$h(G) = \left\{ \begin{array}{ll}
2^n \left[ 3 \left( n + \frac{5}{6} \right) - \frac{73}{12} \right], & p = 2 \\
2^{n-1} \left[ p^2 (1 + p) n^2 + p (3 + p - 4p^2) n + (7p^2 - 3p^2 - 3p + 4) \right] - 4p^3, & p > 2 \end{array} \right.$$ \[Proof: \] For all values of $p$, there exist only one maximal subgroup which is isomorphic to the abelian type: $\ZZ_p \times \ZZ_p \times \ZZ_p$, $p$ of the maximal subgroups are isomorphic to $M_p^n$. We have:

$$\frac{1}{2}h(M_p^n \times C_2) = 2^{n-2}[p(p + 1)(np + 2) - p^3] + h(\ZZ_p \times \ZZ_p \times \ZZ_p)$$

$$\therefore h(M_p^n \times C_2) = 2^{n-1}[p(p + 1)(np + 2) - p^3] + 2h(\ZZ_p \times \ZZ_p \times \ZZ_p).$$

We now have $h(M_p^n \times C_2) = 2^{n-1}[p(p+1)(np+2)-p^3]+2^{n-1}[(3n-5)p+(n^2-5n+8)p^3+4]-4p^3=2^{n-1}[p^2(1+p)n^2+p(3+p-4p^2)n+(7p^2-3p^2-3p+4)]-4p^3$.

And $h(M_p^n \times C_2) = 2^n(3n^2+5n-4)$, for $n > 4, p = 2$.

3. Conclusion

Finally, the general classification for the nilpotent $p$-groups of the specified modular structure given by: $M_p^n \times C_p$ is thus hereby clearly made with the number of distinct fuzzy subgroups explicitly computed for all prime $p$ and every non zero integer $n \geq 3$.

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