Intern. J. Fuzzy Mathematical Archive
Vol. 19, No. 1, 2021, 61-80
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 20 July 2021
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/ijfma.v19n1a05227

International Journal of
Fuzzy Mathematical
Archive

A Study on *m*-polar Fuzzy Sets and *m*-polar Fuzzy Matrix

Purbasha Giri
Department of Applied Mathematics
Vidyasagar University, Midnapore - 721102, India.
E-mail: purbashagiri19@gmail.com

Received 7 June 2021; accepted 19 July 2021

Abstract. Fuzzy matrix is a very important topic of fuzzy algebra. In fuzzy matrix, the elements belong to the unit interval $[0,1]$, each element represents the membership value of an element. In this paper, m-polar fuzzy set, m-polar fuzzy relation, m-polar fuzzy matrix is introduced. In m-polar fuzzy matrix, each element is a vector containing m elements and membership values of each elements lie between 0 and 1 including 0 and 1. In m-polar fuzzy matrix, the membership values of rows and columns are crisp, i.e. rows and columns are certain. But, in many real life situations they are also uncertain. So to model these type of uncertain problems, a new type of uncertain problems, a new type of m-polar fuzzy matrix with m-polar fuzzy rows and columns are defined. For these matrices, null, identity, equality, complement, g-m-polar, complete, density are defined. For these matrices, we checked that whether the matrix is balanced, strictly balanced or not.

Keywords: m-polar fuzzy sets, m-polar fuzzy relation, fuzzy matrix, m-polar fuzzy matrix, equality, null, identity, density, balanced, strictly balanced.

AMS Mathematics Subject Classification (2010): 15B15

1. Introduction

Like classical (crisp) matrices, fuzzy matrices (FMs) are now a very reach topic in modelling uncertain situations occurred in science, automata theory, logic of binary relations, medical diagnosis, etc. In FMs, only the elements are uncertain, while rows and columns are certain. But, in many real life situations we observed that rows and columns may also be uncertain. For example, in a fuzzy graph the vertices and edges both are uncertain. If we represent a fuzzy graph in matrix form where the membership values of vertices and edges represents the membership values of rows and columns and elements represent the membership values of the corresponding edge. That is, in these matrices rows, columns and elements all are uncertain. We call these types of matrices are fuzzy matrices with fuzzy rows and columns (FMFRCs). This is the very new concept in fuzzy matrix theory.

FMs defined first time by Thomson in 1977 [44] and discussed about the convergence of the powers of a fuzzy matrix. The theories of fuzzy matrices were developed by Kim and Rosh [24] as an extension of Boolean matrices. With max-min operation the fuzzy algebra and its matrix theory are considered by many authors [5, 15,

22, 28, 29, 40]. Hashimoto [16] studied the canonical form of a transitive fuzzy matrix. Xin [45] studied controllable fuzzy matrices. Hemashina *et al.* [20] investigated iterates of fuzzy circulant matrices. Determinant theory, powers and nilpotent conditions of matrices over a distributive lattice are considered by Zhang [46] and Tan [43]. The transitivity of matrices over path algebra (*i.e.* additively idempotent semiring) is discussed by Hashimoto [17, 18, 19]. Generalized fuzzy matrices, matrices over an incline and some results about the transitive closer, determinant, adjoint matrices, convergence of powers and conditions for nilpotency are considered by Duan [14] and Lur *et al.* [25]. Dehghan *et al.* [13] give two ideas for finding the inverse of a fuzzy matrix viz. scenario-based and arithmetic-based. Many results are presented for fuzzy matrices in [37].

There are some limitations in dealing with uncertainties by fuzzy set. To overcome these difficulties, Atanassov [4] introduced theory of intuitionistic fuzzy set in 1993 as a generalization of fuzzy set. Based on this concept Pal *et al.* have defined intuitionistic fuzzy determinant in 2001 [28] and intuitionistic fuzzy matrices (IFMs) in 2002 [29]. Bhowmik and Pal [5] introduced some results on IFMs, intuitionistic circulant fuzzy matrix and generalized intuitionistic fuzzy matrix [5, 7, 8, 9, 10, 11, 12]. Shyamal and Pal [39, 41] defined the distances between IFMs and hence defined a metric on IFMs. They also cited few applications of IFMs. In [27], the similarity relations, invertibility conditions and eigenvalues of IFMs are studied. Idempotent, regularity, permutation matrix and spectral radius of IFMs are also discussed. The parameterizations tool of IFM enhances the flexibility of its applications. For other works on IFMs see [1, 2, 3, 26, 31, 32, 40, 41].

Many types of fuzzy matrices are also developed in recent years such as bipolar fuzzy matrices [36], picture fuzzy matrices [6], etc.

Pal [35] introduced the new concepts of fuzzy matrix with fuzzy rows and fuzzy columns. In this fuzzy matrix, we assumed that the rows and columns are uncertain. The same approach is extended to interval-valued fuzzy matrix [34] and intuitionistic fuzzy matrix [38].

The concept of interval-valued fuzzy matrices (IVFMs) as a generalization of fuzzy matrix was introduced and developed in 2006 by Shaymal and Pal [42] by extending the max-min operation in fuzzy algebra. For more works on IVFMs see [30].

Combining IFMs and IVFMs, a new fuzzy matrix called interval-valued intuitionistic fuzzy matrices (IVIFMs) is defined [21]. For other works on IVIFMs, see [10, 12].

2. m-polar fuzzy sets and their properties

Definition 1. (*m*-polar fuzzy set (MPFS))

An ***m-polar fuzzy set (MPFS)*** $\mathcal{M}_{\mathcal{F}}$ *in* X *is an object of the form*

$$\mathcal{M}_{\mathcal{F}} = \{(S, \psi_1(S), \psi_2(S), \dots, \psi_m(S))\}$$

where $\psi_1, \psi_2, \dots, \psi_m : X \to [0,1]$ are m functions represents the membership functions of the element S.

Definition 2. *Let* W *be an* m*-polar fuzzy set on* X *and* $p = (p_1, p_2, \dots, p_m)$, $q = (q_1, q_2, \dots, q_m)$ *be two elements of* W *where* $p_1, p_2, \dots, p_m$ *and* $q_1, q_2, \dots, q_m \in [0,1]$. Then for any $\alpha \in [0,1]$, we define

1. (Maximum)

$$p \vee q = (p_1, p_2, \dots, p_m) \vee (q_1, q_2, \dots, q_m) = (p_1 \vee q_1, p_2 \vee q_2, \dots, p_m \vee q_m)$$

2. (Minimum)

$$p \wedge q = (p_1, p_2, \dots, p_m) \wedge (q_1, q_2, \dots, q_m) = (p_1 \wedge q_1, p_2 \wedge q_2, \dots, p_m \wedge q_m)$$

3. (Ring subtraction)

$$p \ominus q = (p_1, p_2, \dots, p_m) \ominus (q_1, q_2, \dots, q_m) = \begin{cases} (p_1, p_2, \dots, p_m), & if \quad (p_1, p_2, \dots, p_m) > (q_1, q_2, \dots, q_m) \\ (0,0, \dots, 0), & otherwise \end{cases}$$

4. (Upper α-cut $p^{(\alpha)}$)

$$p^{(\alpha)} = \begin{cases} (1,1, \dots, 1), & if \quad (p_1, p_2, \dots, p_m) \geq (\alpha, \alpha, \dots, \alpha) \\ (0,0, \dots, 0), & otherwise \end{cases}$$

5. (Lower α-cut $p^{(\alpha)}$)

$$p_{(\alpha)} = \begin{cases} (p_1, p_2, \dots, p_m), & if \quad (p_1, p_2, \dots, p_m) \geq (\alpha, \alpha, \dots, \alpha) \\ (0,0, \dots, 0), & otherwise \end{cases}$$

6. (Complement)

$$p^c = (1,1, \dots, 1) - (p_1, p_2, \dots, p_m) = (1 - p_1, 1 - p_2, \dots, 1 - p_m)$$

7. (Sum)

$$p + q = (p_1, p_2, \dots, p_m) + (q_1, q_2, \dots, q_m) = (p_1 \vee q_1, p_2 \vee q_2, \dots, p_m \vee q_m)$$

8. (Product)

$$p.q = (p_1, p_2, \dots, p_m).(q_1, q_2, \dots, q_m) = (p_1 \wedge q_1, p_2 \wedge q_2, \dots, p_m \wedge q_m).$$

The MPFS can be represented as

$$\mathcal{M}_{\mathcal{F}} = \{x = (x_1, x_2, \dots, x_m) : x \in X\},$$

where $x_1, x_2, \dots, x_m \in [0,1]$ are the m membership values of $x \in X$ in $\mathcal{M}_{\mathcal{F}}$.

Definition 3. (Equality) *Let* $x, y \in \mathcal{M}_{\mathcal{F}}$, *where* $x = (x_1, x_2, \dots, x_m)$ *and* $y = (y_1, y_2, \dots, y_m)$, *then the equality of two elements of* x *and* y *is denoted by* $x = y$ *and is defined by,* $x_i = y_i$, *for* $i = 1,2, \dots, m$.

Definition 4. *Let* $x, y \in \mathcal{M}_{\mathcal{F}}$, *where* $x = (x_1, x_2, \dots, x_m)$ *and* $y = (y_1, y_2, \dots, y_m)$ *and* $x_1, x_2, \dots, x_m$ *and* $y_1, y_2, \dots, y_m \in [0,1]$ *then,*

1. The **disjunction** of x and y is denoted by $x + y$ and is defined by

$$\begin{aligned} x + y &= (x_1, x_2, \dots, x_m) + (y_1, y_2, \dots, y_m) \\ &= (max\{x_1, y_1\}, max\{x_2, y_2\}, \dots, max\{x_m, y_m\}) \\ &= (x_1 \vee y_1, x_2 \vee y_2, \dots, x_m \vee y_m). \end{aligned}$$

2. The **parallel disjunction** of x and y is denoted by $x.y$ and is defined by

$$\begin{aligned} x.y &= (x_1, x_2, \dots, x_m).(y_1, y_2, \dots, y_m) \\ &= (min\{x_1, y_1\}, min\{x_2, y_2\}, \dots, min\{x_m, y_m\}) \\ &= (x_1 \wedge y_1, x_2 \wedge y_2, \dots, x_m \wedge y_m). \end{aligned}$$

3. The **negation** of x is denoted by $-x$ and is defined by

$$\begin{aligned} -x &= -(x_1, x_2, \dots, x_m) \\ &= (-x_m, -x_{m-1}, \dots, -x_2, -x_1). \end{aligned}$$

4. The **implication** of x and y is denoted by xy and is defined by

$$(xy) = x^c + y$$

5. The **complement** of x is denoted by x^c and is defined by

$$\begin{aligned} x^c &= (x_1, x_2, \dots, x_m)^c \\ &= (x_1^c, x_2^c, \dots, x_m^c) \\ &= (1 - x_1, 1 - x_2, \dots, 1 - x_m). \end{aligned}$$

Definition 5. (Zero element) *The zero element of a MPFS is denoted by*

$$o_m = (0,0, \dots, 0).$$

Definition 6. (Unit element) *The unit element of a MPFS is denoted by* $i_m = (1,1, \dots, 1)$.

Proposition 1. *Let* $\mathcal{M}_{\mathcal{F}}$ *be any MPFS and a,b,c* $\in \mathcal{M}_{\mathcal{F}}$, *where* $a = (a_1, a_2, \dots, a_m)$; $b = (b_1, b_2, \dots, b_m)$ *and* $c = (c_1, c_2, \dots, c_m)$; $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_m \in [0,1]$ *then the following properties are satisfied:*

1. $a + b = b + a$,
   $a.b = b.a$;
2. $a + (b + c) = (a + b) + c$,
   $a.(b.c) = (a.b).c$;
3. $a + o_m = o_m + a = a$,
   $a.i_m = i_m.a = a$;
4. $a.(b + c) = a.b + a.c$;
5. $a - b \neq b - a$;
6. $a.(b - c) \neq a.b - a.c$

Theorem 1. *De Morgan's laws are satisfied on MPFS* $\mathcal{M}_{\mathcal{F}}$. *That is, if* $a = (a_1, a_2, \dots, a_m)$ *and* $b = (b_1, b_2, \dots, b_m) \in \mathcal{M}_{\mathcal{F}}$

1. $(a + b)^c = a^c.b^c$ and
2. $(a.b)^c = a^c + b^c$.

Example 1. *Let* $a = (0.5,0.3,0.8,0.9)$ *and* $b = (0.6,0.1,0.2,0.4)$ *be two elements of* $\mathcal{M}_{\mathcal{F}}$.

Then $a^c = (0.5,0.7,0.2,0.1)$ and $b^c = (0.4,0.9,0.8,0.6)$.

$a + b = (0.6,0.3,0.8,0.9)$, $(a + b)^c = (0.4,0.7,0.2,0.1)$, $a^c.b^c = (0.4,0.7,0.2,0.1)$.

Therefore, $(a + b)^c = a^c.b^c$.

Again, $a.b = (0.5,0.1,0.2,0.4)$

and $(a.b)^c = (0.5,0.9,0.8,0.6)$, $a^c + b^c = (0.5,0.9,0.8,0.6)$.

Therefore, $(a.b)^c = a^c + b^c$.

3. m-polar fuzzy relation

In this section, we define Cartesian product of two MPFSs, and relation. Also several basic properties are investigated.

Definition 7. (Cartesian product of MPFSs) *Let X_1 and X_2 be two universe of discourses and $A = \{x = (x_1, x_2, \dots, x_n): x \in X_1\}$ and $B = \{y = (y_1, y_2, \dots, y_n): y \in X_2\}$ be two MPFSs. The cartesian product of A and B is given by*

$$A \times B = \{(x, y): x \in X_1 \quad and \quad y \in X_2\}.$$

Definition 8. (*m*-polar fuzzy relation) *An m-polar fuzzy relation (MPFR) between two MPFSs A and B is defined as a MPFS in $A \times B$. If R is a relation between A and B, $x \in A$ and $y \in B$ and if $r_1(x, y), r_2(x, y), \dots, r_m(x, y)$ are the m membership values to which x is in relation R with y, then $r = (r_1, r_2, \dots, r_m) \in R$.*

Definition 9. (Inclusion) *Let $\mathcal{M}_{\mathcal{F}}$ be a MPFS over X and let $x, y \in \mathcal{M}_{\mathcal{F}}$ where $x = (x_1, x_2, \dots, x_m)$ and $y = (y_1, y_2, \dots, y_m)$, then $x \leq y$ if and only if $x_i \leq y_i$ for all $i = 1,2, \dots, m$. That is if and only if $x + y = y$.*

Definition 10. *Let $\mathcal{M}_{\mathcal{F}}$ be a MPFS over X and let $x, y \in \mathcal{M}_{\mathcal{F}}$ where $x = (x_1, x_2, \dots, x_m)$ and $y = (y_1, y_2, \dots, y_m)$, then $x < y$ iff $x \leq y$ and $x \neq y$.*

Proposition 2. *The relation '$\leq$' is partial order relation in a MPFS.*

Proof: We have to prove the relation '$\leq$' is partial order relation. That is to prove that the relation '$\leq$' is reflexive, anti-symmetric and transitive. [(i)]

1. Since $a_1 \leq a_1,\ a_2 \leq a_2,\ \dots, a_m \leq a_m$,
   That is $a_i \leq a_i$ for all $i = 1,2, \dots, m$.
   So we write $a \leq a$ for all $a \in \mathcal{M}_{\mathcal{F}}$.
   That is, the relation '$\leq$' is reflexive.
2. Let $a \leq b$ and $b \leq a$ for any $a, b \in M_F$.
   Then $a_1 \leq b_1,\ a_2 \leq b_2,\ \dots, a_m \leq b_m$, and
   also, $b_1 \leq a_1,\ b_2 \leq a_2,\ \dots, b_m \leq a_m$.
   This implies $a_1 = b_1,\ a_2 = b_2,\ \dots, a_m = b_m$.
   That is, $(a_1, a_2, \dots, a_m) = (b_1, b_2, \dots, b_m)$.
   That is, $a = b$.
   Thus, $a \leq b$ and $b \leq a$ implies $a = b$ for any $a, b \in \mathcal{M}_{\mathcal{F}}$.
   That is, the relation '$\leq$' is anti-symmetric.
3. Let $a \leq b$ and $b \leq c$ for any $a, b, c \in \mathcal{M}_{\mathcal{F}}$.
   Then $a_1 \leq b_1,\ a_2 \leq b_2,\ \dots, a_m \leq b_m$ and
   $b_1 \leq c_1,\ b_2 \leq c_2,\ \dots, b_m \leq c_m$.
   or, $a_1 \leq b_1 \leq c_1, a_2 \leq b_2 \leq c_2, \dots, a_m \leq b_m \leq c_m$.
   or, $a_1 \leq c_1,\ a_2 \leq c_2,\ \dots, a_m \leq c_m$
   or, $(a_1, a_2, \dots, a_m) \leq (c_1, c_2, \dots, c_m)$
   That is $a \leq c$.
   Thus $a \leq b$ and $b \leq c$ implies $a \leq c$ for any $a, b, c \in \mathcal{M}_{\mathcal{F}}$.
   That is the relation '$\leq$' is transitive.
   Hence, the relation '$\leq$' in a MPFS is a partial order relation.

Proposition 3. *Let $\mathcal{M}_{\mathcal{F}}$ be a MPFS over X and let $a, b, c \in \mathcal{M}_{\mathcal{F}}$ where, $a = (a_1, a_2, \dots, a_m)$, $b = (b_1, b_2, \dots, b_m)$, $c = (c_1, c_2, \dots, c_m)$, then [(i)]*

1. $o_m \leq x \leq i_m$, for any x,
2. If $a \leq b$ then $a + c \leq b + c$ and $a.c \leq b.c$,
3. $a \leq a + b$ and $b \leq a + b$, $a + b$ is the least upper bound of a and b. In other words, if there is an element c satisfying $a \leq c$ and $b \leq c$ then $a + b \leq c$,
4. $a.b \leq a$ and $a.b \leq b$. That is, $a.b$ is a lower bound of a and b,
5. $a.b.c \leq a.b$.

4. m-polar fuzzy matrix

In order to develop the theory of m-polar fuzzy matrix (MPFM), we begin with the concept of m-polar fuzzy algebra. An m-polar fuzzy algebra is a mathematical system $(\mathcal{M}_{\mathcal{F}}, +, .)$ with two binary operations + and . defined on $\mathcal{M}_{\mathcal{F}}$ satisfying the following properties: [i)]

1. **Idempotent :** $a + a = a \ , \quad a.a = a$
2. **Commutativity :** $a + b = b + a \ , \quad a.b = b.a$
3. **Associativity :** $a + (b + c) = (a + b) + c$
   $a.(b.c) = (a.b).c$
4. **Absorption :** $a + (a.b) = a \ , \quad a.(a + b) = a$
5. **Distributivity :** $a + (b.c) = (a + b).(a + c)$
   $a.(b + c) = (a.b) + (a + c)$
6. **Universal bounds :** $a + o_m = a, \quad a + i_m = i_m,$
   $a.i_m = i_m, \quad i_m.a = a,$

where $a = (a_1, a_2, \dots, a_m) \ , \quad b = (b_1, b_2, \dots, b_m) \ , \quad c = (c_1, c_2, \dots, c_m) \in \mathcal{M}_{\mathcal{F}}$.

Definition 11. (m-polar fuzzy matrix) *An m-**polar fuzzy matrix (MPFM)** is the matrix over the m-polar fuzzy algebra. The zero matrix O_m of order $m \times m$ is the matrix where all the elements are $o_m = (0,0,\dots,0)$ and the identity matrix I_m of order $m \times m$ is the matrix where all the diagonal entries are $i_m = (1,1,\dots,1)$ and all other entries are $o_m = (0,0,\dots,0)$.*

The set of all rectangular MPFMs of order $l \times m$ is denoted by $\mathcal{M}_{lm}$ and that of square MPFMs of order $m \times m$ is denoted by $\mathcal{M}_m$.

From the definition, we conclude that if $A = (a_{ij})_{l\times m} \in \mathcal{M}_{l\times m}$, then $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)}) \in \mathcal{M}_{\mathcal{F}}$ where $a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)} \in [0,1]$ are the membership values of the element a_{ij}, respectively.

Operations on MPFM

Some operations on MPFM are defined as follows:

Definition 12. *Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n} \in \mathcal{M}_{mn}$ be is m-polar fuzzy matrices (MPFMs) where $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)})$, $b_{ij} = (b_{ij}^{(1)}, b_{ij}^{(2)}, \dots, b_{ij}^{(m)})$. Therefore $a_{ij}, b_{ij} \in \mathcal{M}_F$, then*

$$A + B = (a_{ij} + b_{ij})_{m\times n} = (a_{ij}^{(1)} \vee b_{ij}^{(1)}, a_{ij}^{(2)} \vee b_{ij}^{(2)}, \dots, a_{ij}^{(m)} \vee b_{ij}^{(m)})$$

and,

$$A.B = (a_{ij}.b_{ij})_{m\times n} = (a_{ij}^{(1)} \wedge b_{ij}^{(1)}, a_{ij}^{(2)} \wedge b_{ij}^{(2)}, \dots, a_{ij}^{(m)} \wedge b_{ij}^{(m)})$$

Definition 13. *Let* $A = (a_{ij})_{l\times m} \in \mathcal{M}_{lm}$ *and* $B = (b_{ij})_{m\times q} \in \mathcal{M}_{mq}$ *be is m-polar fuzzy matrices (MPFMs), where* $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)})$, $b_{ij} = (b_{ij}^{(1)}, b_{ij}^{(2)}, \dots, b_{ij}^{(m)})$. *Therefore* a_{ij}, $b_{ij} \in \mathcal{M}_F$, *then*

$$A \odot B = \left(\sum_{k=1}^{m} (a_{ik}.b_{kj})\right)_{l\times q} = \left(\max_{k=1}^{m}[\min\{a_{ik}^{(1)}, b_{kj}^{(1)}\}], \max_{k=1}^{m}[\min\{a_{ik}^{(2)}, b_{kj}^{(2)}\}], \dots, \max_{k=1}^{m}[\min\{a_{ik}^{(m)}, b_{kj}^{(m)}\}]\right)_{l\times q},$$

and,

$$A \otimes B = \left(\prod_{k=1}^{m} (a_{ik} + b_{kj})\right)_{l\times q} = \left(\min_{k=1}^{m}[\max\{a_{ik}^{(1)}, b_{kj}^{(1)}\}], \min_{k=1}^{m}[\max\{a_{ik}^{(2)}, b_{kj}^{(2)}\}], \dots, \min_{k=1}^{m}[\max\{a_{ik}^{(m)}, b_{kj}^{(m)}\}]\right)_{l\times q}.$$

Proposition 4. *In the MPFMs* A, B, C *are conformal for corresponding operations, then*

1. $A + B = B + A$, $A.B = B.A$,
2. $A + (B + C) = (A + B) + C$, $A.(B.C) = (A.B).C$,
3. $A.(B + C) = A.B + A.C$, $A + (B.C) = (A + B).(A + C)$,
4. $A + O = O + A = A$, $A.O = O.A = A$, *if* O *be the zero matrix, with appropriate order,*
5. $A \odot B \neq B \odot A$, $A \otimes B \neq B \otimes A$, *in general,*
6. $A \odot (B \odot C) = (A \odot B) \odot C$, $A \otimes (B \otimes C) = (A \otimes B) \otimes C$,
7. $A \odot I = I \odot A = A$, $A \otimes I = I \otimes A = A$, *if* I *be the identity matrix, with appropriate order,*
8. $A \odot (B \otimes C) \neq (A \odot B) \otimes (A \odot C)$, $A \otimes (B.C) \neq (A \otimes B).(A \otimes C)$.

Example 2. *Let*

$$A = \begin{bmatrix} (0.3,0.5,0.8) & (0.1,0.2,0.9) & (0.6,0.7,0.8) \\ (0.4,0.7,0.6) & (0.2,0.3,0.4) & (0.3,0.4,0.1) \\ (0.5,0.3,0.2) & (0.3,0.2,0.1) & (0.6,0.1,0.9) \end{bmatrix}$$

and,

$$B = \begin{bmatrix} (0.9,0.8,0.7) & (0.5,0.4,0.3) & (0.2,0.6,0.4) \\ (0.5,0.6,0.2) & (0.1,0.2,0.3) & (0.5,0.7,0.9) \\ (0.1,0.4,0.7) & (0.2,0.5,0.8) & (0.3,0.6,0.9) \end{bmatrix}$$

then

$$A \odot B = \begin{bmatrix} (0.3,0.5,0.7) & (0.3,0.5,0.8) & (0.3,0.6,0.9) \\ (0.4,0.7,0.6) & (0.4,0.4,0.3) & (0.3,0.6,0.4) \\ (0.5,0.3,0.7) & (0.5,0.3,0.8) & (0.3,0.3,0.9) \end{bmatrix}$$

and,

$$B \odot A = \begin{bmatrix} (0.4,0.5,0.7) & (0.2,0.3,0.7) & (0.6,0.7,0.7) \\ (0.5,0.5,0.3) & (0.3,0.2,0.3) & (0.5,0.6,0.9) \\ (0.3,0.5,0.7) & (0.3,0.3,0.7) & (0.3,0.4,0.9) \end{bmatrix}$$

$$Note \quad that, \quad A \odot B \neq B \odot A.$$

Again,

$$A \otimes B = \begin{bmatrix} (0.5,0.6,0.8) & (0.1,0.2,0.8) & (0.3,0.6,0.8) \\ (0.3,0.4,0.4) & (0.2,0.3,0.4) & (0.3,0.6,0.6) \\ (0.5,0.4,0.2) & (0.3,0.2,0.3) & (0.5,0.6,0.4) \end{bmatrix}$$

and,

$$B \otimes A = \begin{bmatrix} (0.5,0.6,0.4) & (0.3,0.4,0.4) & (0.5,0.4,0.3) \\ (0.5,0.6,0.6) & (0.2,0.3,0.4) & (0.3,0.4,0.3) \\ (0.3,0.5,0.8) & (0.1,0.4,0.8) & (0.3,0.5,0.8) \end{bmatrix}$$

$$Here \quad also, \quad A \otimes B \neq B \otimes A.$$

5. m-polar fuzzy matrix with m-polar fuzzy rows and columns

Definition 14. *Let* $A = (r_A(i))(c_A(j))(a_{ij})_{p\times q}$ *be a* m*-polar fuzzy matrix with* m*-polar fuzzy rows and columns, where* $r_A(i) = (r_A^1(i), r_A^2(i), \dots, r_A^m(i))$; $c_A(j) = (c_A^1(j), c_A^2(j), \dots, c_A^m(j))$; $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)})$.

Here, a_{ij}, $i = 1,2,\dots,p; j = 1,2,\dots,q$ represents the ij^{th} element of A, $r_A(i)$, $c_A(j)$ represent the membership values of i^{th} row and j^{th} column respectively for $i = 1,2,\dots,p$; $j = 1,2,\dots,q$. [(i)]

1. When $r_A(i) = 1$, $i = 1,2,\dots,p$; $c_A(j) = 1$, $j = 1,2,\dots,q$ and $a_{ij} \in [0,1]$, $i = 1,2,\dots,p$; $j = 1,2,\dots,q$, then A is a m-polar fuzzy matrix (MPFM).
2. When $r_A(i) \in [0,1]$, $i = 1,2,\dots,p$; $c_A(j) \in [0,1]$, $j = 1,2,\dots,q$ and $a_{ij} \in [0,1]$, $i = 1,2,\dots,p$; $j = 1,2,\dots,q$, then A is called a m-polar fuzzy matrix with m-polar fuzzy rows and columns.

5.1. Equality of m-polar fuzzy matrix with m-polar fuzzy rows and columns

The equality of two m-polar fuzzy matrix with m-polar fuzzy rows and columns can be defined in three different ways.

Let $A = (r_A(i))(c_A(j))(a_{ij})_{p\times q}$ and $B = (r_B(i))(c_B(j))(b_{ij})_{p\times q}$ be two m-polar fuzzy matrix with m-polar fuzzy rows and columns, where

$$
\begin{aligned}
r_A(i) &= (r_A^1(i), r_A^2(i), \dots, r_A^m(i),\\
r_B(i) &= (r_B^1(i), r_B^2(i), \dots, r_B^m(i),\\
c_A(j) &= (c_A^1(j), c_A^2(j), \dots, c_A^m(j),\\
c_B(j) &= (c_B^1(j), c_B^2(j), \dots, c_B^m(j),\\
a_{ij} &= (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)}),\\
b_{ij} &= (b_{ij}^{(1)}, b_{ij}^{(2)}, \dots, b_{ij}^{(m)}).
\end{aligned}
$$

Type-I

If $r_A(i) = r_B(i)$ [That is, $r_A^1(i) = r_B^1(i), r_A^2(i) = r_B^2(i), \dots, r_A^m(i) = r_B^m(i)$] and $c_A(j) = c_B(j)$ [That is, $c_A^1(j) = c_B^1(j), c_A^2(j) = c_B^2(j), \dots, c_A^m(j) = c_B^m(j)$] for $i = 1,2, \dots, p$; $j = 1,2, \dots, q$. Whatever may be the relation between a_{ij} and b_{ij}, then we say that A and B are RC-equal and it is denoted by $A =_{RC} B$.

If $r_A(i) \neq r_B(i)$ [That is, at least one of $r_A^1(i) \neq r_B^1(i), r_A^2(i) \neq r_B^2(i), \dots, r_A^m(i) \neq r_B^m(i)$] or $c_A(j) \neq c_B(j)$ [That is, at least one of $c_A^1(j) \neq c_B^1(j), c_A^2(j) \neq c_B^2(j), \dots, c_A^m(j) \neq c_B^m(j)$] for at least one i or j, then we say that $A \neq B$ in RC-equal sense.

This is the weak equality between two m-polar fuzzy matrix with fuzzy rows and columns.

Type-II

If $a_{ij} = b_{ij}$ [That is $a_{ij}^1 = b_{ij}^1, a_{ij}^2 = b_{ij}^2, \dots, a_{ij}^m = b_{ij}^m$] for all i and j, whatever may be the values of $r_A(i)$, $r_b(i)$, $c_A(j)$, $c_B(j)$, then A and B are E-equal and it is denoted by $A =_E B$. This type of equality occurs in m-polar fuzzy matrix also.

If $a_{ij} \neq b_{ij}$ [That is $a_{ij}^1 \neq b_{ij}^1, a_{ij}^2 \neq b_{ij}^2, \dots, a_{ij}^m \neq b_{ij}^m$] for at least one i or j, then we say $A \neq_E B$ or $A \neq B$ in E-equal sense.

Type-III

If both $A =_{RC} B$ and $A =_E B$, then we say that A and B are equal and it is denoted as $A = B$. That is, if [i)]

1. $a_{ij} = b_{ij}$ [$a_{ij}^1 = b_{ij}^1, a_{ij}^2 = b_{ij}^2, \dots, a_{ij}^m = b_{ij}^m$] for all i and j,
2. $r_A(i) = r_B(i)$ [$r_A^1(i) = r_B^1(i), r_A^2(i) = r_B^2(i), \dots, r_A^m(i) = r_B^m(i)$] for all i,
3. $c_A(j) = c_B(j)$ [$c_A^1(j) = c_B^1(j), c_A^2(j) = c_B^2(j), \dots, c_A^m(j) = c_B^m(j)$] for all j.

Then A is equal to B and is denoted as $A = B$. If A and B are not equal, then it is denoted by $A \neq B$. That is, $A \neq_{RC} B$ and / or $A \neq_E B$, then we write $A \neq B$.

5.2. Null m-polar fuzzy matrix with m-polar fuzzy rows and columns

Based on the membership values of rows, columns and elements, three types of null m-polar fuzzy matrix with m-polar fuzzy rows and columns are defined.

Type-I

If $r_A(i) = (r_A^1(i), r_A^2(i), \dots, r_A^m(i)) = (0,0, \dots, 0)$, $c_A(j) = (c_A^1(j), c_A^2(j), \dots, c_A^m(j)) = (0,0, \dots, 0)$ and $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)}) = (0,0, \dots, 0)$ for all i and j, then m-polar fuzzy matrix with m-polar fuzzy rows and columns of A is called p-null, denoted by O_p.

For example,

$$\begin{matrix} & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \end{matrix}$$

is a 3×3 order p-null m-polar fuzzy matrix with m-polar fuzzy rows and columns.

Type-II

If $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)}) = (0,0,\dots,0)$ for all i and j, whatever may be the values of $r_A(i)$ and $c_A(j)$, then the m-polar fuzzy matrix with m-polar fuzzy rows and columns A is called E-null and it is denoted by O_E. For example,

$$\begin{matrix} & (0.1,0.5,0.3) & (0.2,0.4,0.7) & (0.9,0.3,0.2) \\ (0.2,0.4,0.6) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0.3,0.5,0.7) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0.8,0.9,0.4) & (0,0,0) & (0,0,0) & (0,0,0) \end{matrix}$$

is a 3×3 order E-null m-polar fuzzy matrix with m-polar fuzzy rows and columns.

Type-III

If $r_A(i) = (r_A^1(i), r_A^2(i), \dots, r_A^m(i)) = (0,0,\dots,0)$, $c_A(j) = (c_A^1(j), c_A^2(j), \dots, c_A^m(j)) = (0,0,\dots,0)$ for all i and j, whatever may be the values of a_{ij}, then the m-polar fuzzy matrix with m-polar fuzzy rows and columns A is called RC-null and it is denoted by O_{RC}. For example,

$$\begin{matrix} & (0.0,0.0,0.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.2,0.5,0.3) & (0.1,0.3,0.5) & (0.8,0.9,0.2) \\ (0.0,0.0,0.0) & (0.4,0.6,0.8) & (0.6,0.7,0.1) & (0.5,0.3,0.1) \\ (0.0,0.0,0.0) & (0.9,0.7,0.5) & (0.5,0.4,0.3) & (0.6,0.4,0.3) \end{matrix}$$

is a 3×3 order RC-null m-polar fuzzy matrix with m-polar fuzzy rows and columns.

5.3. Identity m-polar fuzzy matrix with m-polar fuzzy rows and columns

Two types of identity m-polar fuzzy matrix with m-polar fuzzy rows and columns are defined here.

Type-I

A square m-polar fuzzy matrix with m-polar fuzzy rows and columns of order $n \times n$ is called p-identity m-polar fuzzy matrix with m-polar fuzzy rows and columns, if $r_A(i) = (r_A^1(i), r_A^2(i), \dots, r_A^m(i)) = (1,1,\dots,1)$, $c_A(j) = (c_A^1(j), c_A^2(j), \dots, c_A^m(j)) = (1,1,\dots,1)$ for all i and j and $a_{ii} = (a_{ii}^{(1)}, a_{ii}^{(2)}, \dots, a_{ii}^{(m)}) = (1,1,\dots,1)$ and $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)}) = (0,0,\dots,0)$, $i \neq j$ for all i and j. It is denoted by I_p. For example,

$$\begin{matrix} & (1,1,1) & (1,1,1) & (1,1,1) \\ (1,1,1) & (1,1,1) & (0,0,0) & (0,0,0) \\ (1,1,1) & (0,0,0) & (1,1,1) & (0,0,0) \\ (1,1,1) & (0,0,0) & (0,0,0) & (1,1,1) \end{matrix}$$

is a 3×3 order p-identity m-polar fuzzy matrix with m-polar fuzzy rows and columns.

Type-II

A square m-polar fuzzy matrix with m-polar fuzzy rows and columns of order $n \times n$ is called f-identity m-polar fuzzy matrix with m-polar fuzzy rows and columns, if $a_{ii} = (a_{ii}^{(1)}, a_{ii}^{(2)}, \dots, a_{ii}^{(m)}) = (1,1,\dots,1)$ and $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)}) = (0,0,\dots,0)$, $i \neq j$ for all i and j, whatever may be the values of $r_A(i)$ and $c_A(i)$ and it is denoted by I_f. For example,

$$\begin{array}{c} \\ (0.2,0.4,0.6) \\ (0.5,0.8,0.7) \\ (0.9,0.5,0.3) \end{array} \begin{array}{c} \begin{array}{ccc} (0.5,0.6,0.7) & (0.2,0.3,0.1) & (0.9,0.8,0.7) \end{array} \\ \begin{bmatrix} (1.0,1.0,1.0) & (0.0,0.0,0.0) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (1.0,1.0,1.0) & (0.0,0.0,0.0) \\ (0.0,0.0,0.0) & (0.0,0.0,0.0) & (1.0,1.0,1.0) \end{bmatrix} \end{array}$$

is a 3×3 order f-identity m-polar fuzzy matrix with m-polar fuzzy rows and columns.

Definition 15. *If* $a_{ij}^{(1)} \leq r_A^1(i) \wedge c_A^1(j)$, $a_{ij}^{(2)} \leq r_A^2(i) \wedge c_A^2(j)$, ..., $a_{ij}^{(m)} \leq r_A^m(i) \wedge c_A^m(j)$ *for all* i, j, *then the* m*-polar fuzzy matrix with* m*-polar fuzzy rows and columns* $A = (r_A(i))(C_A(j))(a_{ij})_{p\times q}$ *is called* ***g-*m*-polar fuzzy matrix with* m*-polar fuzzy rows and columns****.*

For example,

$$\begin{array}{c} \\ (0.1,0.5,0.7) \\ (0.2,0.9,0.3) \\ (0.4,0.6,0.3) \end{array} \begin{array}{c} \begin{array}{ccc} (0.1,0.3,0.7) & (0.8,0.5,0.4) & (0.9,0.6,0.3) \end{array} \\ \begin{bmatrix} (0.0,0.2,0.6) & (0.1,0.3,0.2) & (0.0,0.3,0.2) \\ (0.1,0.2,0.1) & (0.1,0.4,0.2) & (0.1,0.5,0.2) \\ (0.0,0.2,0.1) & (0.3,0.2,0.1) & (0.1,0.5,0.2) \end{bmatrix} \end{array}$$

is a 3×3 order g-m-polar fuzzy matrix with m-polar fuzzy rows and columns.

Definition 16. *If* $a_{ij}^{(1)} = r_A^1(i) \wedge c_A^1(j)$, $a_{ij}^{(2)} = r_A^2(i) \wedge c_A^2(j)$, ..., $a_{ij}^{(m)} = r_A^m(i) \wedge c_A^m(j)$ *for all* i, j, *then the* m*-polar fuzzy matrix with* m*-polar fuzzy rows and columns* $A = (r_A(i))(c_A(j))(a_{ij})_{p\times q}$ *is called* ***complete-*m*-polar fuzzy matrix with* m*-polar fuzzy rows and columns****.*

From the definition it is obvious that every complete m-polar fuzzy matrix with m-polar fuzzy rows and columns is g-m-polar fuzzy matrix with m-polar fuzzy rows and columns.

But the converse is not true.

For example,

$$\begin{array}{c} \\ (0.9,0.7,0.5) \\ (0.5,0.3,0.1) \\ (0.2,0.8,0.7) \end{array} \begin{array}{c} \begin{array}{ccc} (0.2,0.4,0.6) & (0.3,0.5,0.7) & (0.1,0.3,0.5) \end{array} \\ \begin{bmatrix} (0.2,0.4,0.5) & (0.3,0.5,0.5) & (0.1,0.3,0.5) \\ (0.2,0.3,0.1) & (0.3,0.3,0.1) & (0.1,0.3,0.1) \\ (0.2,0.4,0.6) & (0.2,0.5,0.7) & (0.1,0.3,0.5) \end{bmatrix} \end{array}$$

is a complete m-polar fuzzy matrix with m-polar fuzzy rows and columns.

Definition 17. *If* $a_{ij}^{(1)} \leq r_A^1(i).c_A^1(j)$, $a_{ij}^{(2)} \leq r_A^2(i).c_A^2(j)$, ..., $a_{ij}^{(m)} \leq r_A^m(i).c_A^m(j)$ *for all* i, j, *then the* m*-polar fuzzy matrix with* m*-polar fuzzy rows and columns* $A =$

$(r_A(i))(C_A(j))(a_{ij})_{p\times q}$ *is called a* ***dot*** m***-polar fuzzy matrix with*** m***-polar fuzzy rows and columns****, where* ***'.'*** *denotes ordinary multiplication.*

For example,

$$A = \begin{matrix} \\ (0.1,0.5,0.7) \\ (0.2,0.9,0.3) \\ (0.4,0.6,0.3) \end{matrix} \begin{matrix} \begin{matrix} (0.1,0.3,0.7) & (0.8,0.5,0.4) & (0.9,0.6,0.3) \end{matrix} \\ \begin{bmatrix} (0.0,0.2,0.6) & (0.1,0.3,0.2) & (0.0,0.3,0.2) \\ (0.1,0.2,0.1) & (0.1,0.4,0.2) & (0.1,0.5,0.2) \\ (0.0,0.2,0.1) & (0.3,0.2,0.1) & (0.1,0.5,0.2) \end{bmatrix} \end{matrix}$$

Lemma 1. *Every dot* m*-polar fuzzy matrix with* m*-polar fuzzy rows and columns is a g-*m*-polar fuzzy matrix with* m*-polar fuzzy rows and columns.*

Proof: Since $0 \le r_A^k(i) \le 1,\ 0 \le c_A^k(j) \le 1$ for all $i,j,k = 1,2,\dots,m.$

$r_A^k(i).c_A^k(j) \le r_A^k(i) \wedge c_A^k(j)$ for all $i,j,k = 1,2,\dots,m.$

If A is a dot m-polar fuzzy matrix with m-polar fuzzy rows and columns, then

$a_{ij}^{(k)} \le r_A^k(i).c_A^k(j)$ for all $i,j,k = 1,2,\dots,m.$

Therefore, for all $i,j.$

$a_{ij}^{(k)} \le r_A^k(i).c_A^k(j) \le r_A^k(i) \wedge c_A^k(j)$, where $k = 1,2,\dots,m.$

That is, $a_{ij}^{(k)} \le r_A^k(i) \wedge c_A^k(j)$; for all $i,j,k = 1,2,\dots,m.$

Hence, A is g-m-polar fuzzy matrix with m-polar fuzzy rows and columns.

Definition 18. *Let* $A = (r_A(i))(c_A(j))(a_{ij})_{p\times q}$ *be a* m*-polar fuzzy matrix with* m*-polar fuzzy rows and columns. Its* ***complement*** *is denoted by* A^c *and it is defined as*

$$A^c = (1 - r_A(i))(1 - c_A(j))(1 - a_{ij})_{p\times q}$$

where,

$$\begin{aligned} r_A(i) &= (r_A^1(i), r_A^2(i), \dots, r_A^m(i), \\ c_A(j) &= (c_A^1(j), c_A^2(j), \dots, c_A^m(j), \\ a_{ij} &= (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)}), \\ 1 &= (1,1,\dots,1). \end{aligned}$$

Definition 19. *A* m*-polar fuzzy matrix with* m*-polar fuzzy rows and columns* A *is called* ***self-complement*** *if* $A^c = A.$

Theorem 2. *If* A *is a* m*-polar fuzzy matrix with* m*-polar fuzzy rows and columns, then* $(A^c)^c = A.$

Proof: Let $A = (r_A(i))(c_A(j))(a_{ij})_{p\times q}$, where $r_A(i) = (r_A^1(i), r_A^2(i), \dots, r_A^m(i));$ $c_A(j) = (c_A^1(j), c_A^2(j), \dots, c_A^m(j));\ a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)}).$

Let $B = A^c.$

Then,

$$\begin{aligned} r_B(i) &= 1 - r_A(i) \\ &= (1,1,\dots,1) - (r_A^1(i), r_A^2(i), \dots, r_A^m(i)) \\ &= (1 - r_A^1(i), 1 - r_A^2(i), \dots, 1 - r_A^m(i)), \end{aligned}$$

$$\begin{aligned} c_B(i) &= 1 - c_A(j) \\ &= (1,1,\dots,1) - (c_A^1(j), c_A^2(j), \dots, c_A^m(j)) \\ &= (1 - c_A^1(j), 1 - c_A^2(j), \dots, 1 - c_A^m(j)) \end{aligned}$$

$$\begin{aligned} b_{ij} &= 1 - a_{ij} \\ &= (1,1,\dots,1) - (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)}) \\ &= (1 - a_{ij}^{(1)}, 1 - a_{ij}^{(2)}, \dots, 1 - a_{ij}^{(m)}) \end{aligned}$$

If $D = B^c = (A^c)^c$.
Then

$$\begin{aligned} d_{ij} &= 1 - b_{ij} \\ &= (1,1,\dots,1) - (b_{ij}^{(1)}, b_{ij}^{(2)}, \dots, b_{ij}^{(m)}) \\ &= (1 - b_{ij}^{(1)}, 1 - b_{ij}^{(2)}, \dots, 1 - b_{ij}^{(m)}) \\ &= (1 - (1 - a_{ij}^{(1)}), 1 - (1 - a_{ij}^{(2)}), \dots, 1 - (1 - a_{ij}^{(m)})) \\ &= (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)}) \\ &= a_{ij}. \end{aligned}$$

Similarly, $r_D(i) = r_A(i)$, $c_D(j) = c_A(j)$ for all i, j.
Hence, $D = A$, i.e. $(A^c)^c = A$.

Theorem 3. *If an m-polar fuzzy matrix with m-polar fuzzy rows and columns* $A = (r_A(i))(c_A(j))(a_{ij})_{p\times q}$, where $r_A(i) = (r_A^1(i), r_A^2(i), \dots, r_A^m(i))$; $c_A(j) = (c_A^1(j), c_A^2(j), \dots, c_A^m(j))$; $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)})$, then $r_A^k(i) = c_A^k(j) = a_{ij}^k = \frac{1}{2}$, for all i, j where $k = 1,2,\dots,m$.

Proof: By the definition of complement,

$$\begin{aligned} a_{ij}^c &= (1,1,\dots,1) - (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)}) \\ &= (1 - a_{ij}^{(1)}, 1 - a_{ij}^{(2)}, \dots, 1 - a_{ij}^{(m)}) \end{aligned}$$

Since A is self complement, therefore, $A^c = A$.
That is, $a_{ij}^c = a_{ij}$
That is, $(1 - a_{ij}^{(1)}, 1 - a_{ij}^{(2)}, \dots, 1 - a_{ij}^{(m)}) = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)})$
This gives, $1 - a_{ij}^{(k)} = a_{ij}^{(k)}$, for all $i, j, k = 1,2,\dots,m$.
$\therefore a_{ij}^{(k)} = \frac{1}{2}$, for all $i, j, k = 1,2,\dots,m$.
Similarly, $r_A^c(i) = c_A^k(j) = \frac{1}{2}$, for all $i, j, k = 1,2,\dots,m$.

6. Density of a m-polar fuzzy matrix with m-polar fuzzy rows and columns

Definition 20. *Let $A = (a_{ij})_{m\times n}$ be a m -polar fuzzy matrix where $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)})$. The **density of** A is denoted by $D(A)$ and is defined as $D(A) = (D^1(A), D^2(A), \dots, D^m(A))$ where*

$$D^k(A) = \frac{1}{mn}\sum_{i,j} a_{ij}^{(k)},\ k = 1,2,\dots,m.$$

From definition it follows that $0 \leq D(A) \leq 1$ for a m-polar fuzzy matrix A. Actually, $D(A)$ represents the average membership of the elements in the MPFM A.

Example 3. *Let*

$$A = \begin{bmatrix} (0.2,0.4,0.6) & (0.7,0.5,0.1) & (0.3,0.2,0.5) \\ (0.8,0.9,0.2) & (0.1,0.3,0.4) & (0.7,0.6,0.5) \\ (0.3,0.2,0.1) & (0.4,0.5,0.7) & (0.8,0.7,0.3) \end{bmatrix}$$

Now,

$$\begin{aligned} D^1(A) &= \tfrac{1}{3\times3}\textstyle\sum_{i,j} a_{ij}^{(1)} \\ &= \tfrac{1}{9}(0.2+0.7+0.3+0.8+0.1+0.7+0.3+0.4+0.8) \\ &= 0.4778 \\ D^2(A) &= \tfrac{1}{3\times3}\textstyle\sum_{i,j} a_{ij}^{(2)} \\ &= \tfrac{1}{9}(0.4+0.5+0.2+0.9+0.3+0.6+0.2+0.5+0.7) \\ &= 0.4778 \\ D^3(A) &= \tfrac{1}{3\times3}\textstyle\sum_{i,j} a_{ij}^{(3)} \\ &= \tfrac{1}{9}(0.6+0.1+0.5+0.2+0.4+0.5+0.1+0.7+0.3) \\ &= 0.3778 \end{aligned}$$

Thus, density of A,
$D(A) = (D^1(A), D^2(A), D^3(A)) = (0.4778,0.4778,0.3778)$.

Using this example, we see that $0 \leq D(A) \leq 1$ for a m-polar fuzzy matrix A (MPFMs). But, in m-polar fuzzy matrix with m-polar fuzzy rows and columns, rows and columns are not certain and hence the density is to be redefined for m-polar fuzzy matrix with m-polar fuzzy rows and columns. The definition is given below.

Definition 21. *Let $A = (r_A(i))(c_A(j))(a_{ij})_{p\times q}$ be a m-polar fuzzy matrix with m-polar fuzzy rows and columns where $r_A(i) = (r_A^1(i), r_A^2(i), \dots, r_A^m(i))$;*
$c_A(j) = (c_A^1(j), c_A^2(j), \dots, c_A^m(j))$; $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, \dots, a_{ij}^{(m)})$.
The density of A is denoted by $D(A)$ and is defined as $D(A) = (D^1(A), D^2(A), \dots, D^m(A))$ where

$$D^k(A) = \frac{\sum_{i,j} a_{ij}^{(k)}}{\sum_{i,j} r_A^k(i) \wedge c_A^k(j)} \quad , \qquad k = 1,2,\dots,m$$

provided $\sum_{i,j} r_A^k(i) \wedge c_A^k(j) \neq 0$.

Example 4.

$$A = \begin{array}{c} \\ (0.2,0.5,0.3) \\ (0.3,0.8,0.1) \\ (0.6,0.8,0.5) \end{array} \begin{array}{c} \begin{array}{ccc} (0.6,0.9,0.5) & (0.5,0.5,0.2) & (0.3,0.2,0.1) \end{array} \\ \begin{bmatrix} (0.5,0.6,0.7) & (0.2,0.3,0.5) & (0.1,0.2,0.4) \\ (0.6,0.4,0.3) & (0.8,0.6,0.4) & (0.9,0.6,0.3) \\ (0.3,0.2,0.3) & (0.5,0.6,0.7) & (0.2,0.1,0.5) \end{bmatrix} \end{array}$$

Now,

$$D^1(A) = \frac{\sum_{i,j} a_{ij}^{(1)}}{\sum_{i,j} r_A^1(i) \wedge c_A^1(j)} = \frac{0.5+0.2+0.1+0.6+0.8+0.9+0.3+0.5+0.2}{0.2+0.2+0.2+0.3+0.3+0.3+0.6+0.5+0.3} = 1.5769$$

$$D^2(A) = \frac{\sum_{i,j} a_{ij}^{(2)}}{\sum_{i,j} r_A^2(i) \wedge c_A^2(j)} = \frac{0.6+0.3+0.2+0.4+0.6+0.6+0.2+0.6+0.1}{0.5+0.5+0.2+0.8+0.5+0.2+0.8+0.5+0.2} = 0.8571$$

$$D^3(A) = \frac{\sum_{i,j} a_{ij}^{(3)}}{\sum_{i,j} r_A^3(i) \wedge c_A^3(j)} = \frac{0.7+0.5+0.4+0.3+0.4+0.3+0.3+0.7+0.5}{0.3+0.2+0.1+0.1+0.1+0.1+0.5+0.2+0.1} = 2.41176$$

Thus, density of A, $D(A) = (D^1(A), D^2(A), D^3(A)) = (1.5769, 0.8571, 2.41176)$.

Note: If A is a m-polar fuzzy matrix with m-polar fuzzy rows and columns, the density of A may be greater than 1, that is the components of $D(A) = (D^1(A), D^2(A), \dots, D^m(A))$ may be greater than 1, that is there is no upper bound.

Theorem 4. *If A is a g-m-polar fuzzy matrix with m-polar fuzzy rows and columns, then $0 \le D(A) \le 1$ that is $(0,0,\dots,0) \le (D^1(A), D^2(A), \dots, D^m(A)) \le (1,1,\dots,1)$.*

Proof: Since A is a g-m-polar fuzzy matrix with m-polar fuzzy rows and columns,

$$0 \le a_{ij}^k \le r_A^k(i) \wedge c_A^k(j) \text{ for all } i, j, k = 1,2,\dots,m.$$

Therefore,

$$0 \le \sum_{i,j} a_{ij}^k \le \sum_{i,j} r_A^k(i) \wedge c_A^k(j) \quad ; \quad where \quad k = 1,2,\dots,m.$$

That is,

$$0 \le \frac{\sum_{i,j} a_{ij}^k}{\sum_{i,j} r_A^k(i) \wedge c_A^k(j)} \le 1 \quad ; \quad where \quad k = 1,2,\dots,m.$$

That is,

$$0 \le D^k(A) \le 1 \quad ; \quad where \quad k = 1,2,\dots,m.$$

Hence, if A is a g-m-polar fuzzy matrix with m-polar fuzzy rows and columns, then $0 \le D(A) \le 1$ that is $(0,0,\dots,0) \le (D^1(A), D^2(A), \dots, D^m(A)) \le (1,1,\dots,1)$.

Example 5. *Let A be a g-m-polar fuzzy matrix with g-m-polar fuzzy rows and columns, where*

$$A = \begin{matrix} \\ (0.2,0.4,0.6) \\ (0.7,0.5,0.3) \\ (0.8,0.9,0.1) \end{matrix} \begin{matrix} \begin{matrix} (0.5,0.6,0.9) & (0.2,0.3,0.5) & (0.7,0.8,0.5) \end{matrix} \\ \begin{bmatrix} (0.1,0.2,0.5) & (0.1,0.2,0.4) & (0.5,0.4,0.3) \\ (0.4,0.3,0.2) & (0.1,0.2,0.2) & (0.5,0.3,0.2) \\ (0.4,0.7,0.1) & (0.1,0.2,0.1) & (0.5,0.7,0.1) \end{bmatrix} \end{matrix}$$

Now,

$$D^1(A) = \frac{\sum_{i,j} a_{ij}^{(1)}}{\sum_{i,j} r_A^1(i) \wedge c_A^1(j)} = \frac{0.1+0.1+0.5+0.4+0.1+0.5+0.4+0.1+0.5}{0.2+0.2+0.2+0.5+0.2+0.7+0.5+0.2+0.7} = 0.7941$$

$$D^2(A) = \frac{\sum_{i,j} a_{ij}^{(2)}}{\sum_{i,j} r_A^2(i) \wedge c_A^2(j)} = \frac{0.2+0.2+0.4+0.3+0.2+0.3+0.7+0.2+0.7}{0.4+0.3+0.4+0.5+0.3+0.5+0.6+0.3+0.8} = 0.7805$$

$$D^3(A) = \frac{\sum_{i,j} a_{ij}^{(3)}}{\sum_{i,j} r_A^3(i) \wedge c_A^3(j)} = \frac{0.5+0.4+0.3+0.2+0.2+0.2+0.1+0.1+0.1}{0.6+0.5+0.5+0.3+0.3+0.3+0.1+0.1+0.1} = 0.7500$$

Thus, density of A, $D(A) = (D^1(A), D^2(A), D^3(A)) = (0.7941, 0.7805, 0.75)$.
Here we see that, $(0,0,\dots,0) \le (D^1(A), D^2(A), D^3(A)) \le (1,1,\dots,1)$.

Theorem 5. *If A is a complete g-m-polar fuzzy matrix with m-polar fuzzy rows and columns, then the density of A, $D(A) = (D^1(A), D^2(A), \dots, D^m(A)) = (1,1,\dots,1) = 1$.*
Proof: Since, A is a complete g-m-polar fuzzy matrix with m-polar fuzzy rows and columns, then $a_{ij}^k = r_A^k(i) \wedge c_A^k(j)$ for all $i, j, k = 1,2,\dots,m$.
Therefore,

$$\sum_{i,j} a_{ij}^k = \sum_{i,j} r_A^k(i) \wedge c_A^k(j) \;\; ; \quad where \quad k = 1,2,\dots,m.$$

That is,

$$\frac{\sum_{i,j} a_{ij}^k}{\sum_{i,j} r_A^k(i) \wedge c_A^k(j)} = 1 \;\; ; \quad where \quad k = 1,2,\dots,m.$$

Hence,

$$D^k(A) = 1 \;\; ; \quad where \quad k = 1,2,\dots,m.$$

Hence, if A is a complete g-m-polar fuzzy matrix with m-polar fuzzy rows and columns, then $D(A) = (D^1(A), D^2(A), \dots, D^m(A)) = (1,1,\dots,1) = 1$.

Example 6. *Let A be a complete g-m-polar fuzzy matrix with m-polar fuzzy rows and columns, where*

$$A = \begin{matrix} \\ (0.1,0.6,0.5) \\ (0.3,0.4,0.6) \\ (0.2,0.5,0.7) \end{matrix} \begin{matrix} \begin{matrix} (0.2,0.5,0.8) & (0.3,0.7,0.8) & (0.5,0.9,0.3) \end{matrix} \\ \begin{bmatrix} (0.1,0.5,0.5) & (0.1,0.6,0.5) & (0.1,0.6,0.3) \\ (0.2,0.4,0.6) & (0.3,0.4,0.6) & (0.3,0.4,0.3) \\ (0.2,0.5,0.7) & (0.2,0.5,0.7) & (0.2,0.5,0.3) \end{bmatrix} \end{matrix}$$

Now,

$$D^1(A) = \frac{\sum_{i,j} a_{ij}^{(1)}}{\sum_{i,j} r_A^1(i) \wedge c_A^1(j)} = \frac{0.1+0.1+0.1+0.2+0.3+0.3+0.2+0.2+0.2}{0.1+0.1+0.1+0.2+0.3+0.3+0.2+0.2+0.2} = 1$$

$$D^2(A) = \frac{\sum_{i,j} a_{ij}^{(2)}}{\sum_{i,j} r_A^2(i) \wedge c_A^2(j)} = \frac{0.5+0.6+0.6+0.4+0.4+0.4+0.5+0.5+0.5}{0.5+0.6+0.6+0.4+0.4+0.4+0.5+0.5+0.5} = 1$$

$$D^3(A) = \frac{\sum_{i,j} a_{ij}^{(3)}}{\sum_{i,j} r_A^3(i) \wedge c_A^3(j)} = \frac{0.5+0.5+0.3+0.6+0.6+0.3+0.7+0.7+0.3}{0.5+0.5+0.3+0.6+0.6+0.3+0.7+0.7+0.3} = 1$$

Thus, density of A, $D(A) = (D^1(A), D^2(A), D^3(A)) = (1,1,1)$.

Definition 22. *An m-polar fuzzy matrix with m-polar fuzzy rows and columns A is called balanced if $D^k(S) \leq D^k(A)$ where $k = 1,2,\dots,m$ and for all sub-m-polar fuzzy matrix with m-polar fuzzy rows and columns S of A.*

Example 7. Let

$$A = \begin{matrix} \\ (0.2,0.4,0.6) \\ (0.1,0.3,0.5) \end{matrix} \begin{matrix} \begin{matrix} (0.8,0.5,0.7) & (0.3,0.9,0.2) \end{matrix} \\ \begin{bmatrix} (0.3,0.2,0.5) & (0.6,0.4,0.3) \\ (0.9,0.2,0.1) & (0.3,0.7,0.5) \end{bmatrix} \end{matrix}$$

Now,

$$D^1(A) = \frac{\sum_{i,j} a_{ij}^{(1)}}{\sum_{i,j} r_A^1(i) \wedge c_A^1(j)} = \frac{0.3+0.6+0.9+0.3}{0.2+0.2+0.1+0.1} = 3.5$$

$$D^2(A) = \frac{\sum_{i,j} a_{ij}^{(2)}}{\sum_{i,j} r_A^2(i) \wedge c_A^2(j)} = \frac{0.2+0.4+0.2+0.7}{0.4+0.4+0.3+0.3} = 1.071$$

$$D^3(A) = \frac{\sum_{i,j} a_{ij}^{(3)}}{\sum_{i,j} r_A^3(i) \wedge c_A^3(j)} = \frac{0.5+0.3+0.1+0.5}{0.6+0.2+0.5+0.2} = 0.933$$

Let S_1, S_2, S_3, S_4 be the sub-3-polar fuzzy matrix with 3-polar fuzzy rows and columns, where

$$S_1 = \begin{matrix} & (0.8,0.5,0.7) \\ (0.2,0.4,0.6) & [(0.3,0.2,0.5)] \end{matrix} \qquad S_2 = \begin{matrix} & (0.3,0.9,0.2) \\ (0.2,0.4,0.6) & [(0.6,0.4,0.3)] \end{matrix}$$

$$S_3 = \begin{matrix} & (0.8,0.5,0.7) \\ (0.1,0.3,0.5) & [(0.9,0.2,0.1)] \end{matrix} \qquad S_4 = \begin{matrix} & (0.3,0.9,0.2) \\ (0.1,0.3,0.5) & [(0.3,0.7,0.5)] \end{matrix}$$

Therefore,

$$D(S_1) = \left(\frac{0.3}{0.2}, \frac{0.2}{0.4}, \frac{0.5}{0.6}\right) = (1.5, 0.5, 0.83)$$
$$D(S_2) = \left(\frac{0.6}{0.2}, \frac{0.4}{0.4}, \frac{0.3}{0.2}\right) = (3, 1, 1.5)$$
$$D(S_3) = \left(\frac{0.9}{0.1}, \frac{0.2}{0.3}, \frac{0.1}{0.5}\right) = (9, 0.67, 0.2)$$
$$D(S_4) = \left(\frac{0.3}{0.1}, \frac{0.7}{0.3}, \frac{0.5}{0.2}\right) = (3, 2.33, 2.5)$$

Note that,

$$D^1(S_3) = 9 > D^1(A)$$
$$D^2(S_4) = 2.33 > D^2(A)$$
$$D^3(S_2) = 1.5 > D^3(A)$$
$$D^3(S_4) = 2.5 > D^3(A)$$

Hence, A is not a balanced m-polar fuzzy matrix with m-polar fuzzy rows and columns.

Definition 23. *An m-polar fuzzy matrix with m-polar fuzzy rows and columns A is called* ***strictly balanced*** *if $D^k(S) = D^k(A)$ where $k = 1,2,\dots,m$ for all sub-m-polar fuzzy matrix with m-polar fuzzy rows and columns S of A.*

Example 8. *Let*

$$A = \begin{matrix} & (0.4,0.6,0.8) & (0.1,0.3,0.6) \\ (0.2,0.4,0.8) & (0.4,0.4,0.0) & (0.2,0.3,0.0) \\ (0.3,0.6,0.9) & (0.6,0.6,0.0) & (0.2,0.3,0.0) \end{matrix}$$

Now,

$$D^1(A) = \frac{\sum_{i,j} a_{ij}^{(1)}}{\sum_{i,j} r_A^1(i) \wedge c_A^1(j)} = \frac{0.4+0.2+0.6+0.2}{0.2+0.1+0.3+0.1} = 2$$
$$D^2(A) = \frac{\sum_{i,j} a_{ij}^{(2)}}{\sum_{i,j} r_A^2(i) \wedge c_A^2(j)} = \frac{0.4+0.3+0.6+0.3}{0.4+0.3+0.6+0.3} = 1$$

$$D^3(A) = \frac{\sum_{i,j} a_{ij}^{(3)}}{\sum_{i,j} r_A^3(i) \wedge c_A^3(j)} = \frac{0.0+0.0+0.0+0.0}{0.8+0.6+0.8+0.6} = 0$$

Let S_1, S_2, S_3, S_4 be the sub-3-polar fuzzy matrix with 3-polar fuzzy rows and columns, where

$$S_1 = \begin{matrix} & (0.4,0.6,0.8) \\ (0.2,0.4,0.8) & [(0.4,0.4,0.0)] \end{matrix} \qquad S_2 = \begin{matrix} & (0.1,0.3,0.6) \\ (0.2,0.4,0.8) & [(0.2,0.3,0.0)] \end{matrix}$$

$$S_3 = \begin{matrix} & (0.4,0.6,0.8) \\ (0.3,0.6,0.9) & [(0.6,0.6,0.0)] \end{matrix} \qquad S_4 = \begin{matrix} & (0.1,0.3,0.6) \\ (0.3,0.6,0.9) & [(0.2,0.3,0.0)] \end{matrix}$$

Therefore,

$$D(S_1) = \left(\frac{0.4}{0.2}, \frac{0.4}{0.4}, \frac{0.0}{0.8}\right) = (2,1,0);$$
$$D(S_2) = \left(\frac{0.2}{0.1}, \frac{0.3}{0.3}, \frac{0.0}{0.6}\right) = (2,1,0);$$
$$D(S_3) = \left(\frac{0.6}{0.3}, \frac{0.6}{0.6}, \frac{0.0}{0.6}\right) = (2,1,0);$$
$$D(S_4) = \left(\frac{0.2}{0.1}, \frac{0.3}{0.3}, \frac{0.0}{0.6}\right) = (2,1,0)$$

Since, $D(S_1) = D(S_2) = D(S_3) = D(S_4) = D(A) = (2,1,0)$.
Hence, A is strictly balanced m-polar fuzzy matrix with m-polar fuzzy rows and columns.

7. Conclusion

It is well known that m-polar fuzzy concept is very important and essential to model a large number of problems that occur in science, engineering, medical science and also on real life. In this paper, we first introduced the concept of fuzzy sets, m-polar fuzzy sets, m-polar fuzzy relation, m-polar fuzzy matrix and their operation based on m-polar fuzzy algebras. Very new kind of m-polar fuzzy matrix with m-polar fuzzy rows and columns has been introduced. Null, equality, identity, complement of m-polar fuzzy matrix with m-polar fuzzy rows and columns. Also, several types of density of m-polar fuzzy matrix with m-polar fuzzy rows and columns are defined. More results can be done about g-m-polar fuzzy matrix with m-polar fuzzy rows and columns, balanced and strictly balanced on m-polar fuzzy matrix with m-polar fuzzy rows and columns.

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