Intern. J. Fuzzy Mathematical Archive
Vol. 19, No. 1, 2021, 49-60
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 29 June 2021
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/ijfma.v19n1a04226

International Journal of
Fuzzy Mathematical
Archive

Some Operations on Cubic Graphs

***Zahra Sadri Irani*[1], *Hossein Rashmanlou*[2]* and *F. Mofidnakhaei*[3]**

[1]Department of Chemistry, Faculty of Science, Falavarjan Branch
Islamic Azad University, Iran
E-mail: sadri@iaufala.ac.ir
[2]Adib Mazandaran Institute of Higher Education, Sari, Iran
E-mail: rashmanlou.1987@gmail.com
[3]Department of Physics, Sari Branch, Islamic Azad University, Sari, Iran
E-mail: Farshid.Mofidnakhaei@gmail.com
[*]Corresponding author.

Received 17 May 2021; accepted 29 June 2021

Abstract. Cubic graph can deal with the uncertainly associated with the inconsistent and indeterminate information of any real-world problem, where fuzzy graphs may fail to reveal satisfactory results. Likewise, cubic graphs are very useful tools for the study of different domains of computer science such as networking, capturing the image, clustering, and also other issues like bioscience, medical science, and traffic plan. Operations are conveniently used in many combinatorial applications; therefore, in this paper, three new operations on cubic graphs, namely, maximal product, rejection, residue product were presented, and some results concerning their degrees were introduced. Different examples are provided to evaluate the validity of the new definitions.

Keywords: Cubic set, cubic graph, maximal product, residue produc

AMS Mathematics Subject Classification (2010): 05C99, 03E72

1. Introduction

A graph basically holds a model of relations, and it is used to depict the real-life problems encompassing the relationships among objects. To represent the objects and the relations between objects, the graph vertices and edges are employed, respectively. Fuzzy graph models are helpful mathematical tools in order to address the combinatorial problems in various fields incorporating research, optimization, algebra, computing, and topology. Due to the natural existence of vagueness and ambiguity, fuzzy graphical models are noticeably better than graphical models. Originally, fuzzy set theory was required to deal with many multifaceted issues, which are replete with incomplete information. In 1965 [41], fuzzy set theory was first suggested by Zadeh. Fuzzy set theory is a highly powerful mathematical tool for solving approximate reasoning related problems. Jun et al. [9] introduced cubic sets. Later on, Muhiuddin et al. [12, 13, 14] applied the notion cubic sets on different aspects. The first description of fuzzy graphs was proposed by Kafmann [10] in 1993, taken from Zadeh's fuzzy relations [42, 43]. However, Rosenfeld [30] described another detailed definition, including fuzzy vertex and fuzzy edges and various fuzzy analogs of graphical theoretical concepts, including paths, cycles, connectedness, etc. Akram et al. [1, 2] presented new definitions of fuzzy graphs. Rashmanlou et al. [17, 18, 19, 20, 21, 22] investigated different concepts on cubic graphs, vague graphs, and bipolar fuzzy graphs. Samanta et al. [33, 34] introduced fuzzy competition graphs and some properties of irregular bipolar fuzzy graphs. Borzooei and Rashmanlou [3, 4, 5, 6] studied new concepts on vague graphs. Gani and Radha [16] recommended regular fuzzy graphs and totally regular fuzzy graphs. Kumaravel and Radha [29] described the concepts of the edge degree and the total edge degree in regular fuzzy graphs. Latha and Gani [15] defined neighborly irregular fuzzy graphs and highly irregular fuzzy graphs. Sunutha et al. [34, 35] presented new concepts for fuzzy graphs. Talebi et al. [37, 38, 39, 40] represented several concepts on interval-valued fuzzy graphs, intuitionistic fuzzy graphs, and bipolar fuzzy graphs. Shoaib et al. [36] given some results on pythagorean fuzzy graphs. Operations are conveniently used in many combinatorial applications. Hence, in this research, three new operations on cubic graphs, namely, maximal product, rejection, residue product were presented, and some results concerning their degrees were introduced. Recently, some research works have been done by the authors in continuation of previous works related to cubic graphs, vague graphs, bipolar fuzzy graphs, and intuitionistic fuzzy graphs which are mentioned in [7, 8, 24, 25, 26, 27, 28, 31].

2. Preliminaries

A fuzzy graph is of the from $G=(\psi,\phi)$ which is a pair of mappings $\psi: V\to[0,1]$ and $\phi: V\times V\to[0,1]$ as is defined as $\phi(m,n)\leq\psi(m)\wedge\psi(n)$, $\forall m,n\in V$, and ϕ is a symmetric fuzzy relation on ψ and $\wedge$ denotes minimum.

Let X be a non-empty set. A function $A: X\to[I]$ is called an interval-valued fuzzy set (shortly, IVF set) in X. Let $[I]^X$ stands for the set of all IVF sets in X. For every $A\in[I]^X$ and $x\in X$, $A(x)=[A^-(x),A^+(x)]$ is called the degree of membership of an element $x\in A$, where $A^-: X\to I$ and $A^+: X\to I$ are fuzzy sets in X which are called a lower fuzzy set and upper fuzzy set in X, respectively. For simplicity, we denote $A=[A^-,A^+]$. For every $A,B\in[I]^X$, we define $A\subseteq B$ if and only if $A(x)\leq B(x)$, for all $x\in X$.

Definition 2.1. *[1] Let $A=[A^-,A^+]$, and $B=[B^-,B^+]$ be two interval valued fuzzy sets in X. Then, we define*

$$r\min\{A(x),B(x)\}=[\min\{A^-(x),B^-(x)\},\min\{A^+(x),B^+(x)\}],$$
$$r\max\{A(x),B(x)\}=[\max\{A^-(x),B^-(x)\},\max\{A^+(x),B^+(x)\}].$$

Definition 2.2. *[9] Let x be a non-empty set. By a cubic set in X, we mean a structure $A=\{\langle x,A(x),\lambda(x): x\in X\rangle\}$ in which A is an interval-valued fuzzy sets in X and λ is a fuzzy set in X. A cubic set $A=\{\langle m,A(m),\lambda(m): m\in X\rangle\}$ is simply denoted by $A=\langle A,\lambda\rangle$. The collection of all cubic sets in X is denoted by $CP(X)$.*

Definition 2.3. *[11] A cubic graph is a triple $\zeta=(G^*,P,Q)$ where $G^*=(V,E)$ is a graph, $P=(\widetilde{\mu_P},\lambda_P)$ is a cubic set on V and $Q=(\widetilde{\mu_Q},\lambda_Q)$ is a cubic set on $V\times V$ so that $\widetilde{\mu_Q}(mn)\leq rmin\{\widetilde{\mu_P}(m),\widetilde{\mu_P}(n)\}$ and $\lambda_Q(mn)\geq max\{\lambda_P(m),\lambda_P(n)\}$.*

3. Some results in cubic graphs

Definition 3.1. *Let* $\zeta_1 = (P_1, Q_1)$ *and* $\zeta_2 = (P_2, Q_2)$ *be two cubic graphs with underlying crisp graphs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$, *respectively.* $\zeta_1 * \zeta_2 = (P, Q)$ *is called maximal cubic graph with underlying crisp graph* $G = (V, E)$, *where* $V = V_1 \times V_2$ *and* $E = \{(m_1, n_1)(m_2, n_2) |\ m_1 = m_2, n_1 n_2 \in E_2 \text{ or } n_1 = n_2, m_1 m_2 \in E_1\}$.

$$(i)\ \begin{cases} \left(\widetilde{\mu_{P_1}} * \widetilde{\mu_{P_2}}\right)(m,n) = r\max\{\widetilde{\mu_{P_1}}(m), \widetilde{\mu_{P_2}}(n)\}, \\ \left(\lambda_{P_1} * \lambda_{P_2}\right)(m,n) = \min\{\lambda_{P_1}(m), \lambda_{P_2}(n)\}, \\ for\ all\ (m,n) \in V = V_1 \times V_2 \end{cases}$$

$$(ii)\ \begin{cases} \left(\widetilde{\mu_{Q_1}} * \widetilde{\mu_{Q_2}}\right)((m_1,n_1)(m_2,n_2)) = r\max\left(\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{Q_2}}(n_1 n_2)\right), \\ \left(\lambda_{Q_1} * \lambda_{Q_2}\right)((m_1,n_1)(m_2,n_2)) = \min\left(\lambda_{P_1}(m_1), \lambda_{Q_2}(n_1 n_2)\right), \\ m_1 = m_2, n_1 n_2 \in E_2, \end{cases}$$

$$(iii)\ \begin{cases} \left(\widetilde{\mu_{Q_1}} * \widetilde{\mu_{Q_2}}\right)((m_1,n_1)(m_2,n_2)) = r\max\left(\widetilde{\mu_{P_2}}(n_1), \widetilde{\mu_{Q_1}}(m_1 m_2)\right), \\ \left(\lambda_{Q_1} * \lambda_{Q_2}\right)((m_1,n_1)(m_2,n_2)) = \min\left(\lambda_{P_2}(n_1), \lambda_{Q_1}(m_1 m_2)\right), \\ m_1 m_2 \in E_1, n_1 = n_2. \end{cases}$$

Example 3.2. *Consider the two cubic graphs* ζ_1 *and* ζ_2 *as shown in Figure 1. Their maximal product* $\zeta_1 * \zeta_2$ *is shown in Figure 2.*

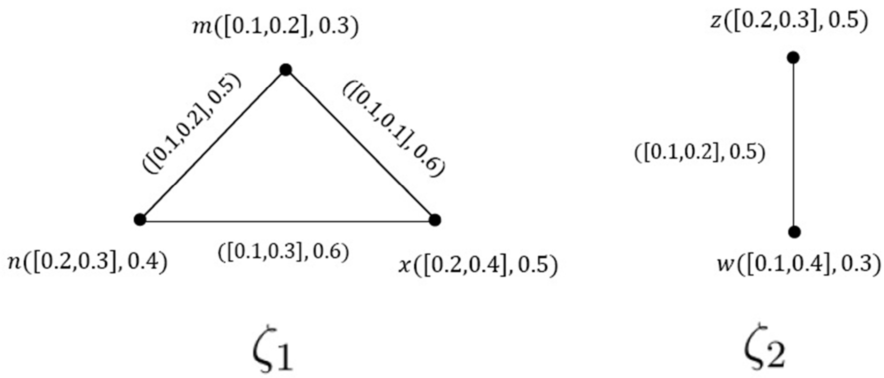

Figure 1: Cubic graph ζ_1 and ζ_2.

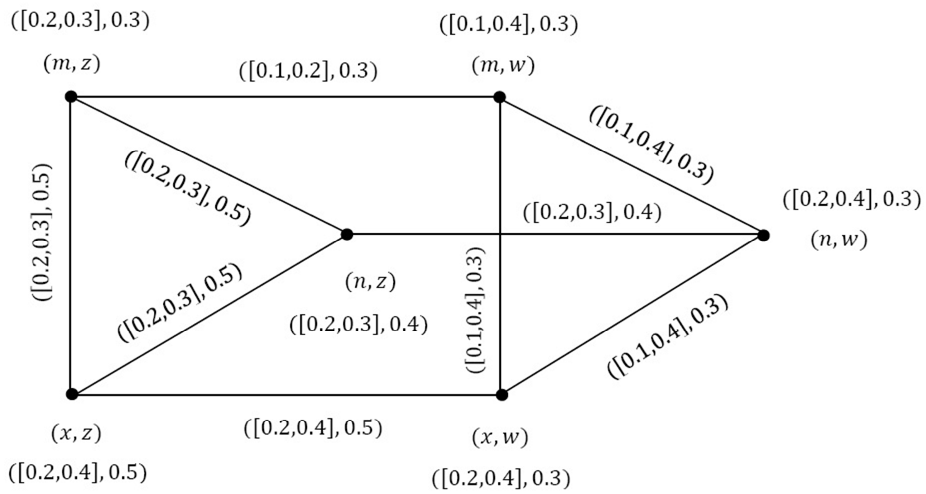

Figure 2: Maximal product of ζ_1 and ζ_2.

For vertex (m,w), we find both membership value and non-membership value as follows:
$\left(\widetilde{\mu_{P_1}} * \widetilde{\mu_{P_2}}\right)(m,w) = r\max\{\widetilde{\mu_{P_1}}(m), \widetilde{\mu_{P_2}}(w)\} = r\max\{[0.1,0.2],[0.1,0.4]\} = [0.1,0.4]$,
$\left(\lambda_{P_1} * \lambda_{P_2}\right)(m,w) = \min\{\lambda_{P_1}(m), \lambda_{P_2}(w)\} = \min\{0.3,0.3\} = 0.3$,
for $m \in V_1$ and $w \in V_2$.

For edge $(m,z)(x,z)$ we have:

$$\left(\widetilde{\mu_{Q_1}} * \widetilde{\mu_{Q_2}}\right)((m,z)(x,z)) = r\max\{\widetilde{\mu_{P_2}}(z), \widetilde{\mu_{Q_1}}(mx)\} = r\max\{[0.2,0.3],[0.1,0.1]\} = [0.2,0.3],$$

$$\left(\lambda_{Q_1} * \lambda_{Q_2}\right)((m,z)(x,z)) = \min\{\lambda_{P_2}(z), \lambda_{Q_1}(mx)\} = \min\{0.5,0.6\} = 0.5.$$

Similarly, we can find both membership and non-membership values for all remaining vertices and edges.

Theorem 3.3. *The maximal product of two cubic graphs ζ_1 and ζ_2, is a cubic graph, too.*

Proof: Let $\zeta_1 = (P_1, Q_1)$ and $\zeta_2 = (P_2, Q_2)$ be two cubic graphs and $((m_1,m_2)(n_1,n_2)) \in E_1 \times E_2$. Then, by Definition 2.3, we have two cases:

(i) $m_1 = n_1 = m$

$$\begin{aligned}\left(\widetilde{\mu_{Q_1}} * \widetilde{\mu_{Q_2}}\right)((m,m_2)(m,n_2)) &= r\max\left(\widetilde{\mu_{P_1}}(m), \widetilde{\mu_{Q_2}}(m_2n_2)\right)\\ &\le r\max\left\{\widetilde{\mu_{P_1}}(m), r\min\left\{\widetilde{\mu_{P_2}}(m_2), \widetilde{\mu_{Q_2}}(n_2)\right\}\right\}\\ &= r\min\left\{r\max\left\{\widetilde{\mu_{P_1}}(m), \widetilde{\mu_{P_2}}(m_2)\right\}, r\max\left\{\widetilde{\mu_{P_1}}(m), \widetilde{\mu_{P_2}}(n_2)\right\}\right\}\\ &= r\min\{\left(\widetilde{\mu_{P_1}} * \widetilde{\mu_{P_2}}\right)(m,m_2), \left(\widetilde{\mu_{P_1}} * \widetilde{\mu_{P_2}}\right)(m,n_2)\},\end{aligned}$$

$$\begin{aligned}\left(\lambda_{Q_1} * \lambda_{Q_2}\right)((m,m_2)(m,n_2)) &= \min\{\lambda_{P_1}(m), \lambda_{P_2}(m_2n_2)\}\\ &\ge \min\left\{\lambda_{P_1}(m), \max\left\{\lambda_{P_2}(m_2), \lambda_{P_2}(n_2)\right\}\right\}\\ &= \max\left\{\min\left\{\lambda_{P_1}(m), \lambda_{P_2}(m_2)\right\}, \min\left\{\lambda_{P_1}(m), \lambda_{P_2}(n_2)\right\}\right\}\end{aligned}$$

$$= \max\{(\lambda_{P_1} * \lambda_{P_2})(m, m_2), (\lambda_{P_1} * \lambda_{P_2})(m, n_2)\}.$$

(i) if $m_2 = n_2 = z$

$$\begin{aligned}(\widetilde{\mu_{Q_1}} * \widetilde{\mu_{Q_2}})((m_1, z)(n_1, z)) &= r\max\{\widetilde{\mu_{Q_1}}(m_1 n_1), \widetilde{\mu_{P_2}}(z)\} \\ &\leq r\max\{r\min\{\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{P_1}}(n_1)\}, \widetilde{\mu_{P_2}}(z)\} \\ &= r\min\left\{r\max\{\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{P_2}}(z)\}, r\max\{\widetilde{\mu_{P_1}}(n_1), \widetilde{\mu_{P_2}}(z)\}\right\} \\ &= r\min\{(\widetilde{\mu_{P_1}} * \widetilde{\mu_{P_2}})(m_1, z), (\widetilde{\mu_{P_1}} * \widetilde{\mu_{P_2}})(n_1, z)\},\end{aligned}$$

$$\begin{aligned}(\lambda_{Q_1} * \lambda_{Q_2})((m_1, z)(n_1, z)) &= \min\{\lambda_{Q_1}(m_1 n_1), \lambda_{P_2}(z)\} \\ &\geq \min\{\max\{\lambda_{P_1}(m_1), \lambda_{P_1}(n_1)\}, \lambda_{P_2}(z)\} \\ &= \max\left\{\min\{\lambda_{P_1}(m_1), \lambda_{P_2}(z)\}, \min\{\lambda_{P_1}(n_1), \lambda_{P_2}(z)\}\right\} \\ &= \max\{(\lambda_{P_1} * \lambda_{P_2})(m_1, z), (\lambda_{P_1} * \lambda_{P_2})(n_1, z)\}.\end{aligned}$$

Hence, $\zeta_1 * \zeta_2$ is a cubic graph.

Definition 3.4. *A Cubic graph $\zeta = (P, Q)$ is strong if:*
$\widetilde{\mu_Q}(mn) = \widetilde{\mu_P}(m) \wedge \widetilde{\mu_P}(n)$, $\lambda_Q(mn) = \lambda_P(m) \vee \lambda_P(n)$, for all $mn \in E$.

Theorem 3.5. *The maximal product of two strong cubic graphs ζ_1 and ζ_2, is a strong cubic graph.*

Proof: Let $\zeta_1 = (P_1, Q_1)$ and $\zeta_2 = (P_2, Q_2)$ be two strong cubic graphs. Then $\widetilde{\mu_{Q_1}}(m_1 m_2) = r\min(\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{P_1}}(m_2))$, $\lambda_{Q_1}(m_1 m_2) = \max(\lambda_{P_1}(m_1), \lambda_{P_2}(m_2))$, for any $m_1 m_2 \in E_1$ and $\widetilde{\mu_{Q_2}}(n_1 n_2) = r\min(\widetilde{\mu_{P_1}}(n_1), \widetilde{\mu_{P_2}}(n_2))$, $\lambda_{Q_2}(n_1 n_2) = \max(\lambda_{P_1}(n_1), \lambda_{P_2}(n_2))$, for any $n_1 n_2 \in E_2$. Then, proceeding according to the definition of maximal product,

(i) if $n_1 = n_2$ and $m_1 m_2 \in E_2$. Then,

$$\begin{aligned}(\widetilde{\mu_{Q_1}} * \widetilde{\mu_{Q_2}})((n_1, m_1)(n_2, m_2)) &= r\max\{\widetilde{\mu_{P_1}}(n_1), \widetilde{\mu_{Q_2}}(m_1 m_2)\} \\ &= r\max\left\{\widetilde{\mu_{P_1}}(n_1), r\min\{\widetilde{\mu_{P_2}}(m_1), \widetilde{\mu_{P_2}}(m_2)\}\right\} \\ &= r\min\left\{r\max\{\widetilde{\mu_{P_1}}(n_1), \widetilde{\mu_{P_2}}(m_1)\}, r\max\{\widetilde{\mu_{P_1}}(n_1), \widetilde{\mu_{P_2}}(m_2)\}\right\} \\ &= r\min\{(\widetilde{\mu_{P_1}} * \widetilde{\mu_{P_2}})(n_1, m_1), (\widetilde{\mu_{P_1}} * \widetilde{\mu_{P_2}})(n_1, m_2)\},\end{aligned}$$

$$\begin{aligned}(\lambda_{Q_1} * \lambda_{Q_2})((n_1, m_1)(n_2, m_2)) &= \min\{\lambda_{P_1}(n_1), \lambda_{Q_2}(m_1 m_2)\} \\ &= \min\left\{\lambda_{P_1}(n_1), \max\{\lambda_{P_2}(m_1), \lambda_{P_2}(m_2)\}\right\} \\ &= \max\left\{\min\{\lambda_{P_1}(n_1), \lambda_{P_2}(m_1)\}, \min\{\lambda_{P_1}(n_1), \lambda_{P_2}(m_2)\}\right\} \\ &= \max\{(\lambda_{P_1} * \lambda_{P_2})(n_1, m_1), (\lambda_{P_1} * \lambda_{P_2})(n_1, m_2)\}.\end{aligned}$$

(ii) if $m_1 = m_2$ and $n_1 n_2 \in E_1$. Then

$$\begin{aligned}(\widetilde{\mu_{Q_1}} * \widetilde{\mu_{Q_2}})((n_1, m_1)(n_2, m_2)) &= r\max\{\widetilde{\mu_{Q_1}}(n_1 n_2), \widetilde{\mu_{P_2}}(m_2)\} \\ &= r\max\{r\min\{\widetilde{\mu_{P_1}}(n_1), \widetilde{\mu_{P_2}}(n_2)\}, \widetilde{\mu_{P_2}}(m_2)\} \\ &= r\min\left\{r\max\{\widetilde{\mu_{P_1}}(n_1), \widetilde{\mu_{P_2}}(m_2)\}, r\max\{\widetilde{\mu_{P_1}}(n_2), \widetilde{\mu_{P_2}}(m_2)\}\right\}\end{aligned}$$

$$= r\min\{(\widetilde{\mu_{P_1}} * \widetilde{\mu_{P_2}})(n_1, m_2), (\widetilde{\mu_{P_1}} * \widetilde{\mu_{P_2}})(n_2, m_2)\},$$

$$\begin{aligned}(\lambda_{Q_1} * \lambda_{Q_2})((n_1, m_1)(n_2, m_2)) &= \min\{\lambda_{Q_1}(n_1 n_2), \lambda_{P_2}(m_2)\}\\ &= \min\{\max\{\lambda_{P_1}(n_1), \lambda_{P_1}(n_2)\}, \lambda_{P_2}(m_2)\}\\ &= \max\left\{\min\{\lambda_{P_1}(n_1), \lambda_{P_2}(m_2)\}, \min\{\lambda_{P_1}(n_2), \lambda_{P_2}(m_2)\}\right\}\\ &= \max\{(\lambda_{P_1} * \lambda_{P_2})(n_1, m_2), (\lambda_{P_1} * \lambda_{P_2})(n_2, m_2)\}.\end{aligned}$$

Therefore, $\zeta_1 * \zeta_2$ is a strong cubic graph.

Example 3.6. *Consider the strong cubic graphs* $\zeta_1 = (P_1, Q_1)$ *and* $\zeta_2 = (P_2, Q_2)$ *as Figure 3. It is easy to see that* $\zeta_1 * \zeta_2 = (P, Q)$ *is a strong cubic graph, too.*

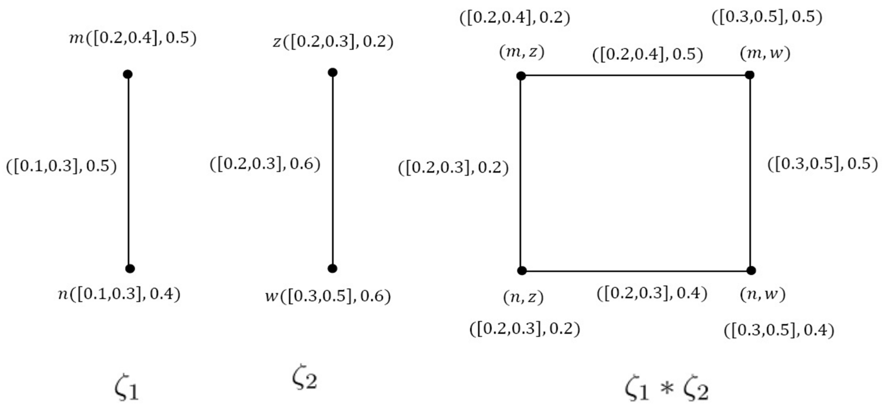

Figure 3: Strong cubic graphs ζ_1, ζ_2 and $\zeta_1 * \zeta_2$.

Remark 3.7 *If the maximal product of two cubic graphs* $(\zeta_1 * \zeta_2)$ *is a strong, then* ζ_1 *and* ζ_2 *need not to be strong, in general.*

Example 3.8 *Consider the cubic graphs* ζ_1, ζ_2 *and* $\zeta_1 * \zeta_2$ *as in Figure 4.*

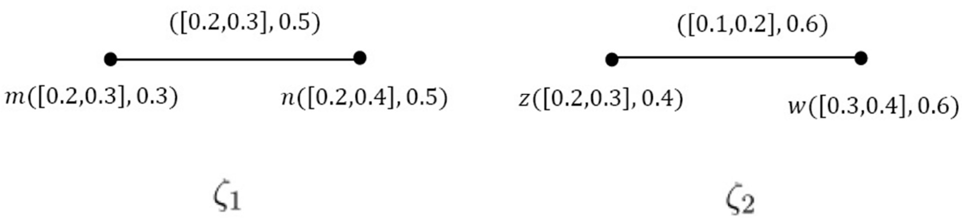

Figure 4: Cubic graphs ζ_1 and ζ_2.

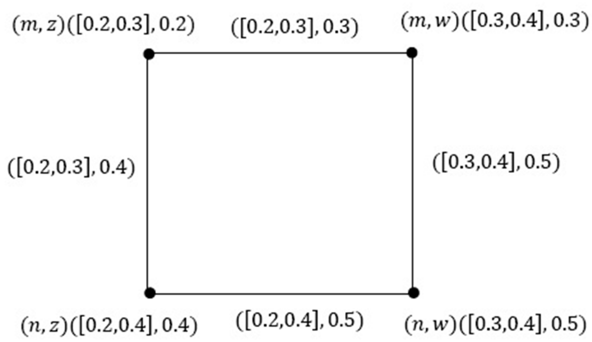

Figure 5: Strong cubic graph $\zeta_1 * \zeta_2$.

Hence, ζ_1 and $\zeta_1 * \zeta_2$ are strong, but ζ_2 is not strong. Since, $\widetilde{\mu_{Q_2}}(zw) = [0.1,0.2]$ but $r\min\{\widetilde{\mu_{P_2}}(z), \widetilde{P_2}(w)\} = r\min\{[0.2,0.3],[0.3,0.4]\} = [0.2,0.3]$.

So, $\widetilde{\mu_{Q_2}}(zw) \neq r\min\{\widetilde{\mu_{P_2}}(z), \widetilde{\mu_{P_2}}(w)\}$.

Definition 3.9. *A cubic graph ζ is called complete if:*
$\widetilde{\mu_Q}(mn) = \widetilde{\mu_P}(m) \wedge \widetilde{\mu_P}(n)$, $\lambda_Q(mn) = \lambda_P(m) \vee \lambda_P(n)$, for all $m,n \in V$.

Remark 3.10. *The maximal product of two complete cubic graphs is not a complete cubic graph, in general. Because we do not include the case $(m_1,m_2) \in E_1$ and $(n_1,n_2) \in E_2$ in the definition of the maximal product of two cubic graphs.*

Remark 3.11. *The maximal product of two complete cubic graphs is strong cubic graph.*

Example 3.12. *Consider the complete cubic graphs ζ_1 and ζ_2 as in Figure 4. A simple calculation concludes that $\zeta_1 * \zeta_2$ is a strong cubic graph.*

Definition 3.13. *The residue product $\zeta_1 \bullet \zeta_2$ of two cubic graphs $\zeta_1 = (P_1,Q_1)$ and $\zeta_2 = (P_2,Q_2)$ is defined as:*

$$(i)\begin{cases}\left(\widetilde{\mu_{P_1}} \bullet \widetilde{\mu_{P_2}}\right)((m_1,m_2)) = \mathrm{r}\max\{\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{P_2}}(m_2)\},\\ \left(\lambda_{P_1} \bullet \lambda_{P_2}\right)((m_1,m_2)) = \min\{\lambda_{P_1}(m_1), \lambda_{P_2}(m_2)\},\\ for\ all\ (m_1,m_2) \in V_1 \times V_2\end{cases}$$

$$(ii)\begin{cases}\left(\widetilde{\mu_{Q_1}} \bullet \widetilde{\mu_{Q_2}}\right)((m_1,m_2)(n_1,n_2)) = \widetilde{\mu_{Q_1}}(m_1n_1),\\ \left(\lambda_{Q_1} \bullet \lambda_{Q_2}\right)((m_1,m_2)(n_1,n_2)) = \lambda_{Q_1}(m_1n_1),\\ for\ all\ m_1n_1 \in E_1, m_2 \neq n_2.\end{cases}$$

Example 3.14. *Consider the cubic graphs ζ_1 and ζ_2 as in Figure 6. The residue product of ζ_1 and ζ_2 ($\zeta_1 \bullet \zeta_2$) shown in Figure 7.*

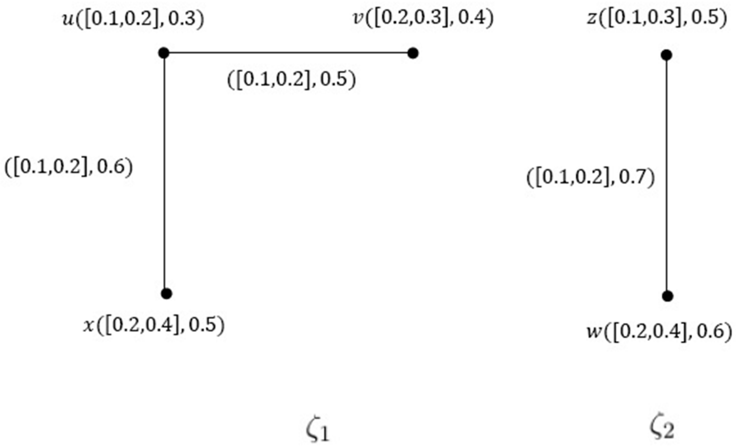

Figure 6: Cubic graphs ζ_1 and ζ_2.

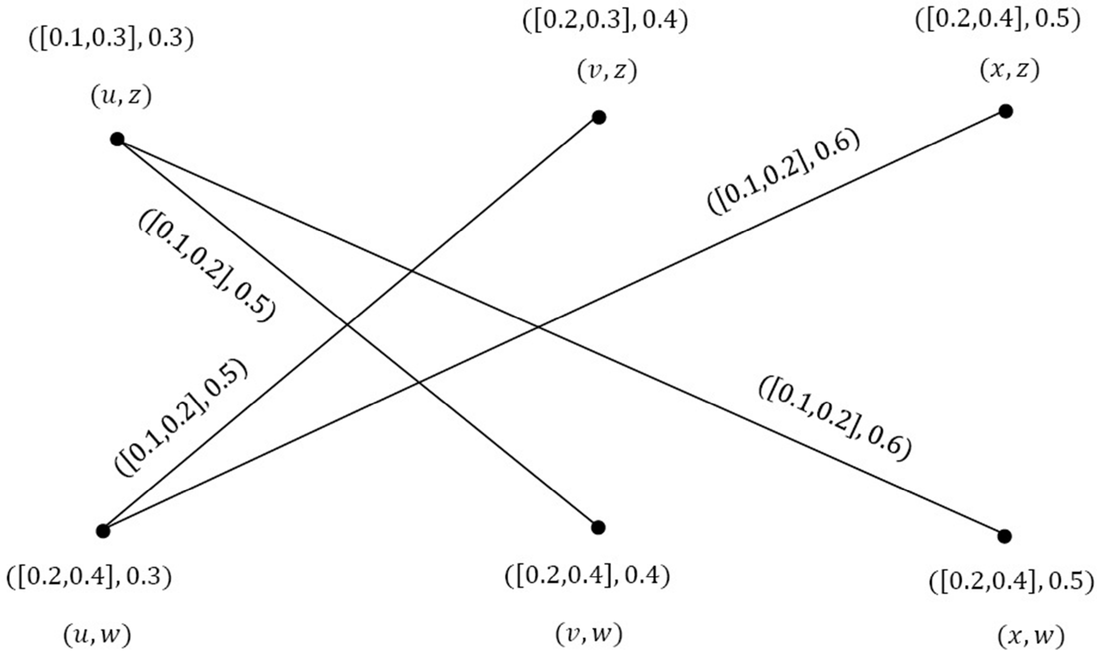

Figure 7: Residue product of two cubic graphs.

Proposition 3.15. *The residue product of two cubic graphs ζ_1 and ζ_2 is a cubic graph.*

Proof: Let $\zeta_1 = (P_1, Q_1)$ and $\zeta_2 = (P_2, Q_2)$ be two cubic graphs and $((m_1, m_2)(n_1, n_2)) \in E_1 \times E_2$. If $m_1 n_1 \in E_1$ and $m_2 \neq n_2$, then we have:

$$\left(\widetilde{\mu_{Q_1}} \bullet \widetilde{\mu_{Q_2}}\right)((m_1, m_2)(n_1, n_2)) = \widetilde{\mu_{Q_1}}(m_1 n_1)$$

$$\leq r\min\{\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{P_1}}(n_1)\}$$
$$\leq r\max\{r\min\{\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{P_1}}(n_1)\}, r\min\{\widetilde{\mu_{P_2}}(m_2), \widetilde{\mu_{P_2}}(n_2)\}\}$$
$$= r\min\{r\max\{\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{P_1}}(n_1)\}, r\max\{\widetilde{\mu_{P_2}}(m_2), \widetilde{\mu_{P_2}}(n_2)\}\}$$
$$= \min\langle(\widetilde{\mu_{P_1}} \bullet \widetilde{\mu_{P_2}})(m_1, m_2), (\widetilde{\mu_{P_1}} \bullet \widetilde{\mu_{P_2}})(n_1, n_2)\},$$

$$(\lambda_{Q_1} \bullet \lambda_{Q_2})((m_1, m_2)(n_1, n_2)) = \lambda_{Q_1}(m_1 n_1)$$
$$\geq \max\{\lambda_{P_1}(m_1), \lambda_{P_1}(n_1)\}$$
$$\geq \min\{\max\{\lambda_{P_1}(m_1), \lambda_{P_1}(n_1)\}, \max\{\lambda_{P_2}(m_2), \lambda_{P_2}(n_2)\}\}$$
$$= r\max\{\min\{\lambda_{P_1}(m_1), \lambda_{P_1}(n_1)\}, \min\{\lambda_{P_2}(m_2), \lambda_{P_2}(n_2)\}\}$$
$$= \max\langle(\lambda_{P_1} \bullet \lambda_{P_2})(m_1, m_2), (\lambda_{P_1} \bullet \lambda_{P_2})(n_1, n_2)\}.$$

Definition 3.16. *The rejection* $\zeta_1 | \zeta_2$ *of two cubic graphs* $\zeta_1 = (P_1, Q_1)$ *and* $\zeta_2 = (P_2, Q_2)$ *is defined as:*

$$(i) \begin{cases} (\widetilde{\mu_{P_1}} | \widetilde{\mu_{P_2}})(m, n) = r\min\{\widetilde{\mu_{P_1}}(m), \widetilde{\mu_{P_2}}(n)\} \\ (\lambda_{P_1} | \lambda_{P_2})(m, n) = \max\{\lambda_{P_1}(m), \lambda_{P_2}(n)\}, \\ for\ all\ (m, n) \in V_1 \times V_2, \end{cases}$$

$$(ii) \begin{cases} (\widetilde{\mu_{Q_1}} | \widetilde{\mu_{Q_2}})((m, m_2)(m, n_2)) = r\min\{\widetilde{\mu_{P_1}}(m), \widetilde{\mu_{P_2}}(m_2), \widetilde{\mu_{P_2}}(n_2)\} \\ (\lambda_{Q_1} | \lambda_{Q_2})((m, m_2)(m, n_2)) = \max\{\lambda_{P_1}(m), \lambda_{P_2}(m_2), \lambda_{P_2}(n_2)\}, \\ for\ all\ m \in V_1\ and m_2 n_2 \in E_2, \end{cases}$$

$$(iii) \begin{cases} (\widetilde{\mu_{Q_1}} | \widetilde{\mu_{Q_2}})((m_1, m)(n_1, m)) = r\min\{\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{P_1}}(n_1), \widetilde{\mu_{P_2}}(m)\} \\ (\lambda_{Q_1} | \lambda_{Q_2})((m_1, m)(n_1, m)) = \max\{\lambda_{P_1}(m_1), \lambda_{P_1}(n_1), \lambda_{P_2}(m)\}, \\ for\ all\ m \in V_2\ and\ m_1 n_1 \notin E_1, \end{cases}$$

$$(iv) \begin{cases} (\widetilde{\mu_{Q_1}} | \widetilde{\mu_{Q_2}})((m_1, m_2)(n_1, n_2)) = r\min\{\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{P_1}}(n_1), \widetilde{\mu_{P_2}}(m_2), \widetilde{\mu_{P_2}}(n_2)\} \\ (\lambda_{Q_1} | \lambda_{Q_2})((m_1, m_2)(n_1, n_2)) = \max\{\lambda_{P_1}(m_1), \lambda_{P_1}(n_1), \lambda_{P_2}(m_2), \lambda_{P_2}(n_2)\}, \\ for\ all\ m_1 n_1 \notin E_1\ and\ m_2 n_2 \notin E_2. \end{cases}$$

Proposition 3.17. *The rejection of two cubic graphs* ζ_1 *and* ζ_2, *is a cubic graph.*

Proof: Let $\zeta_1 = (P_1, Q_1)$ and $\zeta_2 = (P_2, Q_2)$ be two cubic graphs and $((m_1, m_2)(n_1, n_2)) \in E_1 \times E_2$. Then by Definition 3.16 we have:

(i) If $m_1 = n_1$ and $m_2 n_2 \notin E_2$,

$$(\widetilde{\mu_{Q_1}} | \widetilde{\mu_{Q_2}})((m_1, m_2)(n_1, n_2)) = r\min\{\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{P_2}}(m_2), \widetilde{\mu_{P_2}}(n_2)\}$$
$$= r\min\{r\min\{\widetilde{\mu_{P_1}}(m_1), \widetilde{\mu_{P_2}}(m_2)\}, r\min\{\widetilde{\mu_{P_1}}(n_1), \widetilde{\mu_{P_2}}(n_2)\}\}$$
$$= \min\{(\widetilde{\mu_{P_1}} | \widetilde{\mu_{P_2}})(m_1, m_2), (\widetilde{\mu_{P_1}} | \widetilde{\mu_{P_2}})(n_1, n_2)\},$$

$$(\lambda_{Q_1} | \lambda_{Q_2})((m_1, m_2)(n_1, n_2)) = \max\{\lambda_{P_1}(m_1), \lambda_{P_2}(m_2), \lambda_{P_2}(n_2)\}$$
$$= \max\{\max\{\lambda_{P_1}(m_1), \lambda_{P_2}(m_2)\}, \max\{\lambda_{P_1}(n_1), \lambda_{P_2}(n_2)\}\}$$
$$= \max\{(\lambda_{P_1} | \lambda_{P_2})(m_1, m_2), (\lambda_{P_1} | \lambda_{P_2})(n_1, n_2)\}.$$

(ii) If $m_2 = n_2$ and $m_1n_1 \notin E_1$

$$(\widetilde{\mu_{Q_1}}|\widetilde{\mu_{Q_2}})((m_1,m_2)(n_1,n_2)) = r\min\{\widetilde{\mu_{P_1}}(m_1),\widetilde{\mu_{P_1}}(n_1),\widetilde{\mu_{P_2}}(m_2)\}$$
$$= r\min\{r\min\{\widetilde{\mu_{P_1}}(m_1),\widetilde{\mu_{P_2}}(m_2)\},r\min\{\widetilde{\mu_{P_1}}(n_1),\widetilde{\mu_{P_2}}(n_2)\}\}$$
$$= r\min\{(\widetilde{\mu_{P_1}}|\widetilde{\mu_{P_2}})(m_1,m_2),(\widetilde{\mu_{P_1}}|\widetilde{\mu_{P_2}})(n_1,n_2)\},$$
$$(\lambda_{Q_1}|\lambda_{Q_2})((m_1,m_2)(n_1,n_2)) = \max\{\lambda_{P_1}(m_1),\lambda_{P_1}(n_1),\lambda_{P_2}(m_2)\}$$
$$= \max\{\max\{\lambda_{P_1}(m_1),\lambda_{P_2}(m_2)\},\max\{\lambda_{P_1}(n_1),\lambda_{P_2}(n_2)\}\}$$
$$= \max\{(\lambda_{P_1}|\lambda_{P_2})(m_1,m_2),(\lambda_{P_1}|\lambda_{P_2})(n_1,n_2)\}.$$

(iii) If $m_1n_1 \notin E_1$ and $m_2n_2 \notin E_2$

$$(\widetilde{\mu_{Q_1}}|\widetilde{\mu_{Q_2}})((m_1,m_2)(n_1,n_2)) = r\min\{\widetilde{\mu_{P_1}}(m_1),\widetilde{\mu_{P_1}}(n_1),\widetilde{\mu_{P_2}}(m_2),\widetilde{\mu_{P_2}}(n_2)\}$$
$$= r\min\{r\min\{\widetilde{\mu_{P_1}}(m_1),\widetilde{\mu_{P_2}}(m_2)\},r\min\{\widetilde{\mu_{P_1}}(n_1),\widetilde{\mu_{P_2}}(n_2)\}\}$$
$$= r\min\{(\widetilde{\mu_{P_1}}|\widetilde{\mu_{P_2}})(m_1,m_2),(\widetilde{\mu_{P_1}}|\widetilde{\mu_{P_2}})(n_1,n_2)\},$$

$$(\lambda_{Q_1}|\lambda_{Q_2})((m_1,m_2)(n_1,n_2)) = \max\{\lambda_{P_1}(m_1),\lambda_{P_1}(n_1),\lambda_{P_2}(m_2),\lambda_{P_2}(n_2)\}$$
$$= \max\{\max\{\lambda_{P_1}(m_1),\lambda_{P_2}(m_2)\},\max\{\lambda_{P_1}(n_1),\lambda_{P_2}(n_2)\}\}$$
$$= \max\{(\lambda_{P_1}|\lambda_{P_2})(m_1,m_2),(\lambda_{P_1}|\lambda_{P_2})(n_1,n_2)\}.$$

4. Conclusion

Cubic graphs have many applications in different sciences such as topology, natural networks, and operation research. Irregularity is the practical interest in several areas. Operations are conveniently used in many combinatorial applications; hence; in this paper, three new operations on cubic graphs, namely, maximal product, rejection, and residue product were presented, and some results concerning their degrees were introduced. Different examples are provided to evaluate the validity of the new definitions.

Acknowledgement. We would like to provide our cordial thanks to the honorable referees for their valuable comments which help us to enrich the quality of the paper.

REFERENCES

1. M.Akram, Interval-valued fuzzy graphs, *Computers and Mathematics with Applications*, 61 (2) (2011) 289-299.
2. M.Akram and M.Sitara, Certain concepts in intuitionistic neutrosophic graph structures, *Information*, 8 (2017) 154, doi:10.3390/info8040154.
3. R.A.Borzooei, H.Rashmanlou, Ring sum in product intuitionistic fuzzy graphs, *J. Adv. Res. Pure Math*, 7 (2015) 16-31.
4. R.A.Borzooei and H.Rashmanlou, Domination in vague graphs and its applications, *J. Intel-Ligent Fuzzy Syst*, 29 (2015) 1933-1940.
5. R.A.Borzooei, H.Rashmanlou, Degree of vertices in vague graphs, *J. Appl. Math. Inform*, 33 (2015) 545-557.
6. R.A.Borzooei and H.Rashmanlou, S.Samanta, M.Pal, Regularity of vague graphs, *Journal of Intelligent and Fuzzy Systems*, 30 (2016) 3681-3689.
7. R.A.Borzooei and H.Rashmanlou, Semi global domination sets in vague graphs with application, *Journal of Intelligent and Fuzzy Systems*, 7 (2015) 16-31.

8. R.A.Borzooei, H.Rashmanlou, S.Samanta and M.Pal, A Study on fuzzy labeling graphs, *Journal of Intelligent and Fuzzy Systems* 30 (2016) 3349-3355.
9. Y.B.Jun, Ch.S.Kim and K.O.Yang, Cubic sets, *Annals of Fuzzy Mathematicsand Informatics*, 4 (1) (2012) 83-98.
10. A.Kauffman, Introduction a la theories des sous-emsembles 503 ous. Masson Cie (1973) (1) 607–618.
11. P.K.Kishore Kumar, Yahya Talebi, H.Rashmanlou, A.A.Talebi and F.Mofidnakhaei, New concept of cubic graph with application, *J. of Mult.-Valued Logic & Soft Computing*, 33 (2019) 135-154.
12. G.Muhiuddin, S.S.Ahn, C.S.Kim and Y.B.Jun, Stable cubic sets, *Journal of Computational Analysis and Applications*, 23(5) (2017) 802-819.
13. G.Muhiuddin, F.Feng and Y.B.Jun, Subalgebras of BCK/BCI-algebras based on cubic soft sets, *The Scientific World Journal*, 2014 (2014), Article ID 458638, 9.
14. G.Muhiuddin and A.M.Al-roqi, Cubic soft sets with applications in BCK/BCI-algebras, *Annals of Fuzzy Mathematics and Informatics*, 8 (2) (2014) 291-304.
15. A.Nagoorgani and S.R.Latha, On irregular fuzzy graphs, *Appl. Math. Sci*, 6 (2012) 517-523.
16. A.Nagoorgani and K.Radha, On regular fuzzy graphs, *J. Phys. Sci*, 12 (2008) 33-44.
17. H.Rashmanlou and R.A.Borzooei, Product vague graphs and its applications, *Journal of Intelligent & Fuzzy Systems*, (30) (2016) 371-382.
18. H.Rashmanlou and R.A.Borzooei, Vague graphs with application, *J. Intell. Fuzzy Syst*, (30) (2016) 3291-3299.
19. H.Rashmanlou, S.Samanta, M.Pal and R.A.Borzooei, A study on bipolar fuzzy graphs. *J. Intell. Fuzzy Syst*, (28) (2015) 571-580.
20. H.Rashmanlou, Y.B.Jun and R.A.Borzooei, More results on highly irregular bipolar fuzzy graphs. *Ann. Fuzzy Math. Inform*, (8) (2014) 149-168.
21. H.Rashmanlou, R.A.Borzooei, M.Shoaib, Yahya Talebi, M.Taheri and F.Mofidnakhaei, New way for finding shortest path problem in a network, *J. of Mult.-Valued Logic & Soft Computing*, (34) (2020) 451-460.
22. H.Rashmanlou, S.Samanta, M.Pal, R.A.Borzooei, Bipolar fuzzy graphs with categorical properties, *International Journal of Computational Intelligence Systems*, 8(5) (2015) 808-818.
23. H.Rashmanlou, G.Muhiuddin, S.K.Amanathulla, F.Mofidnakhaei and M.Pal, A study on cubic graph with novel application, *Journal of Intelligent and Fuzzy Systems*, (40) (2021) 89-101.
24. H.Rashmanlou, S.Samanta, M.Pal and R.A.Borzooei, Product of bipolar fuzzy graphs and their degree, *International Journal of General Systems*, doi.org/10.1080/03081079.2015.1072521.
25. H.Rashmanlou and Y.B.Jun, Complete interval-valued fuzzy graphs, *Annals of Fuzzy Mathematics and Informatics*, 6 (3) (2013) 677-687.
26. H.Rashmanlou and M.Pal, Antipodal interval-valued fuzzy graphs, *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, 3 (2013) 107-130.
27. H.Rashmanlou and M.Pal, Balanced interval-valued fuzzy graph, *Journal of Physical Sciences*, 17 (2013) 43-57.

28. H.Rashmanlou and M.Pal, Some properties of highly irregular interval-valued fuzzy graphs, *World Applied Sciences Journal*, 27 (12) (2013) 1756-1773.
29. K.Radha and N.Kumaravel, Some properties of edge regular fuzzy graphs, *Jamal Acad. Res. J.,* (2014) 121-127.
30. A.Rosenfeld, Fuzzy graphs, Fuzzy Sets and their Applications; Zadeh, L.A., Fu, K.S., Shimura, M., Eds.; *Academic Press: New York*, NY, USA, (1975)77-95.
31. S.Sahoo, M.Pal, H.Rashmanlou and R.A.Borzooei, Covering and paired domination in intuitionistic fuzzy graphs, *J. Intell. Fuzzy Syst*,, 33 (6) (2017) 4007-4015.
32. S.Samanta and M.Pal, Irregular bipolar fuzzy graphs, *Int. J. Appl. Fuzzy Sets*, 2 (2012) 91-102.
33. S.Samanta and M.Pal, Fuzzy k-competition graphs and pcompetition fuzzy graphs, *Fuzzy Inf. Eng*, 5 (2013) 191-204.
34. M.S.Sunitha and S.Mathew, Fuzzy graph theory: A survey, *Ann. Pure Appl. Math*., (4) (2013) 92-110.
35. M.S.Sunitha and A.Vijayakumar, Complement of a fuzzy graph, *Indian J. Pure Appl. Math*, (33) (2002) 1451-1464.
36. M.Shoaib, S.Kosari, H.Rashmanlou, M.Aslam Malik, Y.Rao, Yahya Talebi and F.Mofidnakhaei, Notion of complex pythagorean fuzzy graph with properties and application, *J. of Mult.-Valued Logic & Soft Computing*, (34) (2020) 553-586.
37. A.A.Talebi, J.Kacprzyk, H.Rashmanlou and S.H.Sadati, A new concept of an intuitionistic fuzzy graph with applications, *J. of Mult.-Valued Logic & Soft Computing*, (35) (2020) 431-454.
38. A.A.Talebi, H.Rashmanlou and S.H.Sadati, Interval-valued intuitionistic fuzzy competition graph, *J. of Mult.-Valued Logic & Soft Computing*, 34 (2020) 335-364.
39. A.A.Talebi, H.Rashmanlou and S.H.Sadati, New concepts on m-polar interval-valued intuitionistic fuzzy graph, *TWMS J. App. and Eng. Math*, 10 (3) (2020) 806-818.
40. A.A.Talebi and H.Rashmanlou, Isomorphic on interval-valued fuzzy graphs, *Annals of fuzzy Mathematics and Informatics*, 6(1) (2013) 47-58.
41. L.A.Zadeh, Fuzzy sets, *Information and Control* , 8 (1965) 338-353.
42. L.A.Zadeh, Similarity relations and fuzzy ordering, *Information Sciences*, 3 (1971) 177-200.
43. L.A.Zadeh, Is there a need for fuzzy logical, *Information Sciences*, 178 (2008) 2751-2779.