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On the Generalized Extended Wright Function

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Abstract. The main object of this paper is to introduce a new generalized extended Wright function, for the new generalized extended Wright function properties such as integral representations, Euler's beta, Mellin, inverse Mellin, Jacobi, Gegenbaur, and Legendre transforms are proposed and investigated. Moreover, some derivative formulas, classical and extended Riemann-Liouville fractional derivative and integral are also proposed.

Keywords: Confluent hypergeometric function, Euler's gamma function, Euler's beta function, Wright function, Mellin transform, Jacobi transform.

AMS Mathematics Subject Classification (2010): 33E12, 33C10, 33B15, 33C15, 33E50

1. Introduction

In this research paper, $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^{-}\mathbb{Z}^{+}$, $\mathbb{R}, \mathbb{R}^{-}, \mathbb{R}^{+}$ denotes the sets of natural numbers, integers, negative integers, positive integers, real numbers, negative real numbers and positive real numbers, respectively. Also $\mathbb{N}_0 = \{0\} \cup \mathbb{N}, \mathbb{Z}_0^{-} = \{0\} \cup \mathbb{Z}^{-}, \mathbb{R}_0^{-} = \{0\} \cup \mathbb{R}^{-}$ and $\mathbb{R}_0^{+} = \{0\} \cup \mathbb{R}^{+}$.

The classical Wright function was first introduced and investigated by English mathematician E.M. Wright (see, [231]-[27]) and is defined as

$$W_{\mu,\Lambda_3}(z) = \sum_{s=0}^{\infty} \frac{z^s}{s!} \frac{1}{\Gamma(\mu s + \Lambda_3)}, \qquad (\mu > -1, \Lambda_3 \in \square), \tag{1}$$

where $W_{\mu,\Lambda_3}(z)$ is also called the generalized Bessel or Bessel-Maitland function with order $(1+\mu)^{-1}$ and it has two auxiliary functions given as

$$M_\mu(z) = W_{-\mu,1-\mu}(-z) = \sum_{s=0}^{\infty} \frac{z^s}{s!} \frac{(-1)^s}{\Gamma(1-\mu s-\mu)}, \qquad (0 < \mu < 1), \tag{2}$$

and

$$F_{\mu}(z)=W_{-\mu,0}(-z)=\sum_{s=0}^{\infty}\frac{z^{s}}{s!}\frac{(-1)^{s}}{\Gamma(-\mu s)}, \qquad (0<\mu<1),$$

the auxiliary function in equation (2) is called Mainardi function (refer to, [11], [15]). Sharma and Devi [17] presented the new extended Wright generalized hypergeometric function as follow:

$${}_{m+1}\psi_{n+1}\left[\begin{matrix}(a_i,\alpha_i)_{1,m} & (\gamma,1)\\ (b_j,\beta_j)_{1,n} & (c,1)\end{matrix}\middle|(z;\wp)\right]=\frac{1}{\Gamma(c-\gamma)}\sum_{s=0}^{\infty}\frac{z^{s}}{s!}\frac{\prod_{i=0}^{m}\Gamma(s\alpha_i+a_i)}{\prod_{j=0}^{n}\Gamma(s\beta_j+b_j)}B_{\wp}(\gamma+s,c-\gamma),$$

$$(\operatorname{Re}(\wp)>0;\operatorname{Re}(c)>\operatorname{Re}(\gamma)>0).$$

Here $B_{\wp}(\widehat{\Im_1},\widehat{\Im_2})$ is the extended beta function introduced by Chaudhry et al., [7] defined by

$$B_{\wp}(\widehat{\Im}_1,\widehat{\Im}_2)=\int_0^1 t^{\widehat{\Im}_1-1}(1-t)^{\widehat{\Im}_2-1}\exp\left(-\frac{\wp}{t(1-t)}\right)dt,$$

$$\left(\operatorname{Re}(\wp)>0,\min\left\{\operatorname{Re}(\widehat{\Im}_1),\operatorname{Re}(\widehat{\Im}_2)\right\}>0\right).$$

El-Shahed and Salem [8] established the extension of classical Wright function in (1) as follow:

$$W_{\mu,\Lambda_3}^{\alpha,\beta}(z)=\sum_{s=0}^{\infty}\frac{z^{s}}{s!}\frac{1}{\Gamma(\mu s+\Lambda_3)}\frac{(\alpha)_s}{(\beta)_s}, \tag{3}$$

$$\left(\alpha,\beta,\mu,\Lambda_3\in\mathbb{C};\mu>-1,\beta\neq 0,-1,-2,\ldots;z\in\mathbb{C}\text{ and }|z|<1\text{ with }\mu\neq -1\right)$$

They also [8] established relationships with some known special functions such as Fox-Wright, H-Fox, Fox-Wright, hypergeometric, Meijer G-, Mittag-Leffler, incomplete gamma and error functions.

With generalized auxiliary functions for order $(0<\mu<1)$ and all complex number except $z\neq 0$ [8]:

$$M_{\mu}^{\alpha,\beta}(z)=W_{-\mu,1-\mu}^{\alpha,\beta}(-z)=\sum_{s=0}^{\infty}\frac{z^{s}}{s!}\frac{(-1)^{s}}{\Gamma(1-\mu s-\mu)}\frac{(\alpha)_s}{(\beta)_s}. \tag{4}$$

And

$$F_{\mu}^{\alpha,\beta}(z)=W_{-\mu,0}^{\alpha,\beta}(-z)=\sum_{s=0}^{\infty}\frac{z^{s}}{s!}\frac{(-1)^{s}}{\Gamma(-\mu s)}\frac{(\alpha)_s}{(\beta)_s}. \tag{5}$$

Khan et al., [12] proposed generalized Wright function using the generalized beta function

$$W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_1;\eta,\lambda}(z;\wp)=\sum_{s=0}^{\infty}\frac{z^s}{s!}\frac{B_{\wp}^{(\eta,\lambda)}(\alpha+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}\frac{1}{\Gamma(\mu s+\Lambda_3)}\frac{(\Lambda_2)_s}{(\beta)_s},$$

$$\left(\mu>-1,\beta\neq 0,-1,-2,\ldots;\mu,\Lambda_3,\alpha,\beta,\Lambda_1,\Lambda_2\in\mathbb{C}\ \text{ with } z\in\mathbb{C}\ \text{ and } |z|<1\right).$$

With the following two Wright type auxiliary functions:

$$M_{\mu}^{c,d,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp)=W_{-\mu,1-\mu}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(-z;\wp)$$

$$=\sum_{s=0}^{\infty}\frac{(-1)^s z^s}{s!}\frac{(\Lambda_2)_s}{(\beta)_s}\frac{(\alpha)_s}{\Gamma(1-\mu s-\mu)(\Lambda_1)_s}.$$

And

$$F_{\mu}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp)=W_{-\mu,0}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(-z;\wp)$$

$$=\sum_{s=0}^{\infty}\frac{(-1)^s z^s}{s!}\frac{(\Lambda_2)_s}{(\beta)_s}\frac{(\alpha)_s}{\Gamma(-\mu s)(\Lambda_1)_s},$$

where $B_{\wp}^{\eta,\lambda}(\widehat{\mathfrak{I}_1},\widehat{\mathfrak{I}_2})$ is extended beta function defined by [14]

$$B_{\wp}^{(\eta,\lambda)}(\mathfrak{I}_1,\mathfrak{I}_2)=\int_0^1 t^{\mathfrak{I}_1-1}(1-t)^{\mathfrak{I}_2-1}\,{}_1F_1\left(\eta;\lambda;-\frac{\wp}{t(1-t)}\right)dt,$$

$$\left(\mathrm{Re}(\wp)>0,\min\left\{\mathrm{Re}(\mathfrak{I}_1),\mathrm{Re}(\mathfrak{I}_2),\mathrm{Re}(\eta),\mathrm{Re}(\lambda)\right\}>0\right),$$

and ${}_1F_1(;)$ is the well-known classical Kumar confluent hypergeometric function.

The Jacobi transform of a given function $f(z)$ is defined as (see, [6])

$$J^{\eta,\lambda}\{f(z);\aleph\}=\int_{-1}^{1}f(z)(1+z)^{\lambda}(1-z)^{\eta}P_{\aleph}^{\eta,\lambda}(z)\,dz, \tag{6}$$

$$W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta)=\frac{2}{B(\Lambda_1-\alpha,\alpha)}\int_0^{\frac{\pi}{2}}\sin^{2\alpha-1}\theta\cos^{2\Lambda_1-2\alpha-1}\,{}_1F_1\left(\eta;\lambda;-\wp\,\mathrm{cosec}^{2\ell}\theta\sec^{2\zeta}\theta\right)$$

$$\times\ W_{\mu,\tau}^{q,\beta}\left(z\sin^2\theta\right)dt.$$

$$\left(\mu,\Lambda_3,\alpha,\beta,\Lambda_1,\Lambda_2\in\mathbb{C},\mathrm{Re}(\Lambda_1)>\mathrm{Re}(\alpha),\mathrm{Re}(\mu)>0;\Lambda_1\in\mathbb{C}_0^-,\Lambda_3\in\mathbb{C}\setminus\mathbb{C}_0^-\right).$$

2. Integral transforms of the new generalized extended Wright function

This section**,** integral transforms: Mellin, inverse Mellin, Jacobi, Gegenbaur and Legendre transforms of the new generalized extended Wright function are discussed

Theorem 4. The Euler's beta transform holds

$$B\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(xz^{\mu};\wp;\ell,\zeta\right):\Lambda_3,\Omega_2\right\}=\Gamma(\Omega_2)W_{\mu,\Lambda_3+\Omega_2}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(x;\wp;\ell,\zeta).$$

Proof: Taking Euler's beta transform (see, [19]) and using equation (10), we obtain

$$B\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(xz^{\mu};\wp;\ell,\zeta\right):\Lambda_3,\Omega_2\right\}=\int_0^1 z^{\Lambda_3-1}(1-z)^{\Omega_2-1}W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(xz^{\mu};\wp;\ell,\zeta\right)dz.$$

On simplifying, we have

$$B\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(xz^{\mu};\wp;\ell,\zeta\right):\Lambda_3,\Omega_2\right\}=\int_0^1 z^{\Lambda_3-1}(1-z)^{\Omega_2-1}\sum_{s=0}^{\infty}\frac{B_{\wp}^{(\eta,\lambda;\ell,\zeta)}(\alpha+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}$$
$$\times\frac{x^s}{s!\ \Gamma(\mu s+\Lambda_3)}\frac{(\Lambda_2)_s}{(\beta)_s}z^{\mu s}dz$$

Swapping the order of integration and summation, yields

$$B\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(xz^{\mu};\wp;\ell,\zeta\right):\Lambda_3,\Omega_2\right\}=\sum_{s=0}^{\infty}\frac{B_{\wp}^{(\eta,\lambda;\ell,\zeta)}(\Lambda_1+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}\frac{x^s}{s!\ \Gamma(\mu s+\Lambda_3)}\frac{(\Lambda_2)_s}{(\beta)_s}$$
$$\times\int_0^1 z^{\Lambda_3+\mu s-1}(1-z)^{\Omega_2-1}dz.$$

Using classical beta function [15], gives

$$B\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(xz^{\mu};\wp;\ell,\zeta\right):\Lambda_3,\Omega_2\right\}=\sum_{s=0}^{\infty}\frac{B_{\wp}^{(\eta,\lambda;\ell,\zeta)}(\alpha+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}\frac{x^s}{s!\ \Gamma(\mu s+\Lambda_3)}\frac{(\Lambda_2)_s}{(\beta)_s}$$
$$\times B(\Omega_2,\Lambda_3+\mu s).$$

On applying equation (10) and algebraic simplifications, we obtain

$$B\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(xz^{\mu};\wp;\ell,\zeta\right):\Lambda_3,\Omega_2\right\}=\Gamma(\Omega_2)W_{\mu,\Lambda_3+\Omega_2}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(x;\wp;\ell,\zeta\right).$$

Corollary 5. The Euler's beta transform holds

$$B\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(x(1-z)^{\mu};\wp;\ell,\zeta\right):\Omega_1,\tau\right\}=\Gamma(\Omega_1)W_{\mu,\Lambda_3+\Omega_1}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(x;\wp;\ell,\zeta\right).$$

Corollary 6. The generalized Euler's beta transform also holds

$$\int_t^x (x-z)^{\Omega_1-1}(z-t)^{\tau-1}W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(z-t)^{\mu};\wp;\ell,\zeta\right)=\Gamma(\Omega_1)(x-t)^{\Lambda_3+\Omega_1-1}$$
$$\times W_{\mu,\Lambda_3+\Omega_1}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(x-t)^{\mu};\wp;\ell,\zeta\right).$$

Theorem 7. The Mellin transform of the generalized extended Wright function is as follows:

$$M\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(z;\wp;\ell,\zeta\right);\Omega_0\right\}=\frac{\Gamma^{(\eta,\lambda)}(\Omega_0)\Gamma(\Lambda_1-\alpha+\ell\Omega_0)\Gamma(\beta)\mathrm{T}(\Lambda_1)}{\Gamma(\alpha)\Gamma(\Lambda_1-\alpha)\Gamma(\Lambda_2)}$$

$$\times {}_2\psi_3\left[\begin{matrix}(\mu,1),\,(\alpha+\ell\Omega_0+1);\\ (\beta,1),(\Lambda_3,\mu),(\Lambda_1+\ell\Omega_0+\zeta\,\Omega_0,1);\end{matrix}\; z\right].(14)$$

where ${}_q\psi_p(;)$ is the generalized Wright function.

Proof: Applying Mellin transform (see, [5]) to equation (10), we have

$$\mathrm{M}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta);\Omega_0\right\}=\int_0^\infty \wp^{\Omega_0-1}W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\sigma;\ell,\zeta)\,d\wp.$$

On simplifying, gives

$$\mathrm{M}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta);\Omega_0\right\}=\frac{1}{B(\Lambda_1-\alpha,\alpha)}\int_0^\infty \wp^{\Omega_0-1}\left\{\int_0^1 t^{\alpha-1}(1-t)^{\Lambda_1-\alpha-1}\,{}_1F_1\left(\eta;\lambda;-\frac{\wp}{t^{\ell}(1-t)^{\zeta}}\right)\right.$$

$$\left.\times\; W_{\mu,\Lambda_3}^{\Lambda_2,\beta}(zt)\,dt\right\}d\wp.$$

On switching the order of integrations, we obtain

$$\mathrm{M}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta);\Omega_0\right\}=\frac{1}{B(\Lambda_1-\alpha,\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\Lambda_1-\alpha-1}W_{\mu,\Lambda_3}^{\Lambda_2,\beta}(zt)$$

$$\times\left\{\int_0^\infty \wp^{\Omega_0-1}\,{}_1F_1\left(\eta;\lambda;-\frac{\wp}{t^{\ell}(1-t)^{\zeta}}\right)d\wp\right\}dt.$$

Putting $\wp=\varpi t^{\ell}(1-t)^{\xi}$, yields

$$\mathrm{M}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\sigma;\ell,\zeta);\Omega_0\right\}=\frac{1}{B(\Lambda_1-\alpha,\alpha)}\left\{\int_0^1 t^{\alpha+\ell\Omega_0-1}(1-t)^{\Lambda_1+\zeta\,\Omega_0-\alpha-1}W_{\mu,\Lambda_3}^{\Lambda_2,\beta}(zt)\,dt\right\}$$

$$\times\left\{\int_0^\infty \varpi^{\Omega_0-1}\,{}_1F_1(\eta;\lambda;-\varpi)\,d\varpi\right\}.$$

Using the extended gamma function in [14], we have

$$\mathrm{M}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta);\Omega_0\right\}=\frac{1}{B(\Lambda_1-\alpha,\alpha)}\left\{\int_0^1 t^{\alpha+\ell\Omega_0-1}(1-t)^{\Lambda_1+\zeta\Omega_0-\alpha-1}W_{\mu,\Lambda_3}^{\Lambda_2,\alpha}(zt)\,dt\right\}\Gamma^{(\eta,\lambda)}(\Omega_0).$$

Rewritten this equation, gives

$$\mathrm{M}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta);\Omega_0\right\}=\frac{\Gamma^{(\eta,\lambda)}(\Omega_0)}{B(\Lambda_1-\alpha,\alpha)}\int_0^1 t^{\alpha+\ell\Omega_0-1}(1-t)^{\Lambda_1+\zeta\Omega_0-\alpha-1}\sum_{s=0}^{\infty}\frac{(tz)^s}{s!\Gamma(\mu s+\tau)}\frac{(\Lambda_2)_s}{(\beta)_s}dt.$$

On exchanging the order integration and summation, yields

$$\mathrm{M}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta);\Omega_0\right\}=\frac{\Gamma^{(\eta,\lambda)}(\Omega_0)}{B(\Lambda_1-\alpha,\alpha)}\sum_{s=0}^{\infty}\frac{(\Lambda_2)_s}{(\beta)_s}\frac{z^s}{s!\Gamma(\mu s+\tau)}\int_0^1 t^{\alpha+s+\ell\Omega_0-1}(1-t)^{\Lambda_1+\zeta\Omega_0-\alpha-1}dt.$$

considering classical Euler beta function in [15], we have

$$\mathrm{M}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp,\ell,\zeta);\Omega_0\right\}=\frac{\Gamma^{(\eta,\lambda)}(\Omega_0)}{B(\Lambda_1-\alpha,\alpha)}\sum_{s=0}^{\infty}\frac{(\Lambda_2)_s}{(\beta)_s}\frac{z^s}{s!\Gamma(\mu s+\Lambda_3)}B(\alpha+s+\ell\Omega_0,p+\zeta\Omega_0-\alpha).$$

On simplifying we get,

$$\mathrm{M}\left\{W_{\mu,\tau}^{\alpha,\beta,p,q;\eta,\lambda}(z;\wp,\ell,\zeta);\Omega_0\right\}=\frac{\Gamma^{(\eta,\lambda)}(\Omega_0)\Gamma(\Lambda_1)}{\Gamma(\alpha)\Gamma(\Lambda_1-\alpha)}\sum_{s=0}^{\infty}\frac{z^s}{s!}\frac{(\Lambda_2)_s}{(\beta)_s}\frac{\Gamma(\alpha+s+\ell\Omega_0)\Gamma(p-\alpha+\zeta\Omega_0)}{\Gamma(\mu s+\Lambda_3)\Gamma(s+pr+\ell\Omega_0+\zeta\Omega_0)}.$$

This can be rewritten as follows:

$$M\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta);\Omega_0\right\}=\frac{\Gamma^{(\eta,\lambda)}(\Omega_0)\Gamma(\Lambda_1-\alpha+\ell\Omega_0)\Gamma(\beta)\Gamma(\Lambda_1)}{\Gamma(\alpha)\Gamma(\Lambda_1-\alpha)\Gamma(\Lambda_2)}$$
$$\times{}_2\psi_3\left[\begin{matrix}(\Lambda_2,1),\,(\alpha+\ell\Omega_0,1);\\(\beta,1),(\tau,\mu),(\Lambda_1+\ell\Omega_0+\zeta\Omega_0,1);\end{matrix}\;z\right].$$

Lemma 8. The extended gamma function given in [14] has the property

$$\Gamma^{(\eta,\lambda)}(1)=\frac{\Gamma(\lambda)\Gamma(\eta-1)}{\Gamma(\eta)\Gamma(\lambda-1)}. \tag{15}$$

Setting $\Omega_0=1$, in equation (14) and using (15), The following Corollary can be obtained.

Corollary 9. The Mellin transform also hold

$$\mathrm{M}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta);1\right\}=\frac{\Gamma(\lambda)\Gamma(\eta-1)}{\Gamma(\eta)\Gamma(\lambda-1)}\frac{\Gamma(\Lambda_1-\alpha+\ell)\Gamma(\beta)\Gamma(\Lambda_1)}{\Gamma(\alpha)\Gamma(\Lambda_1-\alpha)\Gamma(\Lambda_2)}$$
$$\times{}_2\psi_3\left[\begin{matrix}(\Lambda_2,1),\,(\alpha+\ell,1);\\(\beta,1),(\Lambda_3,\mu),(\Lambda_1+\ell+\zeta,1);\end{matrix}\;z\right].$$

Corollary 10. The following inverse Mellin transforms holds:

$$W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma^{(\eta,\lambda)}(\Omega_0)\Gamma(\Lambda_1-\alpha+\ell r)\Gamma(\beta)\Gamma(\Lambda_1)}{\Gamma(\alpha)\Gamma(\Lambda_1-\alpha)\Gamma(\Lambda_2)}$$
$$\times{}_2\psi_3\left[\begin{matrix}(\Lambda_2,1),\,(\alpha+\ell\Omega_0+1);\\(\beta,1),(\Lambda_3,\mu),(\Lambda_1+\ell\Omega_0+\zeta\Omega_0,1);\end{matrix}\;z\right]\sigma^{-\Omega_0}d\Omega_0.$$

Theorem 11. The following Jacobi transform hold:

$$J^{\eta,\lambda}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(xz;\wp;\ell,\zeta);\aleph\right\}=2^{\eta+\lambda+1}\binom{\eta+\aleph}{\aleph}B(\eta+1,\lambda+1)\sum_{s=0}^{\infty}\mathrm{H}_s$$

$$\times F_{1:1:0}^{1:2:1}\left[\begin{matrix}\eta+1; & -\aleph, \eta+\mu+\aleph+1; & 1-\rho-s; \\ \eta+\lambda+2; & \eta+1; & ---;\end{matrix} 1,2\right]\frac{x^s}{s!}, \qquad (16)$$

$$\left(|x|<1; \aleph\in\mathbb{N}_0; \min\{\mathrm{Re}(\eta),\mathrm{Re}(\lambda)\}>-1; \rho\in\mathbb{C}\right),$$

where the coefficients H_s are given by

$$\mathrm{H}_s=\sum_{s=0}^{\infty}\frac{B_{\wp}^{(\eta,\lambda;\ell,\zeta)}(\alpha+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}\frac{(\Lambda_2)_s}{s!\ \Gamma(\mu s+\Lambda_3)(\beta)_s},$$

under the assumptions that the Jacobi transform in (16) exists.

Proof: Using equation (6), we have

$$J^{\eta,\lambda}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(xz;\wp;\ell,\zeta);\aleph\right\}=\int_{-1}^{1}z^{\rho-1}(1+z)^{\lambda}(1-z)^{\mu}P_{\aleph}^{\eta,\lambda}(z)W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(xz,\wp,\ell,\zeta)\,dz.$$

Applying equation (10), gives

$$J^{\eta,\lambda}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(xz;\wp;\ell,\zeta);\aleph\right\}=\int_{-1}^{1}z^{\rho-1}(1+z)^{\lambda}(1-z)^{\mu}P_{\aleph}^{\eta,\lambda}(z)\left(\sum_{s=0}^{\infty}\mathrm{H}_s\frac{(xz)^s}{s!}\right)dz.$$

Commuting the order of integration and summation, yields

$$J^{\eta,\lambda}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(xz;\wp;\ell,\zeta);\aleph\right\}=\sum_{s=0}^{\infty}\mathrm{H}_s\frac{x^s}{s!}\int_{-1}^{1}z^{\rho+s-1}(1+z)^{\lambda}(1-z)^{\mu}P_{\aleph}^{\eta,\lambda}(z)\,dz.$$

Considering equation (7), we find that

$$J^{\eta,\lambda}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(xz;\wp;\ell,\zeta);\aleph\right\}=\sum_{s=0}^{\infty}\mathrm{H}_s\frac{x^s}{s!}2^{\eta+\lambda+1}\binom{\eta+\aleph}{\aleph}B(\eta+1,\lambda+1)$$

$$\times F_{1:1;0}^{1:2:1}\left[\begin{matrix}\eta+1; & -\aleph, & \eta+\mu+\aleph+1; & 1-\rho-s; \\ \eta+\lambda+2; & \eta+1; & ---;\end{matrix} 1,2\right].$$

Rewritten this equation, yields

$$J^{\eta,\lambda}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(xz;\wp;\ell,\zeta);\aleph\right\}=2^{\eta+\lambda+1}\binom{\eta+\aleph}{\aleph}B(\eta+1,\lambda+1)\sum_{s=0}^{\infty}\mathrm{H}_s$$

$$\times F_{1:1;0}^{1:2;1}\left[\begin{matrix}\eta+1; & -\aleph, & \eta+\mu+\aleph+1; & 1-\rho-s; \\ \eta+\lambda+2; & \eta+1; & ---;\end{matrix} 1,2\right]\frac{x^s}{s!}.$$

Using equations (8) and (9), the following corollaries follows:

Corollary 12. The following Gegenbauer transform holds:

$$G^{\upsilon}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(xz;\wp;\ell,\zeta);\aleph\right\}=2^{2\upsilon}\binom{2\upsilon+\aleph-1}{\aleph}B\left(\upsilon+\frac{1}{2},\upsilon+\frac{1}{2}\right)\sum_{s=0}^{\infty}\mathrm{H}_s$$

$$\times F_{1:1;0}^{1:2;1}\left[\begin{matrix} \upsilon+\frac{1}{2}; -\aleph,\ 2\upsilon+\aleph;\ 1-\rho-s; \\ 2\upsilon+1; \quad \upsilon+\frac{1}{2}; \quad ---; \end{matrix} 1,2\right]\frac{x^s}{s!}.$$

Corollary 13. The following Legendre transform holds:

$$L\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(xz,;\wp\ell,\zeta);\aleph\right\}=2\sum_{s=0}^{\infty}\mathrm{H}_s\,F_{1:1;0}^{1:2;1}\left[\begin{matrix}1;\ \aleph,\ \aleph+1;\ 1-\rho-s; \\ 2; \quad 1; \quad ---;\end{matrix}1,2\right]\frac{x^s}{s!}.$$

3. Derivative formulae for the new extended Wright function

Derivative formulae for the new generalized extended Wright function are discussed in this part.

Theorem 14. The following derivative formula hold true.

$$\frac{d}{dz}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp,\ell,\zeta)\right\}=\frac{(\Lambda_2)(\alpha)}{(\Lambda_1)(\beta)}W_{\mu,s\mu+\Lambda_3}^{\alpha+1,\beta+1,\Lambda_1+1,\Lambda_2+1;\eta,\lambda}(z;\wp,\ell,\zeta), \tag{17}$$

$$\left(\mathrm{Re}(\mu)>0,\wp\in\mathbb{R}_0^{+},\mathrm{Re}(\alpha)>\mathrm{Re}(\beta)>0;\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^{-}\right).$$

Proof: Differentiating equation (10) with respect to z, one can received

$$\frac{d}{dz}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta)\right\}=\sum_{s=1}^{\infty}\frac{z^{s-1}}{(s-1)!}\frac{B_{\wp}^{(\eta,\lambda;\ell,\zeta)}(\alpha+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}\frac{1}{\Gamma(\mu s+\Lambda_3)}\frac{(\Lambda_2)_s}{(\beta)_s}$$

Setting $s\to s+1$, we have

$$\frac{d}{dz}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp,\ell,\zeta)\right\}=\sum_{s=0}^{\infty}\frac{z^{s}}{s!}\frac{B_{\wp}^{(\eta,\lambda;\ell,\zeta)}(\alpha+s+1,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}\frac{1}{\Gamma(\mu s+\mu+\tau)}\frac{(\Lambda_2)_{s+1}}{(\beta)_{s+1}}$$

Rewritten this equation, we get

$$\frac{d}{dz}\left\{W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp,\ell,\zeta)\right\}=\frac{(\Lambda_2)(\alpha)}{(\Lambda_1)(\beta)}\sum_{s=0}^{\infty}\frac{z^{s}}{s!}\frac{B_{\sigma}^{(\eta,\lambda;\ell,\zeta)}(\alpha+s+1,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha+1)}\frac{1}{\Gamma(\mu s+\mu+\Lambda_3)}\frac{(\Lambda_2+1)_s}{(\beta+1)_s}.$$

Applying equation (10), yields

$$\frac{d}{dz}\left\{W_{\mu,\Omega_2}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp,\ell,\zeta)\right\}=\frac{(\Lambda_2)(\alpha)}{(\Lambda_1)(\beta)}W_{\mu,\mu+\Lambda_3}^{\alpha+1,\beta+1,\Lambda_1+1,\Lambda_2+1;\eta,\lambda}(z;\wp,\ell,\zeta).$$

Continuing differentiating equation (17) $(s-1)$ times with respect to z, the following Corollary can be obtained.

Corollary 15. The following formula holds:

$$\frac{d^s}{dz^s}\left\{W_{\mu,\Lambda_1}^{\alpha,\beta,\Lambda_1,\Lambda_1;\eta,\lambda}(z;\wp;\ell,\zeta)\right\}=\frac{(\Lambda_2)_s(\alpha)_s}{(\Lambda_1)_s(\beta)_s}W_{\mu,s\mu+\Lambda_3}^{\alpha+s,\beta+s,\Lambda_1+s,\Lambda_2+s;\eta,\lambda}(z;\wp;\ell,\zeta),$$

$$\left(\mathrm{Re}(\mu)>0,\wp\in\mathbb{R}_0^+,\mathrm{Re}(\beta)>\mathrm{Re}(\alpha)>0;\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Theorem 16. The following formula holds.

$$\left(\frac{d}{dz}\right)^s\left[z^{\Lambda_3-1}W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(\omega z^\mu;\wp;\ell,\zeta)\right]=z^{\Lambda_3-n-1}W_{\mu,\Lambda_3-s}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(\omega z^\mu;\wp;\ell,\zeta), \tag{18}$$

$$\left(\mathrm{Re}(\Lambda_3-s)>0,s\in\mathbb{N}\right).$$

Proof: Setting $z\to\omega z^\mu$ in equation (10), we obtain

$$W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(\omega z^\mu;\wp;\ell,\zeta)=\sum_{s=o}^{\infty}\frac{\omega^s}{s!}\frac{B_\sigma^{(\eta,\lambda;\ell,\zeta)}(\alpha+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}\frac{1}{\Gamma(\mu s+\Lambda_3)}\frac{(\Lambda_2)_s}{(\beta)_s}z^{\mu s}.$$

Multiplying both sides by z^{Λ_3-1}, yield

$$z^{\Lambda_3-1}W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(\omega z^\mu;\wp,\ell,\zeta)=\sum_{s=o}^{\infty}\frac{\omega^s}{s!}\frac{B_\wp^{(\eta,\lambda;\ell,\zeta)}(\alpha+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}\frac{1}{\Gamma(\mu s+\Lambda_3)}\frac{(\Lambda_2)_s}{(\beta)_s}z^{\mu s+\Lambda_3-1}.$$

Differentiating $n-$ time with respect to z, we received

$$\left(\frac{d}{dz}\right)^s\left[z^{\Lambda_3-1}W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(\omega z^\mu;\wp,\ell,\zeta)\right]=\sum_{s=0}^{\infty}\frac{\omega^s}{s!}\frac{B_\wp^{(\eta,\lambda;\ell,\zeta)}(\alpha+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}\frac{1}{\Gamma(\mu s+\tau)}\frac{(q)_s}{(\beta)_s}\left(\frac{d}{dz}\right)^s z^{\mu s+\Lambda_3-1}.$$

On simplifying, we obtain

$$\left(\frac{d}{dz}\right)^n\left[z^{\tau-1}W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(\omega z^\mu;\wp,\ell,\zeta)\right]=z^{\tau-n-1}\sum_{s=0}^{\infty}\frac{(\omega z^\mu)^s}{s!}\frac{B_\wp^{(\eta,\lambda;\ell,\zeta)}(\alpha+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}\frac{1}{\Gamma(\mu s+\Lambda_1-n)}\frac{(\Lambda_2)_s}{(\beta)_s}.$$

On using equation (10), we get

$$\left(\frac{d}{dz}\right)^s\left[z^{\Lambda_3-1}W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(\omega z^\mu;\wp;\ell,\zeta)\right]=z^{\tau-n-1}W_{\mu,\Lambda_3-s}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(\omega z^\mu;\wp;\ell,\zeta).$$

4. Fractional derivative and integral

The classical and generalized Riemann-Liouville fractional integral and derivative are studied and investigated here

Lemma 17. The Riemann-Liouville fractional integral I_{o+}^{Λ} and derivative D_{o+}^{Λ} are given in ([10], [16]) as follows:

$$\left(I_{o+}^{\Lambda}f\right)(z)=\frac{1}{\Gamma(\Lambda)}\int_0^z f(t)(z-t)^{\Lambda-1}dt, \tag{19}$$

and

$$\left(D_{o+}^{\Lambda} f\right)(z) = \left(\frac{d}{dz}\right)^{r} \left(I_{o+}^{r-\Lambda} f\right)(z), \tag{20}$$

$$\left(\Lambda \in \mathbb{C}, \mathrm{Re}(\Lambda) > 0,\ r = \left[\mathrm{Re}(\Lambda)\right] + 1\right).$$

Theorem 18. Let $o \in \mathbb{R}^{+}, \Lambda, \omega \in \mathbb{C}, Re(\mu) > 0, Re(\Lambda_1) > Re(\alpha) > 0$, then

$$\left(I_{o+}^{\Lambda}\left\{(t-o)^{\Lambda_3-1} W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right)\right\}\right)(z) = (z-o)^{\Lambda_3+o-1} W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right).$$

Proof: By virtue of equations (10) and (19), we obtain

$$\left(I_{o+}^{\Lambda}\left\{(t-o)^{\Lambda_3-1} W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right)\right\}\right)(z) = \sum_{s=0}^{\infty} \frac{\omega^{s}}{s!} \frac{B_{\wp}^{(\eta,\lambda;\ell,\zeta)}(\alpha+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}$$

$$\times \frac{1}{\Gamma(\mu s+\Lambda_3)} \frac{(\Lambda_2)_s}{(\beta)_s} \left(I_{o+}^{v}\left\{(t-o)^{\mu s+\Lambda_3-1}\right\}\right)(z).$$

Applying equation (19) and algebraic simplifications, gives

$$\left(I_{o+}^{\Lambda}\left\{(t-o)^{\tau-1} W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right)\right\}\right)(z) = (z-o)^{\Lambda_3+o-1} \sum_{s=o}^{\infty} \frac{B_{\wp}^{(\eta,\lambda;\ell,\zeta)}(\alpha+s,\Lambda_1-\alpha)}{B(\Lambda_1-\alpha,\alpha)}$$

$$\times \frac{\left(\omega(z-o)^{\mu}\right)^{s}}{s!\,\Gamma(\mu s+v+\Lambda_3)} \frac{(\Lambda_2)_s}{(\beta)_s}.$$

Owing to equation (10), we obtain

$$\left(I_{o+}^{\Lambda}\left\{(t-o)^{\Lambda_3-1} W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right)\right\}\right)(z) = (z-o)^{\Lambda_3+o-1} W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right).$$

Theorem 19. Let $o \in \mathbb{R}^{+}, \Lambda, \omega \in \mathbb{C}, Re(\mu) > 0, Re(\Lambda_1) > Re(\alpha) > 0$, then

$$\left(D_{o+}^{\Lambda}\left\{(t-o)^{\Lambda_3-1} W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right)\right\}\right)(z) = (z-o)^{\Lambda_3-o-1} W_{\mu,\Lambda_3-\Lambda}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right).$$

Proof: In view equations (10), (18), (19) and (20), we have

$$\left(D_{o+}^{\Lambda}\left\{(t-o)^{\Lambda_3-1} W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right)\right\}\right)(z)$$

$$= \left(\frac{d}{dz}\right)^{s} \left(I_{o+}^{s-\Lambda}\left\{(t-o)^{\Lambda_3-1} W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_3;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right)\right\}\right)(z)$$

$$= \left(\frac{d}{dz}\right)^{s} \left\{(t-o)^{\Lambda_3+s-\Lambda-1} W_{\mu,\Lambda_3+r-\Lambda}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right)\right\}$$

$$= (z-o)^{\Lambda_3-\Lambda-1} W_{\mu,\Lambda_3-\Lambda}^{\alpha,\beta,\Lambda_1,\Lambda_1;\eta,\lambda}\left(\omega(t-o)^{\mu};\wp,\ell,\zeta\right).$$

Lemma 20. The extended Riemann-Liouville fractional derivative is defined in [3] as follows:

$$R_z^{\Lambda,\wp;\eta,\lambda}\left\{f(z)\right\}=\frac{1}{\Gamma(-\Lambda)}\int_0^z (z-t)^{-\Lambda-1} f(t)\, {}_1F_1\left(\eta;\lambda;-\frac{\wp z^{\ell+\zeta}}{t^{\ell}(z-t)^{\zeta}}\right)dt, \tag{21}$$

$$\left(\operatorname{Re}(\eta)>0, \operatorname{Re}(\lambda)>0, \operatorname{Re}(\wp)>0, \operatorname{Re}(\Lambda)<0\right).$$

Theorem 21. The following extended Riemann-Liouville fractional derivative holds:

$$R_z^{\Lambda_1-\alpha,\wp;\eta,\lambda}\left\{W_{\mu,\Lambda_3}^{\Lambda_2,\beta}(z)\, z^{\Lambda_1-1}\right\}=z^{\alpha-1}\frac{\Gamma(\Lambda_1)}{\Gamma(\alpha)}W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta). \tag{22}$$

Proof: Using (10) and (21), we obtain

$$R_z^{\Lambda_1-\alpha,\wp;\eta,\lambda}\left\{W_{\mu,\Lambda_3}^{\Lambda_2,\beta}(z)\, z^{\Lambda_1-1}\right\}=\frac{1}{\Gamma(\Lambda_1-\alpha)}\int_0^z (z-t)^{\Lambda_3-\Lambda_1-1}\, {}_1F_1\left(\eta;\lambda;-\frac{\wp z^{\ell+\zeta}}{t^{\ell}(z-t)^{\zeta}}\right)t^{\Lambda_1-1}W_{\mu,\Lambda_3}^{\Lambda_2,\beta}(t)\,dt.$$

Setting $t=\varpi z$, we obtain

$$R_z^{\Lambda_1-\alpha,\wp;\eta,\lambda}\left\{W_{\mu,\Lambda_3}^{\Lambda_2,\beta}(z)\, z^{\Lambda_1-1}\right\}=\frac{z^{\alpha-1}}{\Gamma(\alpha-\Lambda_1)}\int_0^z \varpi^{\Lambda_1-1}(1-\varpi)^{\alpha-\Lambda_1-1}\, {}_1F_1\left(\eta;\lambda;-\frac{\wp}{\varpi^{\ell}(1-\varpi)^{\zeta}}\right)W_{\mu,\Lambda_3}^{\Lambda_2,\beta}(\varpi z)\,dt.$$

Applying equation (13), we have

$$R_z^{\Lambda_1-\alpha,\wp;\eta,\lambda}\left\{W_{\mu,\Lambda_3}^{\Lambda_2,\beta}(z)\, z^{\Lambda_1-1}\right\}=\frac{1}{\Gamma(\alpha-\Lambda_1)}\left\{B(\Lambda_1-\alpha,\Lambda_1)W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp,\ell,\zeta)\right\}z^{\alpha-1}.$$

On simplifying, we get

$$R_z^{\Lambda_1-\alpha,\wp;\eta,\lambda}\left\{W_{\mu,\Lambda_3}^{\Lambda_2,\beta}(z)\, z^{\Lambda_1-1}\right\}=\frac{\Gamma(\Lambda_1)}{\Gamma(\alpha)}W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;\ell,\zeta)\, z^{\alpha-1}.$$

5. Conclusion

In this article, we investigated new generalized extended Wright function and its integral representations, Euler's beta, Mellin, inverse Mellin, Jacobi, Gegenbaur and Legendre transforms. Furthermore, some derivative formulas, classical and generalized Riemann-Liouville fractional derivative and integral are also presented.

The new generalized extended Wright function can be reducing to the well-known classical and extended Wright functions as illustrated bellow:

For $\ell=\xi=1, \Lambda_1=\Lambda_2, \alpha=\beta$ and $\wp=0$, then

$$W_{\mu,\Lambda_3}(z)=W_{\mu,\Lambda_3}^{\alpha,\alpha,\Lambda_1,\Lambda_1;\eta,\lambda}(z;0;1,1).$$

If $\ell=\xi=1, \Lambda_1=\Lambda_2$ and $\wp=0$, then

$$W_{\mu,\Lambda_3}^{\alpha,\beta}(z)=W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_1;\eta,\lambda}(z;0;1,1).$$

Setting $\Lambda_1=\Lambda_2$, then

$$W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1;\eta,\lambda}(z;\wp;\ell,\zeta)=W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_1;\eta,\lambda}(z;\wp;\ell,\zeta).$$

Substituting $\ell=\xi=1$, then

$$W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp)=W_{\mu,\Lambda_3}^{\alpha,\beta,\Lambda_1,\Lambda_2;\eta,\lambda}(z;\wp;1,1).$$

This generalized extended Wright function can be used to study generalized special functions (refer to, [1], [2], [4] and [13]]) and fractional integral and differential calculus such as Riemann-Liouville and Caputo fractional derivative and integral (see for example, [18]).

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