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# Some Operations on Complex Vague Graphs with Application

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***Abstract.*** Vague graphs has found its importance in most recent years. It has played the vital role in many real life applications. The mathematical approach of blending different aspects of vague graphs gives a more generalized approach. As a result of it, we introduce the concepts of Complex vague graphs (CVG). In this chapter, we introduce certain notions including union, join, complement and composition of complex vague graphs. This helps in manipulating the real life applications in decision making problems. We also investigate on the ideas of homomorphisms of complex vague graphs. Finally we provide an application of CVG in launching a water reservoir in different locations.

***Keywords:*** Vague graph, complex vague graph, operations on vague graphs

***AMS Mathematics Subject Classification (2010):*** 05C72

## 1. Introduction

We divide this section into four main paragraphs. In the first paragraph, we provide some details about fuzzy sets. In the second paragraph, we give the details of complex fuzzy sets which is an extension of fuzzy sets. In third paragraph the introduction of fuzzy graphs and its types were discussed. The fourth paragraph describes the concepts of vague graph and combines the approach of second and third paragraph leading to the complex vague graphs.

Fuzzy set theory was first introduced by Zadeh [17] in the year 1965 to solve the problems with uncertainties. Then after many researchers came out with different ideas to enhance the concepts of fuzzy sets and their properties. This helped them in many ways to solve the real life problems involving uncertain and ambiguous environment. Atanassov

[3] proposed the concepts of a different fuzzy set called intuitionistic fuzzy set(IF set) by adding a new component which is said to be an extended form of fuzzy set later. The idea of this fuzzy set was introduced by Atanassov with the presence of degree of truth and false membership functions. There are various applications broadly studied under this IF set such as pattern recognition, image processing and so on. Gau and Buehrer [6] proposed the concept of vague sets in 1993 by replacing the value of an element in a set having the sub-interval of [0,1]. A true membership function $t_v(x)$ and a false membership function $f_v(x)$ are used to understand the boundaries of the membership degree. These two boundaries forms a subinterval as $[t_v(x),1-f_v(x)]$ in [0,1]. The vague set theory is very much used to deal with uncertain, inexact and vague knowledge.

Buckley and Nguyen et al., [4] combined complex numbers with fuzzy sets. After their introduction, Ramot et al., [12] extended the range of membership to unit circle in a complex plane where others limited this range to [0,1]. Some of the applications of complex fuzzy sets are considered in image restoration, decision making and reasoning schemes. Zhang et al., [18] analysed some properties on operations of complex fuzzy sets resulting in a concept called $\delta$ -inequalities of complex fuzzy sets. These concepts have also been studied in intuitionisitic fuzzy sets. Ali et al., [1] introduced the concepts of complex intuitionistic fuzzy classes. Alkhouri and salleh [2] studied some operations on complex intuitionistic fuzzy sets.

Rosenfeld [14] discussed the concept of fuzzy graphs whose ideas are implemented by Kauffman [8] in 1973. The fuzzy relation between fuzzy sets were also considered by Rosenfeld who developed the structure of fuzzy graphs, obtaining various analagous results of several graph theoretical concepts. Bhattacharya [5] gave some remarks of fuzzy graphs. The complement of fuzzy graphs was introduced by Mordeson [10]. The concepts of intuitionistic fuzzy graphs was introduced by Atanassov. Shannon and Atannasov [15] investigated some of the properties of intuitionistic fuzzy graphs. Thirunavukarasu et al., [16] analysed the concept of complex fuzzy graphs. Ramakrishna[10] introduced the concepts of vague graphs and analysed some of their properties. Recently, new papers have published about fuzzy graphs in [7,9,11].

Finally in this paper we discuss the concepts of complex vague graphs (CVG) and its operations such as union, join, composition and complement. An application of CVG in launching water reservoirs in different locations is discussed.

## 2. Preliminaries

**Definition 2.1.** *A complex fuzzy set(CFS) defined on a universe of discourse $\chi$ is an object of the form $A=\{(x,u_A(x)e^{i\omega_A(x)}]\}$ where $i=\sqrt{-1}$, $u_A(x)\in[0,1]$ and $0\leq\omega_A(x)\leq 2\pi$.*

**Definition 2.2.** *A complex intuitionistic fuzzy set(cif set) A defined on the universe of discourse $\chi$ is an object of the form $A=\{(x,\mu_A(x)e^{i\alpha_A(x)},\nu_A(x)e^{i\beta_A(x)})\}$ where $i=\sqrt{-1}$, $\mu_A(x),\nu_A(x)\in[0,1]$, $\alpha_A(x),\beta_A(x)\in[0,2\pi]$ and $0\leq\mu_A(x)+\nu_A(x)\leq 2\pi$*

**Definition 2.3.** *A complex vague set A defined on the universe of discourse $\chi$ is an object of the form $A=\{(x,t_A(x)e^{i\alpha_A(x)},f_A(x)e^{i\beta_A(x)})\}$ where $i=\sqrt{-1}$, $t_A(x),f_A(x)\in[0,1]$,*

$\alpha_A(x), \beta_A(x) \in [0,2\pi]$ *and* $0 \le t_A(x) + f_A(x) \le 2\pi$

**Definition 2.4.** *Let A and B be two cif sets in* $\chi$ *where* $A = \{(x, \mu_A(x)e^{i\alpha_A(x)}, \nu_A(x)e^{i\beta_A(x)}): x \in \chi\}$ *and* $B = \{(x, \mu_B(x)e^{i\alpha_B(x)}, \nu_A(x)e^{i\beta_B(x)}): x \in \chi\}$.

Then $A \cup B = \{(x, \mu_{A\cup B}(x)e^{i\alpha_{A\cup B}(x)}, \nu_{A\cup B}(x)e^{i\beta_{A\cup B}(x)}): x \in \chi$ where $\mu_{A\cup B}(x)e^{i\alpha_{A\cup B}(x)} = [\mu_A(x) \wedge \mu_B(x)]e^{i\{\alpha_A(x)\wedge\alpha_B(x)\}}]$ , $\nu_{A\cup B}(x)e^{i\alpha_{A\cup B}(x)} = [\nu_A(x) \cap \nu_B(x)]e^{i\{\alpha_A(x)\cap\alpha_B(x)\}}]$

**Definition 2.5.** *Let A and B be two cif sets in* $\chi$ *where* $A = \{(x, \mu_A(x)e^{i\alpha_A(x)}, \nu_A(x)e^{i\beta_A(x)}): x \in \chi\}$ *and* $B = \{(x, \mu_B(x)e^{i\alpha_B(x)}, \nu_A(x)e^{i\beta_B(x)}): x \in \chi\}$.

Then for all $x \in \chi$

(1) $A \subset B$ if and only if $\mu_A(x) < \mu_B(x), \nu_A(x) > \nu_B(x)$ for amplitude terms and $\alpha_A(x) < \alpha_A(x)$, $\beta_A(x) > \beta_B(x)$ for phase terms. (2) $A = B$ if and only if $\mu_A(x) = \mu_B(x), \nu_A(x) = \nu_B(x)$ for amplitude terms and $\alpha_A(x) = \alpha_A(x)$, $\beta_A(x) = \beta_B(x)$ for phase terms.

**Definition 2.6.** *Let A and B be two complex vague sets in* $\chi$ *where* $A = \{(x, t_A(x)e^{i\alpha_A(x)}, f_A(x)e^{i\beta_A(x)}): x \in \chi\}$ *and* $B = \{(x, t_B(x)e^{i\alpha_B(x)}, f_A(x)e^{i\beta_B(x)}): x \in \chi\}$. *Then for all* $x \in \chi$

(1) $A \subset B$ if and only if $t_A(x) < \mu_B(x), f_A(x) > \nu_B(x)$ for amplitude terms and $\alpha_A(x) < \alpha_A(x)$, $\beta_A(x) > \beta_B(x)$ for phase terms.

(2) $A = B$ if and only if $t_A(x) = \mu_B(x), f_A(x) = \nu_B(x)$ for amplitude terms and $\alpha_A(x) = \alpha_A(x)$, $\beta_A(x) = \beta_B(x)$ for phase terms.

## 3. Complex vague graphs

In this section, we provide definition and some operations on complex vague graphs.

**Definition 3.1.** *A complex vague graph(CVG) with an underlying set* $V$ *is defined to be a pair* $G = (A, B)$, *where* $A$ *is a complex vague set on* $V$ *and* $B$ *is a complex vague set on* $E \subseteq V \times V$ *such that*

$$t_B(xy)e^{i\alpha_B(xy)} \le min\{t_A(x), t_A(y)\}e^{imin\{\alpha_A(x),\alpha_A(y)\}}$$
$$f_B(xy)e^{i\beta_B(xy)} \ge max\{f_A(x), f_A(y)\}e^{imax\{\beta_A(x),\beta_A(y)\}} \qquad (1)$$

for all $x, y \in V$

**Definition 3.2.** *Let* $G = (A, B)$ *be a CVG.* The order of a CVG is defined by

$$O(G) = (\textstyle\sum_{x\in V} \; t_A(x)e^{\sum_{x\in V} \; \alpha_A(x)}, \sum_{x\in V} \; f_A(x)e^{\sum_{x\in V} \; \beta_A(x)})$$

The degree of a vertex $x$ in $G$ is defined by

$$deg(x) = (\textstyle\sum_{x\in E} \; t_B(xy)e^{\sum_{x\in E} \; \alpha_B(xy)}, \sum_{x\in E} \; f_B(xy)e^{\sum_{x\in E} \; \beta_B(xy)})$$

**Example 3.3.** *Consider a graph* $G^* = (V, E)$ *such that* $V = a, b, c, d$, $E = ab, ad, bc, cd$. *Let* $A$ *be a complex vague subset of* $V$ *and* $B$ *be a complex vague subset of* $E \subseteq V \times V$, *as given:*

$$A = \left(\frac{0.2e^{i1.3\pi}, 0.4e^{i0.6\pi}}{a}, \frac{0.8e^{i1.2\pi}, 0.1e^{i0.7\pi}}{b}, \frac{0.7e^{i0.4\pi}, 0.2e^{i1.5\pi}}{c}, \frac{1.0e^{i1.6\pi}, 0.0e^{i0.4\pi}}{c}\right)$$

$$B = \left(\frac{0.2e^{i1.1\pi}, 0.4e^{i0.8\pi}}{ab}, \frac{0.2e^{i0.8\pi}, 0.4e^{i0.7\pi}}{ad}, \frac{0.5e^{i0.3\pi}, 0.3e^{i1.7\pi}}{bc}, \frac{0.6e^{i0.3\pi}, 0.3e^{i1.6\pi}}{cd}\right)$$

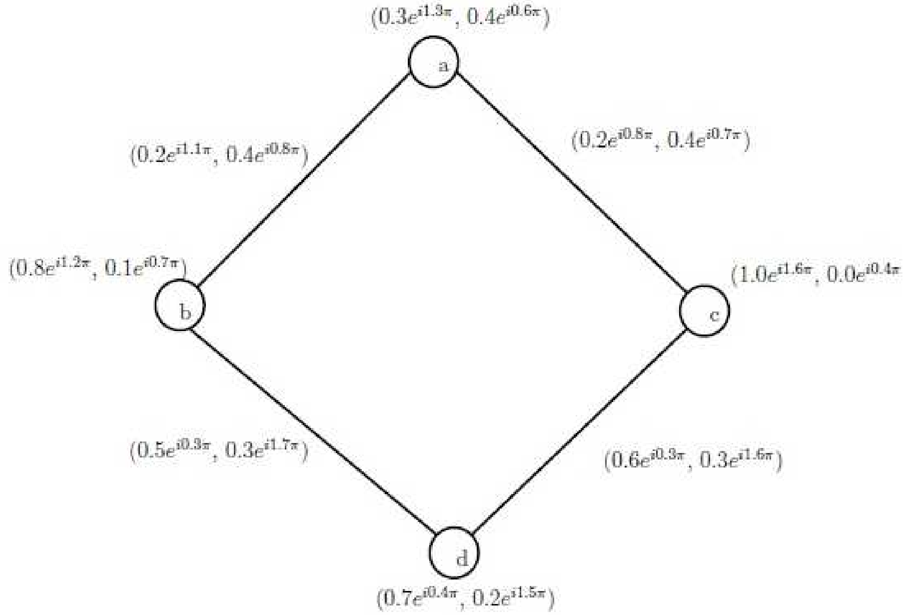


**Figure 1:** Complex vague graph G

(i) By usual computations, it can be observed that the graph shown in figure 1 is a complex vague graph.

(ii) Order of the complex vague graph is $O(G) = (2.4e^{i3.6\pi}, 0.7e^{i3.2\pi})$

(iii) Degree of each vertex $G$ is

$$deg(a) = (0.4e^{i1.9\pi}, 0.7e^{i1.4\pi}),\ deg(b) = (0.7e^{i1.4\pi}, 0.7e^{i2.5\pi}),$$
$$deg(c) = (1.1e^{i0.9\pi}, 0.6e^{i3.3\pi}),\ deg(d) = (0.8e^{i1.1\pi}, 0.7e^{i2.3\pi})$$

**Definition 3.4.** *The cartesian product* $G_1 \times G_2$ *of two complex vague graphs is defined as a pair* $G_1 \times G_2 = (A_1 \times A_2, B_1 \times B_2)$, *such that:*

1. $t_{A_1 \times A_2}(x_1, x_2)e^{i\alpha_{A_1 \times A_2}(x_1 \times x_2)} = min\{t_{A_1}(x_1), t_{A_2}(x_2)\}e^{imin(\alpha_{A_1}(x_1), \alpha_{A_2}(x_2))}$
$f_{A_1 \times A_2}(x_1, x_2)e^{i\beta_{A_1 \times A_2}(x_1 \times x_2)} = max\{f_{A_1}(x_1), f_{A_2}(x_2)\}e^{imax(\beta_{A_1}(x_1), \beta_{A_2}(x_2))}$ for all $x_1, x_2 \in V$

2. $t_{B_1 \times B_2}((x, x_2)(x, y_2)e^{i\alpha_{B_1 \times B_2}((x,x_2)(x,y_2)} = min\{t_{A_1}(x), t_{B_2}(x_2y_2)\}e^{imin(\alpha_{A_1}(x), \alpha_{B_2}(x_2y_2)}$
$f_{B_1 \times B_2}((x, x_2)(x, y_2)e^{i\beta_{B_1 \times B_2}((x,x_2)(x,y_2)} = min\{t_{A_1}(x), t_{B_2}(x_2y_2)\}e^{imax(\beta_{A_1}(x), \beta_{B_2}(x_2y_2)}$ for all $x \in V_1$ and $x_2y_2 \in E_2$,

3. $t_{B_1 \times B_2}((x_1, z)(y_1, z)e^{i\alpha_{B_1 \times B_2}((x_1,z)(y_1,z)} = min\{t_{B_1}(x_1y_1), t_{A_2}(z)\}e^{imin(\alpha_{B_1}(x_1y_1), \alpha_{A_2}(z)}$
$f_{B_1 \times B_2}((x_1, z)(y_1, z)e^{i\beta_{B_1 \times B_2}((x_1,z)(y_1,z)} = max\{f_{B_1}(x_1y_1), f_{A_2}(z)\}e^{imax(\beta_{B_1}(x_1y_1), \beta_{A_2}(z)}$

for all $z \in V_2$ and $x_1y_1 \in E_1$

**Example 3.5.** *Consider the two complex vague graphs $G_1$ and $G_2$, then their corresponding cartesian product $G_1 \times G_2$ is shown in the below figure.*

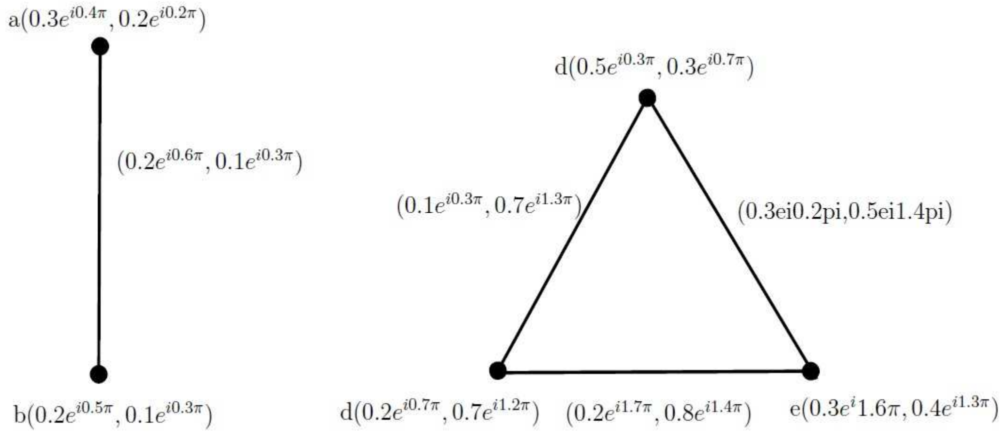


**Figure 2:** $G_1$ and $G_2$

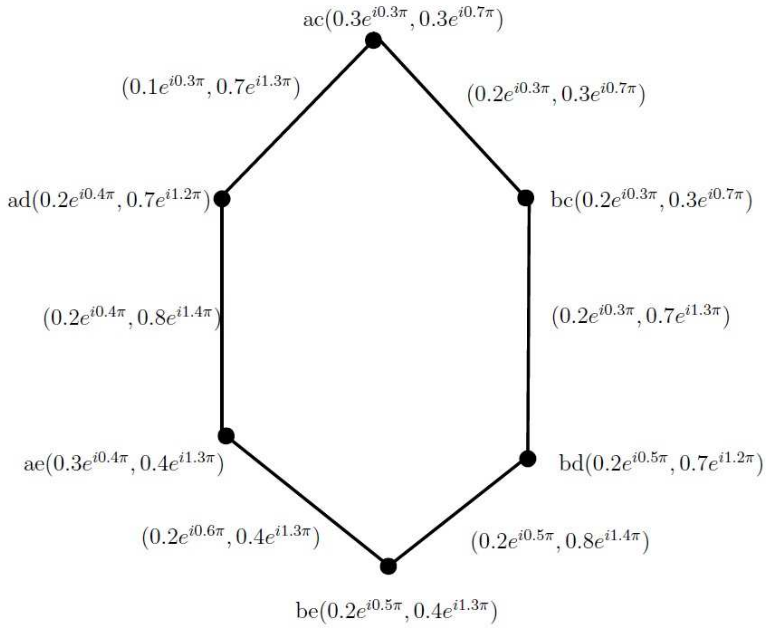


**Figure 3:** $G_1 \times G_2$

**Proposition 3.6.** *The cartesian product of two complex vague graphs is a complex vague graph.*

**Proof:** The conditions of $A_1 \times A_2$ are obvious and therefore it is easy to verify the conditions for $B_1 \times B_2$. Let $x \in V_1$, and $x_2y_2 \in E_2$. Then,

$$t_{A_1\times A_2}(x,x_2)(x,y_2)e^{i\alpha_{B_1\times B_2}((x,x_2)(x,y_2))}$$
$$= min\{t_{A_1}(x), t_{B_2}(x_2y_2)\}e^{imin(\alpha_{A_1}(x),\alpha_{B_2}(x_2y_2)}$$
$$\leq min\{t_{A_1}(x), min\{t_{A_2}(x_2), t_{A_2}(y_2)\}\}e^{imin\{\alpha_{A_1}(x),min\{\alpha_{A_2}(x_2),\alpha_{A_2}(y_2)\}\}}$$
$$= \min\{\min\{t_{A_1}(x), t_{A_2}(x_2)\}, \min\{t_{A_1}(x), t_{A_2}(y_2)\}\}$$
$$e^{imin\{min\{\alpha_{A_1}(x),\alpha_{A_2}(x_2)\},min\{\alpha_{A_1}(x),\alpha_{A_2}(y_2)\}\}}$$
$$= min\{t_{A_1\times A_2}(x,x_2), t_{A_1\times A_2}(x,y_2)\}e^{imin\{\alpha_{A_1\times A_2}(x,x_2),\alpha_{A_1\times A_2}(x,y_2)\}}$$
$$f_{B_1\times B_2}((x,x_2)(x,y_2))e^{i\gamma_{B_1\times B_2}((x,x_2)(x,y_2))}$$
$$= max\{f_{A_1}(x), f_{B_2}(x_2y_2)\}e^{imax(\gamma_{A_1}(x),\gamma_{B_2}(x_2y_2)}$$
$$\geq max\{f_{A_1}(x), max\{f_{A_2}(x_2), f_{A_2}(y_2)\}\}e^{imax\{\gamma_{A_2}(x),max\{\gamma_{A_2}(x_2),\gamma_{A_2}(y_2)\}\}}$$
$$= \max\{\max\{f_{A_1}(x), f_{A_2}(x_2)\}, \max\{f_{A_1}(x), f_{A_2}(y_2)\}\}$$
$$e^{imax\{max\{\gamma_{A_1}(x),\gamma_{A_2}(x_2)\},max\{\gamma_{A_1}(x),\gamma_{A_2}(y_2)\}\}}$$
$$= max\{f_{A_1\times A_2}(x,x_2), f_{A_1\times A_2}(x,y_2)\}e^{imax\{\gamma_{A_1\times A_2}(x,x_2),\gamma_{A_1\times A_2}(x,y_2)\}}$$

**Definition 3.7.** *The composition $G_1 \circ G_2$ of two complex vague graphs is defined as a pair $G_1 \circ G_2 = (A_1 \circ A_2), B_1 \circ B_2)$, such that:*

1. $t_{A_1\circ A_2}(x_1,x_2)e^{i\alpha_{A_1\circ A_2}(x_1,x_2)}$
$= \min\{t_{A_1}(x_1), t_{A_2}(x_2)\}\, e^{imin\{\alpha_{A_1}(x_1),\alpha_{A_2}(x_2)\}} f_{A_1\circ A_2}(x_1,x_2)e^{i\beta_{A_1\circ A_2}(x_1,x_2)}$
$= \max\{t_{A_1}(x_1), t_{A_2}(x_2)\}\, e^{imax\{\beta_{A_1}(x_1),\beta_{A_2}(x_2)\}} forall x_1, x_2 \in V.$
2. $t_{B_1\circ B_2}(x,x_2)(x,y_2)e^{i\alpha_{B_1\circ B_2}(x,x_2)(x,y_2))}$
$= \min\{t_{A_1}(x), t_{A_2}(x_2y_2)\}\, e^{imin\{\alpha_{A_1}(x),\alpha_{B_2}(x_2y_2)\}} f_{B_1\circ B_2}(x,x_2)(x,y_2)e^{i\beta_{B_1\circ B_2}(x,x_2)(x,y_2))}$
$= \max\{t_{A_1}(x), t_{A_2}(x_2y_2)\}\, e^{imax\{\beta_{A_1}(x),\beta_{B_2}(x_2y_2)\}} forall x \in V_1, x_2y_2 \in E_2$
3. $t_{B_1\circ B_2}(x_1,z)(y_1,z)e^{i\beta_{B_1\circ B_2}(x_1,z)(y_1,z))}$
$= \min\{t_{B_1}(x_1y_1), t_{A_2(z)}\}\, e^{imin\{\alpha_{B_1}(x_1y_1),\alpha_{A_2}(z)}$ $f_{B_1\circ B_2}(x_1,z)(y_1,z)e^{i\beta_{B_1\circ B_2}(x_1,z)(y_1,z))}$
$= \max\{f_{B_1}(x_1y_1), f_{A_2(z)}\}\, e^{imax\{\alpha_{B_1}(x_1y_1),\alpha_{A_2}(z)}$ 4. $t_{B_1\circ B_2}(x_1,x_2)(y_1,y_2)e^{i\alpha_{B_1\circ B_2}(x_1,x_2)(y_1,y_2)}$
$= \min\{t_{A_2}(x_2), t_{A_2}(y_2), t_{B_1}(x_1y_1)\}\, e^{imin\{\alpha_{A_2}(x_2),\alpha_{A_2}(y_2),\alpha_{B_1}(x_1y_1)\}}$
$f_{B_1\circ B_2}(x_1,x_2)(y_1,y_2)e^{i\beta_{B_1\circ B_2}(x_1,x_2)(y_1,y_2)}$
$= \max\{t_{A_2}(x_2), t_{A_2}(y_2), t_{B_1}(x_1y_1)\}\, e^{imax\{\beta_{A_2}(x_2),\beta_{A_2}(y_2),\beta_{B_1}(x_1y_1)\}}$
$\quad for\ all\ x_2, y_2 \in V_2, x_2 \neq y_2\ and\ x_1y_1 \in E_1$

**Definition 3.8.** *Let $G_1$ and $G_2$ be two complex vague graphs. The degree of a vertex in $G_1 \circ G_2$ can be defined as, for any vertex $(x_1,x_2) \in V_1 \times V_2$,*

$$d_{G_1\circ G_2}(x_1,x_2) = (\sum_{(x_1,x_2),(y_1,y_2)\in E} t_{B_1\circ B_2}((x_1,x_2)(y_1,y_2)e^{\sum_{(x_1,x_2),(y_1,y_2)\in E}\alpha_{B_1\circ B_2}}((x_1,x_2),(y_1,y_2)),$$
$$\sum\nolimits_{(x_1,x_2),(y_1,y_2)\in E} f_{B_1\circ B_2}((x_1,x_2)(y_1,y_2)e^{\sum_{(x_1,x_2),(y_1,y_2)\in E}\beta_{B_1\circ B_2}}((x_1,x_2),(y_1,y_2)))$$

**Proposition 3.9.** *The composition of two complex vague graphs is a complex vague graph.*

**Definition 3.10.** *The union* $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ *of two complex vague graphs is defined as*

1. $t_{A_1 \cup A_2}(x) e^{i\alpha_{A_1 \cup A_2}(x)} = t_{A_1}(x) e^{i\alpha_{A_1}(x)}$,
$f_{A_1 \cup A_2}(x) e^{i\beta_{A_1 \cup A_2}(x)} = f_{A_1}(x) e^{i\beta_{A_1}(x)}, for x \in V_1 and x \notin V_2$.

2. $t_{A_1 \cup A_2}(x) e^{i\alpha_{A_1 \cup A_2}(x)} = t_{A_1}(x) e^{i\alpha_{A_2}(x)}$,
$f_{A_1 \cup A_2}(x) e^{i\beta_{A_1 \cup A_2}(x)} = f_{A_2}(x) e^{i\beta_{A_2}(x)}, for x \in V_2 and x \notin V_1$.
3. $t_{A_1 \cup A_2}(x) e^{i\alpha_{A_1 \cup A_2}(x)} = max\{t_{A_1}(x), t_{A_2}(x)\} e^{imax\{\alpha_{A_1}(x), \alpha_{A_2}(x)\}}$,
$f_{A_1 \cup A_2}(x) e^{i\beta_{A_1 \cup A_2}(x)} = \min\{f_{A_1}(x), f_{A_2}(x)\} e^{imin\{\beta_{A_1}(x), \beta_{A_2}(x)\}}, for x \in V_1 \cup V_2$.
4. $t_{B_1 \cup B_2}(xy) e^{i\alpha_{B_1 \cup B_2}}(xy) = t_{B_1}(xy) e^{i\alpha_{B_1}(xy)}$,
$f_{B_1 \cup B_2}(xy) e^{i\beta_{B_1 \cup B_2}}(xy) = f_{B_1}(xy) e^{i\beta_{B_1}(xy)} for xy \in E_1$ and $xy \notin E_2$.
5. $t_{B_1 \cup B_2}(xy) e^{i\alpha_{B_1 \cup B_2}}(xy) = t_{B_2}(xy) e^{i\alpha_{B_2}(xy)}$,
$f_{B_1 \cup B_2}(xy) e^{i\beta_{B_1 \cup B_2}}(xy) = f_{B_2}(xy) e^{i\beta_{B_2}(xy)} for xy \in V_2 and xy \notin V_1$.
6. $t_{B_1 \cup B_2}(xy) e^{i\alpha_{B_1 \cup B_2}(xy)} = max\{t_{B_1}(x), t_{B_2}(xy)\} e^{imax\{\alpha_{B_1}(xy), \alpha_{B_2}(xy)\}}$,
$f_{B_1 \cup B_2}(xy) e^{i\beta_{B_1 \cup B_2}(xy)} = min\{f_{B_1}(xy), f_{B_2}(xy)\} e^{imin\{\beta_{B_1}(xy), \beta_{B_2}(xy)\}}, for$
$xy \in V_1 \cap V_2$.

**Proposition 3.11.** *The union of two complex vague graphs is a complex vague graph.*

**Definition 3.12.** *The join* $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ *of two complex vague graphs, where* $V_1 \cup V_2 = \emptyset$ *is defined as*

1. $t_{A_1 + A_2}(x) e^{i\alpha_{A_1 + A_2}(x)} = t_{A_1 \cup A_2}(x) e^{i\alpha_{A_1 \cup A_2}(x)}$
$f_{A_1 + A_2}(x) e^{i\beta_{A_1 + A_2}(x)} = f_{A_1 \cup A_2}(x) e^{i\beta_{A_1 \cup A_2}(x)}$
2. $t_{B_1 + B_2}(x) e^{i\alpha_{B_1 + B_2}(x)} = t_{B_1 \cup B_2}(xy) e^{i\alpha_{B_1 \cup B_2}(x)}$
$f_{B_1 + B_2}(x) e^{i\beta_{B_1 + B_2}(x)} = f_{B_1 \cup B_2}(xy) e^{i\beta_{B_1 \cup B_2}(x)}$
3. $t_{B_1 + B_2}(xy) e^{i\beta_{B_1 + B_2}(xy)} = min\{t_{A_1}(x), t_{A_2}(y)\} e^{imin\{\alpha_{A_1}(x), \alpha_{A_2}(y)\}}$
$f_{B_1 + B_2}(xy) e^{i\beta_{B_1 + B_2}(xy)} = max\{f_{A_1}(x), f_{A_2}(y)\} e^{imax\{\beta_{A_1}(x), \beta_{A_2}(y)\}}$

if $xy \in E$, where E is the set of all edges joining the vertices of $V_1$ and $V_2$.

**Proposition 3.13.** *The join of two complex vague graphs is a complex vague graphs.*

**Proposition 3.14.** *Let* $G_1 = (A_1, A_2)$ *and* $G_2 = (B_1, B_2)$ *be the complex vague graphs of the graph* $G_1^*$ *and* $G_2^*$ *and let* $V_1 \cap V_2 = \emptyset$*. Then the union* $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ *is a complex vague graph of* $G^*$ *if and only if* $G_1$ *and* $G_2$ *are complex vague graphs of* $G_1^*$ *and* $G_2^*$ *respectively.*

**Proof:** Consider $G_1 \cup G_2$ is a complex vague graph. Let $xy \in E_1$. Then, $xy \notin E_2$ and $x, y \in V_1 - V_2$. Therefore,

$$t_{B_1}(xy) e^{i\alpha_{B_1}(x)} = t_{B_1 \cap B_2}(xy) e^{i\alpha_{B_1 \cap B_2}(x)}$$

$$\leq min\{t_{A_1\cap A_2}(x), t_{A_1\cap A_2}(y)e^{imin(\alpha_{A_1\cap A_2}(x),\alpha_{A_2\cap A_2}(y))}$$
$$= min(t_{A_1}(x), t_{A_1}(y))e^{imin(\alpha_{A_2}(x),\alpha_{A_1}(y))}$$
$$f_{B_1}e^{i\beta_{B_1}(x)} = f_{B_1\cap B_2}(xy)e^{i\beta_{B_1\cap B_2}(x)}$$
$$\geq max\{t_{A_1\cap A_2}(x), t_{A_1\cap A_2}(y)e^{imax(\beta_{A_1\cap A_2}(x),\beta_{A_2\cap A_2}(y))}$$
$$= max(f_{A_1}(x), f_{A_1}(y))e^{imax(\beta_{A_2}(x),\beta_{A_1}(y))}.$$

This shows that $G_1 = (A_1, B_1)$ is a complex vague graphs. Also we can show that $G_2 = (A_2, b_2)$ is a complex vague graph. The converse of this proposition is obvious. Hence the proof.

**4. Application of complex vague graph**

In this section we will consider a real time situation and implement the concepts using complex vague graph. Vague sets are the generalization of the fuzzy sets. We implement the concepts of complex vague set with the graph theory. Complex vague graphs have wide applications in neural networks, decision making problems etc., Consider the Water Desalination plant in gulf which plans to launch the minimum number of plants in the city, such that it covers the maximum number of houses connected with the pipe lines directly. This way of approach benefits the users to have a drinking water facility 24/7 without any interruption. For this purpose, the following parameters are taken into consideration namely,

- Suitable place to launch the water plant.
- Users
- Connectivity with the main power plant unit.
- Connectivity to Urban location and hill areas.
- Available resources which acts as a carrier in transporting water.
- Expenditure and income by launching such a water desalination plant.

Suppose a team selected five places where they are interested in launching a water plant, so that they can cover maximum number of houses in those places. The following are the observations taken care before launching:

1. Launching a water plant exactly at a location which will cover all these five places.

2. Launching a water plant between any two of the selected five places. In case of situation 1, we could proceed as follows:

Let $V = \{C_1, C_2, C_3, C_4, C_5\}$ be the set of places the team is attracted to launch a water plant. This is considered as a vertex set. Suppose that $70\%$ of the experts on the team believes that $C_1$ should have a water plant and $20\%$ of the expert believe that there is no need to fix a water plant in that location after observing various parameters and constraints.

The amplitude term for both the membership and non-membership functions are well defined. Now we consider the phase term that has to ensure the period. Let $35\%$ of the experts believe that in a particular time frame $C_1$ can attact the maximum number of end-users to get benefitted and $40\%$ of the experts have to think on the other end oppositely. We frame this into a model with the information as $\langle C_1: 0.7e^{i0.35\pi}, 0.2e^{i0.4\pi}\rangle$.

Thus the expert team finalises the opinion on $C_1$. They now move onto $C_2$. After taking care on this observation, they model this information as $\langle C_2: 0.6e^{i0.4\pi}, 0.2e^{i0.3\pi}\rangle$.

This means $60\%$ of the experts are in the favour of $C_2$, although it will produce $30\%$ will produce the profit. Similarly, they visit other places and model it as $\langle C_3: 0.5e^{i0.6\pi}, 0.7e^{i0.4\pi}\rangle$, $\langle C_4: 0.4e^{i0.4\pi}, 0.8e^{i0.2\pi}\rangle$ and $\langle C_5: 0.7e^{i0.6\pi}, 0.5e^{i0.7\pi}\rangle$ respectively.

We denote this models as follows:

$$A = \begin{pmatrix} \langle C_1: 0.7e^{i0.35\pi}, 0.2e^{i0.4\pi}\rangle \\ \langle C_2: 0.3e^{i0.4\pi}, 0.5e^{i0.3\pi}\rangle \\ \langle C_3: 0.6e^{i0.6\pi}, 0.1e^{i0.4\pi}\rangle \\ \langle C_4: 0.4e^{i0.4\pi}, 0.8e^{i0.2\pi}\rangle \\ \langle C_5: 0.7e^{i0.6\pi}, 0.5e^{i0.7\pi}\rangle \end{pmatrix}$$

Here the complex membership of the vertices defines the positive characteristics and the non complex membership of the vertices defines the negative characteristics of a parameter for a certain place. We will now find the absolute values as

$$|C_1| = (0.7,0.2)$$
$$|C_2| = (0.3,0.5)$$
$$|C_3| = (0.6,0.1)$$
$$|C_4| = (0.4,0.8)$$
$$|C_5| = (0.7,0.5)$$

We now find the optimal choice by using a score function of the absolute values of each location $C_1, C_2, C_3, C_4, C_5$ as follows:

$$S(C_1) = 0.7 - 0.2 = 0.5$$
$$S(C_2) = 0.3 - 0.5 = -0.2$$
$$S(C_3) = 0.6 - 0.1 = 0.5$$
$$S(C_4) = 0.4 - 0.8 = -0.4$$
$$S(C_5) = 0.7 - 0.5 = 0.2$$

Since the scores of $C_1$ and $C_3$ are equal, we will find the accuracies of $C_1$ and $C_2$ as $H(C_1) = 0.7 + 0.2 = 0.9$ and $H(C_3) = 0.6 + 0.1 = 0.7$, thus $C_1 > C_3$, which is the best choice to launch a water plant. We can see this application of complex vague graph with out any edges mentioned as shown below.

Now, for situation 2,we proceed as follows:

If a water plant is launched between places $C_1$ and $C_3$, it will represent the edge $C_1C_3$ of the vertex $C_1, C_3$.

To find the model of $C_1C_3$ , we use definition and find that $\langle C_1C_3: 0.5e^{0.3\pi}, 0.2e^{0.5\pi}$. Similarly, we find the other edges and we denote this as

$$B = \begin{pmatrix} \langle C_1C_2: 0.2e^{i0.3\pi}, 0.5e^{i0.4\pi}\rangle \\ \langle C_1C_4: 0.3e^{i0.3\pi}, 0.8e^{i0.5\pi}\rangle \\ \langle C_1C_5: 0.6e^{i0.3\pi}, 0.5e^{i0.7\pi}\rangle \\ \langle C_1C_3: 0.5e^{i0.3\pi}, 0.2e^{i0.45\pi}\rangle \\ \langle C_2C_5: 0.2e^{i0.2\pi}, 0.6e^{i0.7\pi}\rangle \\ \langle C_2C_4: 0.3e^{i0.3\pi}, 0.4e^{i0.4\pi}\rangle \\ \langle C_2C_3: 0.3e^{i0.3\pi}, 0.6e^{i0.4\pi}\rangle \\ \langle C_3C_4: 0.3e^{i0.2\pi}, 0.8e^{i0.6\pi}\rangle \\ \langle C_3C_5: 0.2e^{i0.6\pi}, 0.6e^{i0.8\pi}\rangle \\ \langle C_4C_5: 0.2e^{i0.3\pi}, 0.5e^{i0.7\pi}\rangle \end{pmatrix}$$

If we consider the edge $\langle C_1C_3: 0.5e^{0.3\pi}, 0.2e^{0.5\pi}$ we have, the amplitude term shows that $50\%$ of the experts believe that there should be a tower between these two places and $20\%$ of the experts believe the opposite. The phase terms show that $30\%$ of the experts believe that in a certain time if a tower is fixed between these two places it will produce maximum profit, while $50\%$ of the experts believe the opposite. Also the absolute values of the edges are:

$$|C_1C_2| = (0.2,0.5), |C_1C_4| = (0.3,0.8)$$
$$|C_1C_5| = (0.6,0.5), |C_1C_3| = (0.5,0.2)$$
$$|C_2C_5| = (0.2,0.6), |C_2C_4| = (0.3,0.8)$$
$$|C_2C_3| = (0.3,0.6), |C_3C_4| = (0.3,0.8)$$
$$|C_3C_5| = (0.5,0.6), |C_4C_5| = (0.3,0.8)$$

To get the optimal choice, we find the score function of the absolute values of the edges. Hence we get,

$$S(C_1C_2) = -0.3, S(C_1C_4) = -0.5$$
$$S(C_1C_5) = 0.1, S(C_1C_3) = 0.3$$
$$S(C_2C_5) = -0.4, S(C_2C_4) = -0.5$$
$$S(C_2C_3) = -0.3, S(C_3C_4) = -0.5$$
$$S(C_3C_5) = -0.1, S(C_4C_5) = -0.5$$

We see that $S(C_1C_3) = 0.3$ is the greater value and hence the suitable choice to launch the water plant.

## 5. Conclusion and future work

In this work we have defined the concepts of complex vague graphs and accomplished the concepts of union, join, composition and cartesian product of complex vague graphs. Also we analysed an applicaton related to various parameters and since soft set is widely used to handle more parameters in real time, in future we will define the concepts of complex vague soft graphs and its applications.

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