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Generalized Fuzzy Continuous Maps and Generalized Fuzzy Closed Maps in Fuzzy Biclosure Space

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Abstract. The aim of this paper is to introduce the notion of generalized fuzzy continuous maps and generalized fuzzy closed maps in fuzzy biclosure spaces by using the concept of generalized fuzzy closed set briefly a g-fuzzy closed set. In this paper study and investigation of some important properties and characterizations of generalized fuzzy continuous maps and generalized fuzzy closed maps in fuzzy biclosure spaces has also been done.

Keywords: Fuzzy closure space, fuzzy biclosure space, g-fuzzy closed set, g-fuzzy continuous map, g-fuzzy closed map.

AMS Mathematics Subject Classification (2010): 54A40

1. Introduction

In order to deal with fuzziness, the concept of fuzzy sets was introduced by Zadeh [13] in 1965 as an extension of the classical notion of set. The fuzzy set theory provides a natural foundation for building new branches of fuzzy mathematics. Fuzzy mathematics is a kind of mathematical theory that contains wider content than classical theory. Also, it has found numerous applications in different fields such as information technology, knowledge-based systems, computer vision, control systems, risk analysis, linguistics, robotics, pattern recognition, expert systems, military control, artificial intelligence, psychology and economics, etc. The theory of general topology is based on the set operations of union, intersection and complementation. Fuzzy sets were assumed to have set-theoretic behavior almost identical to that of ordinary sets. It is therefore natural to extend the concept of point-set topology to fuzzy sets. The introduction of the notion of a fuzzy set has inspired many mathematicians and researchers in this field to generalize the mathematical concepts and structure into the framework of fuzzy sets. Fuzzy set theory has become important with application in almost all areas of mathematics of which one is the area of topology. Inspired by these observations Chang [4] extended the concepts of point-set topology to fuzzy sets and laid the foundation of the fuzzy topology.

In topology, closure spaces are defined in two ways depending upon the two well – known concepts of closure operators due to Birkhoff [7] and Čech [5]. Here we have considered the closure spaces which were introduced by Eduard Čech [5] known as Čech closure spaces. Closure spaces were introduced by E.Čech [5] and then studied by many authors Chvalina [10] and Slapal [11]. In Čech's approach, the operator satisfies idemponent condition among Kuratowski axioms. This condition need not hold for every set A of. When this condition is also true, the operator becomes the topological closure operator. Thus Čech closure space or simply closure space is a generalization of the concept of topological space. Closure functions that are more general than the topological ones have been studied already by Day [15]. A thorough discussion on closure functions is due to Hammer [19] and more recently by Gnilka [20,21]. Today, the theory of closure space is one of the most popular theory of mathematics which finds many interesting applications in the areas of fuzzy sets, combinatorics, genetics or quantum mechanics.

Fuzzy topological spaces do not constitute a natural boundary for the validity of theorems, but many results can be extended to what are called fuzzy closure spaces. In 1985 fuzzy closure spaces were first studied by Mashhour and Ghanim [2,14] as a generalization of fuzzy topological spaces. Later many researchers and recently Zahan and Nasrin [8] has contributed in this field.

The concept of bitopological spaces was introduced by Kelly [9] in 1963. Kandil [1] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. After that an extensive work on fuzzy bitopological has been carried out by many researchers.

After the introduction of closure space, Chawalit Boonpok [3] introduced the notion of biclosure spaces as an extension of closure space. Such spaces are equipped with two arbitrary closure operators. He extended some of the standard results of closure spaces to biclosure spaces.

In 2011, Tapi and Navalakhe [22-24] studied the concept of biclosure spaces in the framework of fuzzy set theory and introduced the new type of biclosure space called fuzzy biclosure space as an extension of fuzzy closure space.

The generalized closed set is the most common but important and interesting concepts in topological spaces as well as fuzzy topological spaces. The concept of generalized closed set in general topological space was first investigated by Levine [17] in 1970, which has been extensively used as an excellent tool for studying different concepts in the said space. Balachandran et al. [12] introduced the notion of generalized continuous mapping, briefly g-continuous mapping by using g-closed sets. In fuzzy setting, the concept of generalized fuzzy closed set and generalized fuzzy continuous mapping was initiated by Balasubramanian et al. [6] in 1997.

Latter generalized closed sets, generalized continuous mapping and generalized closed mapping in biclosure spaces were studied by Boonpok [3].

In this paper the concept of g-fuzzy continuous maps and g-fuzzy closed maps in fuzzy biclosure spaces has been introduced using g-fuzzy closed set. Various properties and characterization of g-fuzzy continuous maps and g-fuzzy closed maps in fuzzy biclosure spaces has been proved.

2. Preliminaries

Definition 2.1. [22] A function $u : I^X \to I^X$ defined on the family I^X of all fuzzy sets of X is called a fuzzy closure operator on X and the pair (X , u) is called fuzzy closure space, if the following conditions are satisfied

1) $u\,\phi = \phi$
2) $A \le u\ (A)$ for all A $\in I^X$.
3) $u(A \vee B) = u(A) \vee u(B)$ for all $A, B \in I^X$.

Definition 2.2. [22] Let (X,u) and (Y,v) be fuzzy closure spaces. A map $f:(X,u) \to (Y,v)$ is said to be fuzzy continuous if $f(uA) \le vf(A)$ for every fuzzy subset $A \le X$. In other words a map $f:(X,u) \to (Y,v)$ is fuzzy continuous if and only if $uf^{-1}(B) \le f^{-1}v(B)$ for every fuzzy subset $B \le Y$. Clearly, if map $f:(X,u) \to (Y,v)$ is fuzzy continuous, then $f^{-1}(F)$ is a fuzzy closed subset of (X,u) for every fuzzy closed subset F of (Y,v).

Definition 2.3. [22,16,18] A *fuzzy biclosure space* is a triple (X, u_1, u_2) where X is a set and u_1, u_2 are two fuzzy closure operators on X which satisfy the following properties:
(i) $u_1\phi = \phi$ and $u_2\phi = \phi$
(ii) $A \le u_1 A$ and $A \le u_2 A$ for all $A \le I^X$
(iii) $u_1(A \vee B) = u_1 A \vee u_1 B$ and $u_2(A \vee B) = u_2 A \vee u_2 B$ for all $A, B \le I^X$.

Definition 2.4. [22] A subset A of a fuzzy biclosure space (X, u_1, u_2) is called fuzzy closed if $u_1 u_2 A = A$. The complement of fuzzy closed set is called fuzzy open.

Definition 2.5. [22] The product of a family $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ fuzzy closure spaces denoted by $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$ is the fuzzy closure space $\left(\prod_{\alpha\in J} X_\alpha, u\right)$ where $\prod_{\alpha\in J} X_\alpha$ denotes the Cartesian product of fuzzy sets $X_\alpha, \alpha \in J$ and u is the fuzzy closure operator defined by $u\mu = \prod_{\alpha\in J} u_\alpha \pi_\alpha(\mu)$ for each $\mu \le \prod_{\alpha\in J} X_\alpha$.

The following statement is evident.

Proposition 2.6. 22 Let $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ be a family of fuzzy closure spaces and let. Then the projection map $\pi_\beta : \prod_{\alpha \in J}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is fuzzy continuous and fuzzy closed for every $\beta \in J$.

Proof. Obvious.

Proposition 2.7. [22] Let $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ be a family of fuzzy closure spaces and let $\beta \in J$. Then $\eta \le X_\beta$ is a fuzzy closed subset of (X_β, u_β) if and only if $\eta \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$ is a fuzzy closed subset of $\prod_{\alpha \in J}(X_\alpha, u_\alpha)$.

3. Generalized-fuzzy closed sets

In this section, we introduce generalized fuzzy closed (generalized fuzzy open) sets in fuzzy biclosure spaces and study some of their properties in fuzzy biclosure spaces.

Definition 3.1. A fuzzy set μ in fuzzy biclosure space (X, u_1, u_2) is called generalized fuzzy closed briefly g -fuzzy closed set, if $u_1\mu \le \upsilon$ wheneve υ is a fuzzy open set in (X, u_2) with $\mu \le \upsilon$. The complement of a g -fuzzy closed set is called g -fuzzy open.

Clearly, if μ is a fuzzy closed subset of a fuzzy biclosure space (X, u_1, u_2), then μ is g -fuzzy closed.

The intersection of all generalized fuzzy closed sets containing μ is called generalized fuzzy closure of μ and is denoted by $u_1\mu$ or $u_2\mu$.

Theorem 3.2. If μ is a g -fuzzy closed set in (X, u_1, u_2) and $\mu \le \upsilon \le u_1\mu$, then υ is g -fuzzy closed set in (X, u_1, u_2).

Proof: Let η be a g -fuzzy open set such that $\upsilon \le \eta$ then $\mu \le \eta$ and since μ is g -fuzzy closed, $u_1\mu \le \eta$. Now $\upsilon \le u_1\mu \Rightarrow u_1\upsilon \le u_1u_1\mu = u_1\mu \le \eta$. Consequently, υ is g -fuzzy closed.

Proposition 3.3. Let (X, u_1, u_2) be a fuzzy biclosure space. Then $\mu \le X$ is a g -fuzzy open subset of (X, u_1, u_2) if and only if $\gamma \le 1_X - u_1(1_X - \mu)$ for every γ which is fuzzy closed subset of (X, u_2) with $\gamma \le \mu$.

Proof: Assume that μ is g -fuzzy open and let γ be a fuzzy closed subset of (X, u_2) such that $\gamma \le \mu$. Then $1_X - \mu \le 1_X - \gamma$. Since $1_X - \mu$ is g -fuzzy closed and $1_X - \gamma$ is a fuzzy open subset of (X, u_2), $u_1(1_X - \mu) \le 1_X - \gamma$. Therefore, $\gamma \le 1_X - u_1(1_X - \mu)$

Conversely, let η be a fuzzy open subset of (X,u_2) such that $1_X-\mu\le\eta$. Then $1_X-\eta\le\mu$. Since $1_X-\eta$ is a fuzzy closed subset of (X,u_2), $1_X-\eta \le 1_X-u_1(1_X-\mu)$. Consequently, $u_1(1_X-\mu)\le\eta$. Hence, $1_X-\mu$ is g -fuzzy closed and so μ is g -fuzzy open set in (X,u_1,u_2).

Proposition 3.4. Let $\{(X_\alpha,u_\alpha^1,u_\alpha^2):\alpha\in J\}$ be a family of fuzzy biclosure spaces and let $\beta\in J$ Then $\gamma\le X_\beta$ is a g -fuzzy open subset of $(X_\beta,u_\beta^1,u_\beta^2)$ if and only if $\gamma\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha$ is a g-fuzzy open subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha^1,u_\alpha^2)$.

Proof: Let η be a fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha^1)$ such that $\eta<\gamma\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha$. Then $\pi_\beta(\eta)\le\gamma$. Since $\pi_\beta(\eta)$ is a fuzzy closed subset of (X_β,u_β^1), $\pi_\beta(\eta)\le 1_\beta-u_\beta^2(1_\beta-\gamma)$. Therefore,

$$\eta\le\pi_\beta^{-1}\left(1_\beta-u_\beta^2(1_\beta-\gamma)\right)=\prod_{\alpha\in J}X_\alpha-\prod_{\alpha\in J}u_\alpha^2\pi_\alpha\left(\prod_{\alpha\in J}X_\alpha-\gamma\times\prod_{\alpha\in J}X_\alpha\right).$$

By Proposition 3.3, $\gamma\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha$ is a g -fuzzy open subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha^1,u_\alpha^2)$.

Conversely, let η be a fuzzy closed subset of (X_β,u_β^1) such that $\eta\le\gamma$. Then $\eta\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha\le\gamma\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha$. Since $\eta\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha$ is fuzzy closed and $\gamma\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha$ is g -fuzzy open, $\eta\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha\le\prod_{\alpha\in J}X_\alpha-\prod_{\alpha\in J}u_\alpha^2\pi_\alpha\left(\prod_{\alpha\in J}X_\alpha-\ \gamma\times\prod_{\substack{\alpha\in J\\ \alpha\neq\beta}}X_\alpha\right)$ by Proposition 3.3.

Therefore, $\prod_{\alpha\in J}u_\alpha^2\pi_\alpha\left((1_\beta-\gamma)\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha\right)\le\prod_{\alpha\in J}X_\alpha-\eta\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha=(1_\beta-\eta)\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha$.

Consequently, $u_\beta^2(1_\beta-\gamma)\le 1_\beta-\eta$ implies $\eta\le 1_\beta-u_\beta^2(1_\beta-\gamma)$. Hence, γ is a g -fuzzy open subset of $(X_\beta,u_\beta^1,u_\beta^2)$.

Proposition 3.5. Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in J\}$ be a family of fuzzy biclosure spaces and let $\beta \in J$. Then $\eta \le X_\beta$ is a g-fuzzy closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$ if and only if $\eta \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$ is a g-fuzzy closed subset of $\prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$.

Proof: Let η be a g-fuzzy closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$. Then $1_\beta - \eta$ is a g-fuzzy open subset of $(X_\beta, u_\beta^1, u_\beta^2)$. By Proposition 3.4,
$(1_\beta - \eta) \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha = \prod_{\alpha \in J} X_\alpha - \eta \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$ is a g-fuzzy open subset of $\prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$.
Hence, $\eta \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$ is a g-fuzzy closed subset of $\prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$.

Conversely, let γ be a fuzzy open subset of (X_β, u_β^1) such that $\eta \le \gamma$. Then $\eta \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha \le \gamma \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$. Since $\eta \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$ is a g-fuzzy closed and $\gamma \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$ is fuzzy open in $\prod_{\alpha \in J}(X_\alpha, u_\alpha^1)$, $\prod_{\alpha \in J} u_\alpha^2 \pi_\alpha \left(\eta \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\beta \right) \le \gamma \times \prod_{\alpha \in J} X_\alpha$. Consequently, $u_\beta^2 \eta \le \gamma$.
Hence, η is g-fuzzy closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$.

Proposition 3.6. Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in J\}$ be a family of fuzzy biclosure spaces. For each $\beta \in J$, let $\pi_\beta : \prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2) \to (X_\beta, u_\beta^1, u_\beta^2)$ be the projection map. Then

(i) If η is a g-fuzzy closed subset of $\prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$, then $\pi_\beta(\eta)$ is a g-fuzzy closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$.

(ii) If η is a g-fuzzy closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$, then $\pi_\beta^{-1}(\eta)$ is a g-fuzzy closed subset of $\prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$.

Proof: **(i)** Let η be g-fuzzy closed subset of $\prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$ and let γ be a fuzzy open subset of (X_β, u_β^1) such that $\pi_\beta(\eta) \le \gamma$. Then $\eta \le \pi_\beta^{-1}(\gamma) = \gamma \times \prod_{\alpha \in J} X_\alpha$. Since $\gamma \times \prod_{\substack{\alpha \in J \\ \alpha \ne \beta}} X_\alpha$ is a fuzzy open subset of $\prod_{\alpha \in J}(X_\alpha, u_\alpha^1)$, $\prod_{\alpha \in J} u_\alpha^2 \pi_\alpha(\eta) \le \gamma \times \prod_{\substack{\alpha \in J \\ \alpha \ne \beta}} X_\alpha$.

Consequently, $u_\beta^2\pi_\beta(\eta)\le\gamma$. Hence, $\pi_\beta(\eta)$ is a g-fuzzy closed subset of $(X_\beta,u_\beta^1,u_\beta^2)$.

(ii) Let η be a g-fuzzy closed subset of $(X_\beta,u_\beta^1,u_\beta^2)$. Then $\pi_\beta^{-1}(\eta)=\eta\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha$. By Proposition 3.5, $\eta\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_\alpha$ is a g-fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha^1,u_\alpha^2)$. Hence $\pi_\beta^{-1}(\eta)$ is a g-fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha^1,u_\alpha^2)$.

4. Generalized fuzzy continuous maps

In this section, we introduce the concept of generalized fuzzy continuous maps by using g-fuzzy closed sets in fuzzy biclosure spaces. We also study and investigate some important characterizations of generalized fuzzy continuous maps in fuzzy biclosure spaces.

Definition 4.1. Let (X,u_1,u_2) and (Y,v_1,v_2) be fuzzy biclosure spaces. A map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is said to be g-fuzzy continuous if $f^{-1}(\mu)$ is a g-fuzzy closed subset of (X,u_1,u_2) for every fuzzy closed subset μ of (Y,v_1,v_2).

Clearly, a map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is g-fuzzy continuous if and only if $f^{-1}(\upsilon)$ is a g-fuzzy open subset of (X,u_1,u_2) for every fuzzy open subset υ of (Y,v_1,v_2).

Remark 4.2. The following implications hold for any map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$:
f is fuzzy continuous $\Rightarrow$ f is g-fuzzy continuous.

Definition 4.3. Let (X,u_1,u_2) and (Y,v_1,v_2) be fuzzy biclosure spaces. A map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is called g-fuzzy irresolute if $f^{-1}(\upsilon)$ is a g-fuzzy closed set in (X,u_1,u_2) for every g-fuzzy closed set υ in (Y,v_1,v_2).

Clearly, a map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is g-fuzzy irresolute if and only if $f^{-1}(\upsilon)$ is a g-fuzzy open set in (X,u_1,u_2) for every g-fuzzy open set υ in (Y,v_1,v_2).

The following statement is obvious.

Proposition 4.4. Let (X,u_1,u_2) and (Y,v_1,v_2) be fuzzy biclosure spaces. If $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is g-fuzzy irresolute, then f is g- fuzzy continuous.

Proposition 4.5. Let (X,u_1,u_2) (Y,v_1,v_2) and (Z,w_1,w_2) be fuzzy biclosure spaces. Let $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ and $h:(Y,v_1,v_2)\to(Z,w_1,w_2)$ be maps. Then

(i) If h is fuzzy continuous and f is g-fuzzy continuous, then $h\circ f$ is g-fuzzy continuous.

(ii) If f and h are g-fuzzy irresolute, then $h\circ f$ is g-fuzzy irresolute.

(iii) if h is g-fuzzy continuous and f is g-fuzzy irresolute, then $(h\circ f)^{-1}(\eta)$ is g-fuzzy continuous.

Proof: Let η be a fuzzy closed subset of (Z,w_1,w_2). Then η is a fuzzy closed subset of (Z,w_1) and (Z,w_2) respectively. Since the map h is fuzzy continuous, $h^{-1}(\eta)$ is a fuzzy closed subset of (Y,v_1) and (Y,v_2) respectively. Consequently, $h^{-1}(\eta)$ is a fuzzy closed subset of (Y,v_1,v_2). Since the map f is g-fuzzy continuous, $f^{-1}\left(h^{-1}(\eta)\right)$ is a g-fuzzy closed subset of (X,u_1,u_2). Therefore $(h\circ f)^{-1}(\eta)$ is a g-fuzzy closed subset of (X,u_1,u_2). Hence, the map $h\circ f$ is g-fuzzy continuous.
The proofs of (ii) and (iii) are similar.

Proposition 4.6. Let $\left\{\left(X_\alpha,u_\alpha^1,u_\alpha^2:\alpha\in J\right)\right\}$ be a family of fuzzy biclosure spaces. Then for each $\beta\in J$, the projection map $\pi_\beta:\prod_{\alpha\in J}\left(X_\alpha,u_\alpha^1,u_\alpha^2\right)\to\left(X_\beta,u_\beta^1,u_\beta^2\right)$ is g-fuzzy continuous.
Proof: It is obvious.

Proposition 4.7. Let (X,u_1,u_2) be fuzzy biclosure space and let $\left\{\left(Y_\alpha,v_\alpha^1,v_\alpha^2:\alpha\in J\right)\right\}$ be a family of fuzzy biclosure spaces. Let $f:X\to\prod_{\alpha\in J}Y_\alpha$ be a map. If $f:(X,u_1,u_2)\to\prod_{\alpha\in J}\left(Y_\alpha,v_\alpha^1,v_\alpha^2\right)$ is g-fuzzy continuous, then $\pi_\alpha\circ f:(X,u_1,u_2)\to\left(Y_\alpha,v_\alpha^1,v_\alpha^2\right)$ is g-fuzzy continuous for each $\alpha\in J$.
Proof: Let f be g-fuzzy continuous. Since π_α is fuzzy continuous for each $\alpha\in J$, therefore it follows that $\pi_\alpha\circ f$ is g-fuzzy continuous for each $\alpha\in J$.

Proposition 4.8. Let $\left\{\left(X_{\alpha},u_{\alpha}^{1},u_{\alpha}^{2}:\alpha\in J\right)\right\}$ and $\left\{\left(Y_{\alpha},v_{\alpha}^{1},v_{\alpha}^{2}:\alpha\in J\right)\right\}$ be families of fuzzy biclosure spaces. For each $\alpha\in J$, let $f_{\alpha}:X_{\alpha}\to Y_{\alpha}$ be a map and $f:\prod_{\alpha\in J}X_{\alpha}\to\prod_{\alpha\in J}Y_{\alpha}$ be defined by $f\left((x_{\alpha})_{\alpha\in I}\right)=\left(f_{\alpha}(x_{\alpha})\right)_{\alpha\in I}$. If $f:\prod_{\alpha\in J}\left(X_{\alpha},u_{\alpha}^{1},u_{\alpha}^{2}\right)\to\prod_{\alpha\in J}\left(Y_{\alpha},v_{\alpha}^{1},v_{\alpha}^{2}\right)$ is g-fuzzy continuous, then $f_{\alpha}:\left(X_{\alpha},u_{\alpha}^{1},u_{\alpha}^{2}\right)\to\left(Y_{\alpha},v_{\alpha}^{1},v_{\alpha}^{2}\right)$ is g-fuzzy continuous for each $\alpha\in J$.

Proof: Let η be a fuzzy closed subset of $\left(Y_{\beta},v_{\beta}^{1},v_{\beta}^{2}\right)$. Then $\eta\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}Y_{\alpha}$ is a fuzzy closed subset of $\prod_{\alpha\in J}\left(Y_{\alpha},v_{\alpha}^{1},v_{\alpha}^{2}\right)$. Since the map f is g-fuzzy continuous $f^{-1}\left(\eta\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}Y_{\alpha}\right)=f_{\beta}^{-1}(\eta)\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}}X_{\alpha}$ is a g-fuzzy closed subset of $\prod_{\alpha\in J}\left(X_{\alpha},u_{\alpha}^{1},u_{\alpha}^{2}\right)$. By Proposition 4.7, $f_{\beta}^{-1}(\eta)$ is a g-fuzzy closed subset of $\left(X_{\beta},u_{\beta}^{1},u_{\beta}^{2}\right)$. Hence, the map f_{β} is g-fuzzy continuous for each $\beta\in J$.

5. Generalized fuzzy closed maps

In this section, we introduce the notion of g eneralized fuzzy closed map and g eneralized fuzzy open map in fuzzy biclosure spaces and study some of their properties.

Definition 4.1. Let $\left(X,u_1,u_2\right)$ and $\left(Y,v_1,v_2\right)$ be fuzzy biclosure spaces. A map $f:\left(X,u_1,u_2\right)\to\left(Y,v_1,v_2\right)$ is called g-fuzzy closed (resp. g-fuzzy open) if $f(\mu)$ is a g-fuzzy closed (resp. g-fuzzy open) set in $\left(Y,v_1,v_2\right)$ whenever μ is a fuzzy closed (resp. fuzzy open) set in $\left(X,u_1,u_2\right)$.

Every fuzzy closed map is g-fuzzy closed.

The following statement is evident.

Proposition 4.2. Let $\left(X,u_1,u_2\right)$, $\left(Y,v_1,v_2\right)$ and $\left(Z,w_1,w_2\right)$ be fuzzy biclosure spaces. If the map $f:\left(X,u_1,u_2\right)\to\left(Y,v_1,v_2\right)$ is fuzzy closed(resp. fuzzy open) and the map $h:\left(Y,v_1,v_2\right)\to\left(Z,w_1,w_2\right)$ is g-fuzzy closed(resp. g-fuzzy open), then the map $h\circ f:\left(X,u_1,u_2\right)\to\left(Z,w_1,w_2\right)$ is g-fuzzy closed (resp. g-fuzzy open).

Proposition 4.3. Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be fuzzy biclosure spaces. If the map $h\circ f:(X,u_1,u_2)\to(Z,w_1,w_2)$ is g-fuzzy closed and the map $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is surjective and fuzzy continuous, then the map $h:(Y,v_1,v_2)\to(Z,w_1,w_2)$ is g-fuzzy closed.

Proof: Let η be a fuzzy closed subset of (Y,v_1,v_2). Then η is a fuzzy closed subset of (Y,v_1) and (Y,v_2) respectively. Since f is fuzzy continuous, $f^{-1}(\eta)$ is a fuzzy closed subset of (X,u_1) and (X,u_2) respectively. Consequently, $f^{-1}(\eta)$ is a fuzzy closed subset of (X,u_1,u_2). Since $h\circ f$ is g-fuzzy closed and f is surjective, $h\circ f\left(f^{-1}(\eta)\right)=h(\eta)$ is a g-fuzzy closed subset of (Z,w_1,w_2). Therefore, the map h is g-fuzzy closed.

Proposition 4.4. Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be fuzzy biclosure spaces. If $h\circ f:(X,u_1,u_2)\to(Z,w_1,w_2)$ is fuzzy closed and $h:(Y,v_1,v_2)\to(Z,w_1,w_2)$ is injective and g-fuzzy continuous, then $f:(X,u_1,u_2)\to(Y,v_1,v_2)$ is g-fuzzy closed.

Proof: Let η is a fuzzy closed subset of (X,u_1,u_2). Then η is a fuzzy closed subset of (X,u_1) and (X,u_2) respectively. Since $h\circ f$ is fuzzy closed, $h\circ f(\eta)$ is a fuzzy closed subset of (Z,w_1) and (Z,w_2) respectively. Consequently, $h\circ f\ (\eta)$ is a fuzzy closed subset of (Z,w_1,w_2). Since h is g-fuzzy continuous and injective, $h^{-1}\left(h\circ f(\eta)\right)=f(\eta)$ is a g-fuzzy closed subset of (Y,v_1,v_2). Therefore, the map f is g-fuzzy closed.

Proposition 4.5. Let $\left\{\left(X_\alpha,u_\alpha^1,u_\alpha^2:\alpha\in J\right)\right\}$ be a family of fuzzy biclosure spaces. Then for each $\beta\in J$, the projection map $\pi_\beta:\prod_{\substack{\alpha\in J\\ \alpha\neq\beta}}\left(X_\alpha,u_\alpha^1,u_\alpha^2\right)\to\left(X_\beta,u_\beta^1,u_\beta^2\right)$ is g-fuzzy closed.

Proof. Let η be a fuzzy closed subset of $\prod_{\alpha\in J}\left(X_\alpha,u_\alpha^1,u_\alpha^2\right)$ and let γ be a g-fuzzy open subset of $\left(X_\beta,u_\beta^1\right)$ such that $\pi_\beta(\eta)\le\gamma$. Then $\eta\le\pi_\beta^{-1}(\gamma)=\gamma\times\prod_{\substack{\alpha\in J\\ \alpha\neq\beta}}Y_\alpha$.

Since $\gamma\times\prod_{\substack{\alpha\in J\\ \alpha\neq\beta}} X_\alpha$ is a g-fuzzy open subset of $\prod_{\alpha\in J}\left(X_\alpha, u_\alpha^1\right)$ and η is a fuzzy closed subset of $\prod_{\alpha\in J}\left(X_\alpha, u_\alpha^1, u_\alpha^2\right)$, $\prod_{\alpha\in J} u_\alpha^2\pi_\alpha(\eta) \le \gamma\times\prod_{\substack{\alpha\in J\\ \alpha\neq\beta}} X_\alpha$. Consequently, $u_\beta^2\pi_\beta(\eta)\le\gamma$. Therefore, $\pi_\beta(\eta)$ is a g-fuzzy closed subset of $\left(X_\beta, u_\beta^1, u_\beta^2\right)$. Hence, the map π_β is g-fuzzy closed.

Proposition 4.6. Let (X, u_1, u_2) be a fuzzy biclosure space, $\left\{\left(Y_\alpha, v_\alpha^1, v_\alpha^2 : \alpha\in J\right)\right\}$ be a family of fuzzy biclosure spaces and $f: X\to\prod_{\alpha\in J} Y_\alpha$ be a map. Then

$$f:(X, u_1, u_2)\to\prod_{\alpha\in J}\left(Y_\alpha, v_\alpha^1, v_\alpha^2\right)$$

is g-fuzzy closed if and only if $\pi_\alpha\circ f:(X, u_1, u_2)\to\left(Y_\alpha, v_\alpha^1, v_\alpha^2\right)$ is g-fuzzy closed for each $\alpha\in J$.

Proof: Let $\beta\in J$. Let η be a fuzzy closed subset of (X, u_1, u_2) and let γ. Then $f(\eta)\le\pi_\beta^{-1}(\gamma)=\gamma\times\prod_{\substack{\alpha\in J\\ \alpha\neq\beta}} Y_\alpha$. Since $\gamma\times\prod_{\substack{\alpha\in J\\ \alpha\neq\beta}} Y_\alpha$ is a g-fuzzy open subset of $\prod_{\alpha\in J}\left(Y_\alpha, v_\alpha^1\right)$ and $f(\eta)$ is a fuzzy closed subset of $\prod_{\alpha\in J}\left(Y_\alpha, v_\alpha^1, v_\alpha^2\right)$, $\prod_{\alpha\in J} v_\alpha^2\pi_\alpha f(\eta)\le\gamma\times\prod_{\substack{\alpha\in J\\ \alpha\neq\beta}} Y_\alpha$. Consequently, $v_\beta^2\pi_\beta f(\eta)\le\gamma$. Therefore, $\pi_\beta f(\eta)$ is a g-fuzzy closed subset of $\left(Y_\beta, v_\beta^1, v_\beta^2\right)$. Hence, the map $\pi_\beta\circ f$ is a g-fuzzy closed map.

Conversely, let the map $\pi_\alpha\circ f$ be a g-fuzzy closed for each $\alpha\in J$. Suppose that the map f is not g-fuzzy closed. Then there exists a fuzzy closed subset η of (X, u_1, u_2) such that $f(\eta)$ is not g-fuzzy closed subset of $\prod_{\alpha\in J}\left(Y_\alpha, v_\alpha^1, v_\alpha^2\right)$. Therefore, there exists $\beta\in J$ such that $\pi_\beta\left(f(\eta)\right)$ is not g-fuzzy closed subset of $\left(Y_\beta, v_\beta^1, v_\beta^2\right)$. But the map $\pi_\beta\circ f$ is g-fuzzy closed, hence $\pi_\beta\left(f(\eta)\right)$ is a g-fuzzy closed subset of $\left(Y_\beta, v_\beta^1, v_\beta^2\right)$. This is a contradiction. Therefore, the map f is g-fuzzy closed.

Proposition 4.7. Let $\left\{\left(X_\alpha, u_\alpha^1, u_\alpha^2 : \alpha \in J\right)\right\}$ and $\left\{\left(Y_\alpha, v_\alpha^1, v_\alpha^2 : \alpha \in J\right)\right\}$ be families of fuzzy biclosure spaces. For each $\alpha \in J$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a surjection and let $f : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha$ be defined by $f\left(\left(x_\alpha\right)\right)_{\alpha \in I} = \left(f_\alpha\left(x_\alpha\right)\right)_{\alpha \in I}$.

Then $f : \prod_{\alpha \in J}\left(X_\alpha, u_\alpha^1, u_\alpha^2\right) \to \prod_{\alpha \in J}\left(Y_\alpha, v_\alpha^1, v_\alpha^2\right)$ is g-fuzzy closed if and only if

$f_\alpha : \prod_{\alpha \in J}\left(X_\alpha, u_\alpha^1, u_\alpha^2\right) \to \prod_{\alpha \in J}\left(Y_\alpha, v_\alpha^1, v_\alpha^2\right)$ is g-fuzzy closed for each $\alpha \in J$.

Proof: Let $\beta \in J$. Let η be a fuzzy closed subset of $\left(X_\beta, u_\beta^1, u_\beta^2\right)$. Then $\eta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is a fuzzy closed subset of $\prod_{\alpha \in J}\left(X_\alpha, u_\alpha^1, u_\alpha^2\right)$. Since the map f is g-fuzzy closed,

$f\left(\eta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha\right)$ is a g-fuzzy closed subset of $\prod_{\alpha \in J}\left(Y_\alpha, v_\alpha^1, v_\alpha^2\right)$.

But $f\left(\eta \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha\right) = f_\beta(\eta) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} Y_\alpha$, hence $f_\beta(\eta) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} Y_\alpha$ is a g-fuzzy closed subset of $\prod_{\alpha \in J}\left(Y_\alpha, v_\alpha^1, v_\alpha^2\right)$. By Proposition 4.6, $f_\beta(\eta)$ is a g-fuzzy closed subset of $\left(Y_\beta, v_\beta^1, v_\beta^2\right)$. Hence, the map f_β is g-fuzzy closed for each $\beta \in J$.

Conversely, let f_β be g-fuzzy closed for each $\beta \in J$. Suppose that f is not g-fuzzy closed. Then there exists a fuzzy closed subset η of $\prod_{\alpha \in J}\left(X_\alpha, u_\alpha^1, u_\alpha^2\right)$ such that $f(\eta)$ is not a g-fuzzy closed subset of $\prod_{\alpha \in J}\left(Y_\alpha, v_\alpha^1, v_\alpha^2\right)$. Therefore, there exists $\beta \in J$ such that $f_\beta\left(\pi_\beta(\eta)\right)$ is not a g-fuzzy closed subset of $\left(Y_\beta, v_\beta^1, v_\beta^2\right)$. But $\pi_\beta(\eta)$ is a fuzzy closed subset of $\left(X_\beta, u_\beta^1, u_\beta^2\right)$ and f_β is g-fuzzy closed, $f_\beta\left(\pi_\beta(\eta)\right)$ is a g-fuzzy closed subset of $\left(Y_\beta, v_\beta^1, v_\beta^2\right)$. This is a contradiction. Therefore, the map f is g-fuzzy closed.

5. Conclusion

In this paper, using the concept of g-fuzzy closed set the notion of generalized fuzzy continuous maps and generalized fuzzy closed maps in fuzzy biclosure spaces has been introduced. Apart from the introduction of concept of generalized fuzzy continuous maps and generalized fuzzy closed maps in fuzzy biclosure spaces, several important properties, results and characterization of g-fuzzy continuous maps and g-fuzzy closed maps in fuzzy biclosure spaces has been proved.

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