*Intern. J. Fuzzy Mathematical Archive*
*Vol. 19, No. 2, 2021, 149-159*
*ISSN: 2320 –3242 (P), 2320 –3250 (online)*
*Published on 27 December 2021*
*www.researchmathsci.org*
*DOI: http://dx.doi.org/10.22457/ijfma.v19n2a05234*

International Journal of Fuzzy Mathematical Archive

# On Intuitionistic Fuzzy r-Regular Spaces

***Md. Saiful Islam*[1], *Md. Sahadat Hossain*[2*] and *Md. Aman Mahbub*[3]**

[1]Department of Computer Science and Engineering
Jatiya Kabi Kazi Nazrul Islam University, Trishal, Mymensingh, Bangladesh
E-mail: saifulmath@yahoo.com
[2]Department of Mathematics, University of Rajshahi, Rajshahi, Bangladesh
[3]Department of Mathematics, Comilla University, Comilla, Bangladesh
E-mail: rinko.math@gmail.com
[*]Corresponding author. Email: sahadat@ru.ac.bd

*Received 1 November 2021; accepted 26 December 2021*

***Abstract.*** The purpose of this paper is to give some new notions of intuitionistic fuzzy regular spaces and investigate the relationship among them. Also the relations between our notions and other given notion of intuitionistic fuzzy regular spaces have been investigated. Moreover it is shown that all these notions satisfy hereditary property of regular spaces. Finally it is shown that under some conditions image and preimage preserve intuitionistic fuzzy regular spaces.

***Keywords:*** Fuzzy set, intuitionistic fuzzy set, intuitionistic fuzzy topological space, intuitionistic fuzzy regular space

***AMS Mathematics Subject Classification (2010):*** 08A72

## 1. Introduction

Fuzzy topology is an important research field in fuzzy mathematics which has been established by Chang [1] in 1968 based on Zadeh's [2] concept of fuzzy sets. Later, the notion of an intuitionistic fuzzy set was introduced by Atanassov [3] in 1986 which take into account both the degrees of membership and nonmembership subject to the condition that their sum does not exceed 1. Coker and coworker [5,6,7,] introduced the basic concepts of the theory of intuitionistic fuzzy topological spaces. Since then, Coker et al. [8], Singh et al. [9], Lee et al. [10], Saadati et al. [11], Ahmed et al. [12,13,14,15,16] , Mahabub et al. [18,19,20] and Saiful et al. [23,24,25] subsequently initiated a study of intuitionistic fuzzy topological spaces by using intuitionistic fuzzy sets. Various researchers work particularly on intuitionistic fuzzy regular spaces [17]. Currently Alamgir [21] and Islam [23] are working in the field fuzzy logic. In this paper, we investigate some properties and features of intuitionistic fuzzy regular Space.

## 2. Notations and preliminaries

In this section we recall some basic definitions and known results of fuzzy sets, intuitionistic sets, intuitionistic fuzzy sets, topology and their mappings.

Through this paper, $X$ is a nonempty set, r and $s$ are constants in $(0,1)$. $T$ is a topology, t is a fuzzy topology, $\mathcal{T}$ is an intuitionistic topology and $\tau$ is an intuitionistic fuzzy topology. $\lambda$ and $\mu$ are fuzzy sets, $A = (\mu_A, \nu_A)$ is intuitionistic fuzzy set. By $\underline{0}$ and $\underline{1}$, we denote constant fuzzy sets taking values 0 and 1 respectively.

**Definition 2.1.** [1]**.** Let $X$ be a non empty set. A family $t$ of fuzzy sets in $X$ is called a fuzzy topology (FT, in short) on $X$ if the following conditions hold.

(1) $\underline{0}, \underline{1} \in t$,

(2) $\lambda \cap \mu \in t$, for all $\lambda, \mu \in t$,

(3) $\cup \lambda_j \in t$, for any arbitrary family $\{\lambda_j \in t,\ j \in J\}$.

The above definition is in the sense of C. L. Chang. The pair $(X, t)$ is called a fuzzy topological space (FTS, in short), members of $t$ are called fuzzy open sets (FOS, in short) in $X$ and their complements are called fuzzy closed sets (FCS, in short) in $X$.

**Definition 2.2.** [17]. Suppose $X$ is a non empty set. An intuitionistic set $A$ on $X$ is an object having the form $A = (X, A_1, A_2)$ where $A_1$and $A_2$ are subsets of $X$ satisfying $A_1 \cap A_2 = \phi$. The set $A_1$ is called the set of member of $A$ while $A_2$ is called the set of non-member of $A$. In this paper, we use the simpler notation $A = (A_1, A_2)$ instead of $A = (X, A_1, A_2)$ for an intuitionistic set.

**Remark 2.1.** [17]. Every subset $A$ of a nonempty set $X$ may obviously be regarded as an intuitionistic set having the form $A = (A, A^c)$ where $A^c = X - A$.

**Definition 2.3.** [17]**.** Let the intuitionistic sets $A$ and $B$ in $X$ be of the forms $A = (A_1, A_2)$ and $B = (B_1, B_2)$ respectively. Furthermore, let $\{A_j, j \in J\}$ be an arbitrary family of intuitionistic sets in $X$, where $A_j = \left(A_j^{(1)}, A_j^{(2)}\right)$. Then

(a) $A \subseteq B$ if and only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$,

(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,

(c) $\bar{A} = (A_2, A_1)$, denotes the complement of $A$,

(d) $\cap A_j = (\cap A_j^{(1)}, \cup A_j^{(2)})$,

(e) $\cup A_j = (\cup A_j^{(1)}, \cap A_j^{(2)})$,

(f) $\phi_\sim = (\phi, X)$ and $X_\sim = (X, \phi)$.

**Definition 2.4. [**15]. Let $X$ be a non empty set. A family $\mathcal{T}$ of intuitionistic sets in $X$ is called an intuitionistic topology (IT, in short) on $X$ if the following conditions hold.

(1) $\phi_\sim, X_\sim \in \mathcal{T}$,

(2) $A \cap B \in \mathcal{T}$ for all $A, B \in \mathcal{T}$,

(3) $\cup A_j \in \mathcal{T}$ for any arbitrary family $\{A_j \in \mathcal{T}, j \in J\}$.

The pair $(X, \mathcal{T})$ is called an intuitionistic topological space (ITS, in short), members of $\mathcal{T}$ are called intuitionistic open sets (IOS, in short) in $X$ and their complements are called intuitionistic closed sets (ICS, in short) in $X$.

**Definition 2.5.** [3]**.** Let $X$ be a non empty set. An intuitionistic fuzzy set $A$ (IFS, in short) in $X$ is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, where $\mu_A$ and $\nu_A$ are fuzzy sets in $X$ denote the degree of membership and the degree of non- membership respectively with $\mu_A(x) + \nu_A(x) \leq 1$.

Throughout this paper, we use the simpler notation $A = (\mu_A, \nu_A)$ instead of $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ for intuitionistic fuzzy sets.

**Remark 2.2.** Obviously every fuzzy set $\lambda$ in $X$ is an intuitionistic fuzzy set of the form $(\lambda, 1 - \lambda) = (\lambda, \lambda^c)$ and every intuitionistic set $A = (A_1, A_2)$ in $X$ is an intuitionistic fuzzy set of the form $(1_{A_1}, 1_{A_2})$.

**Definition 2.6.** [3]. Let $X$ be a nonempty set and $A, B$ are intuitionistic fuzzy sets on $X$ be given by $(\mu_A, \nu_A)$ and $(\mu_B, \nu_B)$ respectively, then

(a) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$,
(b) $A = B$ if $A \subseteq B$ and $B \subseteq A$,
(c) $\bar{A} = (\nu_A, \mu_A)$,
(d) $A \cap B = (\mu_A \cap \mu_B, \nu_A \cup \nu_B)$,
(e) $A \cup B = (\mu_A \cup \mu_B, \nu_A \cap \nu_B)$.

**Definition 2.7.** [5]. Let $\{A_j = \left(\mu_{A_j}, \nu_{A_j}\right), j \in J\}$ be an arbitrary family of IFSs in $X$. Then

(a) $\cap A_j = (\cap \mu_{A_j}, \cup \nu_{A_j})$,
(b) $\cup A_j = (\cup \mu_{A_j}, \cap \nu_{A_j})$,
(c) $0_{\sim} = (\underline{0}, \underline{1})$, $1_{\sim} = (\underline{1}, \underline{0})$.

**Definition 2.8.** [5]. An intuitionistic fuzzy topology (IFT, in short) on a nonempty set $X$ is a family $\tau$ of IFSs in $X$, satisfying the following axioms:

(1) $0_{\sim}, 1_{\sim} \in \tau$,
(2) $A \cap B \in \tau$, for all $A, B \in \tau$,
(3) $\cup A_j \in \tau$ for any arbitrary family $\{A_j \in \tau, j \in J\}$.

The pair $(X, \tau)$ is called an intuitionistic fuzzy topological space (IFTS, in short), members of $\tau$ are called intuitionistic fuzzy open sets (IFOS, in short) in $X$, and their complements are called intuitionistic fuzzy closed sets (IFCS, in short) in $X$.

**Remark 2.3.** [16]. Let $X$ be a non empty set and $A \subseteq X$, then the set $A$ may be regarded as a fuzzy set in $X$ by its characteristic function $1_A : X \to \{0,1\} \subset [0,1]$ which is defined by

$$1_A(x) = \begin{cases} 1 & if\ x \in A \\ 0\ if\ x \notin A, i.e., & if\ x \in A^c \end{cases}$$

Again, we know that a fuzzy set $\lambda$ in $X$ may be regarded as an intuitionistic fuzzy set by $(\lambda, 1 - \lambda) = (\lambda, \lambda^c)$. So every sub set $A$ of $X$ may be regarded as intuitionistic fuzzy set by $(1_A, 1 - 1_A) = (1_A, 1_{A^c})$.

**Theorem 2.1**. Let $(X, T)$ be a topological space. Then $(X, \tau)$ is an IFTS where $\tau = \left\{(1_{A_j}, 1_{A_j^c}), j \in J : A_j \in T\right\}$.

**Proof:** The proof is obvious.

**Note 2.1.** Above $\tau$ is the corresponding intuitionistic fuzzy topology of $T$.

**Theorem 2.2**. Let $(X, t)$ be a fuzzy topological space. Then $(X, \tau)$ is an IFTS where $\tau = \{(\lambda,\ \lambda^c) : \lambda \in t\}$.
**Proof:** The proof is obvious.

**Note 2.2.** Above $\tau$ is the corresponding intuitionistic fuzzy topology of t.

**Theorem 2.3.** Let $(X, \mathcal{T})$ be an intuitionistic topological space. Then $(X, \tau)$ is an intuitionistic fuzzy topological space where

$$\tau = \left\{\left(1_{A_{j1}},\ 1_{A_{j2}}\right), j \in J :\ A_j = \left(A_{j1}, A_{j2}\right) \in \mathcal{T}\right\}.$$

**Proof:** The proof is obvious.

**Note 2.3.** Above $\tau$ is the corresponding intuitionistic fuzzy topology of $\mathcal{T}$.

**Definition 2.9.** [3] Let $X$ and $Y$ be two nonempty sets and $f: X \to Y$ be a function. If $A = \{(x, \mu_A(x),\ \nu_A(x)) : x \in X\}$ and $B = \{(y, \mu_B(y),\ \nu_B(y)) : y \in Y\}$ are IFSs in $X$ and $Y$ respectively, then the pre image of $B$ under $f$, denoted by $f^{-1}(B)$ is the IFS in $X$ defined by

$f^{-1}(B) = \left\{\left(x, (f^{-1}(\mu_B))(x), (f^{-1}(\nu_B))(x)\right) : x \in X\right\} = \{(x, \mu_B(f(x)), \nu_B(f(x))) : x \in X\}$

and the image of $A$ under $f$, denoted by $f(A)$ is the IFS in $Y$ defined by $f(A) = \left\{\left(y,\ (f(\mu_A))(y), (f(\nu_A))(y)\right) : y \in Y\right\}$, where for each $y \in Y$

$$(f(\mu_A))(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x) & if\ f^{-1}(y) \neq \phi \\ 0 & otherwise \end{cases}$$

$$(f(\nu_A))(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \nu_A(x) & if\ f^{-1}(y) \neq \phi \\ 1 & otherwise \end{cases}$$

**Definition 2.10.** [5] Let $(X, \tau)$ and $(Y, \delta)$ be IFTSs. A function $f: X \to Y$ is called continuous if $f^{-1}(B) \in \tau$ for all $B \in \delta$ and $f$ is called open if $f(A) \in \delta$ for all $A \in \tau$.

**Definition 2.11.** [20]**.** A fuzzy topological space $(X, t)$ is called fuzzy regular if for any fuzzy point $x_\alpha \in X$ and closed set $\lambda$ in $X$ with $x_\alpha \notin \lambda$ there exists $u, v \in t$ such that $x_\alpha \in u,\ \ \lambda \subset v$ and $u \cap v = \underline{0}$.

**Definition 2.12.** [8]. Let $A = (\mu_A, \nu_A)$ be a IFS in $X$ and $U$ be a non empty subset of $X$. The restriction of $A$ to $U$ is a IFS in $U$, denoted by $A|U$ and defined by $A|U = (\mu_A|U, \nu_A|U)$.

**Definition 2.13.** Let $(X, \tau)$ be an intuitionistic fuzzy topological space and $U$ is a non empty sub set of $X$ then $\tau_U = \{A|U : A \in \tau\}$ is an intuitionistic fuzzy topology on $U$ and $(U, \tau_U)$ is called sub space of $(X, \tau)$.

**Definition 2.14.** [8] Let $\alpha, \beta \in [0,1]$ and $+\beta \leq 1$ . An intuitionistic fuzzy point (IFP in short) $x_{(\alpha,\beta)}$ of $X$ is an intuitionistic fuzzy set in $X$ define by

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & if\ y = x \\ (0.1) & if\ y \neq x \end{cases}$$

An IFP $x_{(\alpha,\beta)}$ is said to belong to an IFS $A = (\mu_A, \nu_A)$ if $\alpha < \mu_A(x)$ and $\beta > \nu_A(x)$.

**Definition 2.15.** [17] An intuitionistic fuzzy topological space $(X, \tau)$ is called IF-Regular if for all intuitionistic fuzzy point $x_{(\alpha,\beta)}$ and all intuitionistic fuzzy closed set $F = (\mu_F, \nu_F)$ with $x_{(\alpha,\beta)} \notin F$ there exists intuitionistic fuzzy sets $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $x_{(\alpha,\beta)} \in A$, $F \subset B$ and $A \cap B = 0_\sim$.

## 3. Intuitionistic fuzzy regular spaces

In this section, we introduce four new notions of intuitionistic fuzzy regular spaces and form an implication among them. Also, we discuss the various features and properties of these concepts.

**Definition 3.1.** Let $r \in (0,1)$. An intuitionistic fuzzy topological space $(X, \tau)$ is called.

(1) IF-Regular(r-i) if for all $x \in X$ and closed set $F = (\mu_F, \nu_F)$ with $\mu_F(x) < \nu_F(x)$, there exists $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $\mu_A(x) > r, \nu_A(x) < r$ and $F \subset B$ with $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$.

(2) IF-Regular(r-ii) if for all $x \in X$ and closed set $F = (\mu_F, \nu_F)$ with $\mu_F(x) < \nu_F(x)$, there exists $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $\mu_A(x) > r, \nu_A(x) = 0$ and $F \subset B$ with $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$.

(3) IF-Regular(r-iii) if for all $x \in X$ and closed set $F = (\mu_F, \nu_F)$ with $\mu_F(x) < \nu_F(x)$, there exists $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $\mu_A(x) > r, \nu_A(x) < r$ and $F \subset B$ with $(\mu_A \cap \mu_B) = \underline{0}$ and $(\nu_A \cup \nu_B)(y) > 0$ for all $y \in X$.

(4) IF-Regular(r-iv) if for all $x \in X$ and closed set $F = (\mu_F, \nu_F)$ with $\mu_F(x) < \nu_F(x)$, there exists $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $\mu_A(x) > r, \nu_A(x) = 0$ and $F \subset B$ with $(\mu_A \cap \mu_B) = \underline{0}$ and $(\nu_A \cup \nu_B)(y) > 0$ for all $y \in X$.

**Theorem 3.1.** Let $(X, T)$ be a topological space and $(X, \tau)$ be its corresponding IFTS where $\tau = \left\{(1_{A_j}, 1_{A_j}^{c}), j \in J : A_j \in T\right\}$. Then $(X, T)$ is Regular $\Leftrightarrow$ $(X, \tau)$ is IF-Regular(r-k) for any $k \in \{i, ii, iii, iv\}$.

**Proof:** Suppose $(X, T)$ is regular space. Let $x \in X$ and $(1_F, 1_{F^c})$ is closed in $(X, \tau)$ with $1_F(x) < 1_{F^c}(x)$. Now clearly $1_F(x) = 0$ and $1_{F^c}(x) = 1$. So $x \notin F$. Again since $(1_F, 1_{F^c})$ is closed in $(X, \tau)$, then $F$ is closed in $(X, T)$.
Since $(X, T)$ is regular, then there exists $A, B \in T$ such that $x \in A$, $F \subset B$ and $A \cap B = \emptyset$.
By the definition of $\tau$, we get $(1_A, 1_{A^c}), (1_B, 1_{B^c}) \in \tau$ as $A, B \in T$.
Clearly $1_A(x) = 1$, $1_{A^c}(x) = 0$ and $(1_F, 1_{F^c}) \subset (1_B, 1_{B^c})$.
Now for any $y \in X$ we get $(1_A \cap 1_B)(y) = 1_{A \cap B}(y) = 1_\phi(y) = 0$.
and $(1_{A^c} \cup 1_{B^c})(y) = 1_{A^c \cup B^c}(y) = 1_{(A \cap B)^c}(y) = 1_{\phi^c}(y) = 1_X(y) = 1$. i.e., $(1_A \cap 1_B) \subset (1_{A^c} \cup 1_{B^c})$
Therefore for any $r \in (0,1)$, we get $1_A(x) > r, 1_{A^c}(x) < r$ and $(1_F, 1_{F^c}) \subset (1_B, 1_{B^c})$ with $(1_A \cap 1_B) \subset (1_{A^c} \cup 1_{B^c})$. So $(X, \tau)$ is IF-Regular(r-i).
Conversly, suppose $(X, \tau)$ is IF-Regular(r-i). Let $x \in X$ , $F$ is closed in $(X, T)$ with $x \notin F$. Clearly $(1_F, 1_{F^c})$ is closed in $(X, \tau)$. Since $x \notin F$, $1_F(x) = 0$ and $1_{F^c}(x) = 1$. i.e., $1_F(x) < 1_{F^c}(x)$. Since $(X, \tau)$ is IF-Regular(r-i), then there exists $(1_A, 1_{A^c}), (1_B, 1_{B^c}) \in \tau$

such that $1_A(x) > r, 1_{A^c}(x) < r$; $\mu_B(x) < r, \nu_B(x) > r$ and $(1_F, 1_{F^c}) \subset (1_B, 1_{B^c})$ with $(1_A \cap 1_B) \subset (1_{A^c} \cup 1_{B^c})$.
By the definition of $\tau$, it is clear that $A, B \in T$
Now $1_A(x) > r, 1_{A^c}(x) < r$ $1_A(x) = 1, 1_{A^c}(x) = 0 \Rightarrow x \in A$
Again $(1_F, 1_{F^c}) \subset (1_B, 1_{B^c}) \Rightarrow 1_F \subset 1_B \Rightarrow F \subset B$.
Now for any $y \in X$, $(1_A \cap 1_B)(y) < (1_{A^c} \cup 1_{B^c})(y) \Rightarrow$ $(1_A \cap 1_B)(y) = 0$ and $(1_{A^c} \cup 1_{B^c})(y) = 1$
Now $(1_A \cap 1_B)(y) = 0 \Rightarrow (1_{A\cap B})(y) = 0 \Rightarrow y \notin A \cap B \Rightarrow A \cap B = \phi$.
Therefore for any $x \in X$ , $F$ is closed in $(X, T)$ with $x \notin F$ there exists $A, B \in T$, $x \in A$, $F \subset B$ and $A \cap B = \phi$.
So $(X, T)$ is regular.
Similarly we can prove this theorem for $k = ii, iii, iv$.

**Theorem 3.2.** Let $(X, t)$ be a fuzzy topological space and $(X, \tau)$ be its corresponding IFTS where $\tau = \{(\lambda, \lambda^c) : \lambda \in t\}$. Then $(X, t)$ is fuzzy regular $\Rightarrow (X, \tau)$ is IF-Regular(r-k) for any $k \in \{i, ii, iii, iv\}$. where $r \in (0,1)$.
**Proof:** Suppose $(X, t)$ is fuzzy regular. Let $x \in X$ and $(\lambda, \lambda^c)$ is closed in $(X, \tau)$ with $\lambda(x) < \lambda^c(x)$.
Consider fuzzy point $x_1$ in $X$. Since $\lambda(x) < \lambda^c(x)$, then $\lambda(x) \neq 1$. So $x_1 \notin \lambda$ and clearly $\lambda$ is closed in $(X, t)$ as $(\lambda, \lambda^c)$ is closed $(X, \tau)$.
Now since $(X, t)$ is fuzzy regular, then there exists $u, v \in t$ such that $x_1 \in u$, $\lambda \subset v$ and $u \cap v = \underline{0}$.
By the definition of $\tau$, it is clear that $(u, u^c), (v, v^c) \in \tau$.
Now $x_1 \in u$ implies $u(x) = 1, u^c(x) = 0$. Therefore $u(x) > r$, $u^c(x) < r$.
Again $\lambda \subset v$ implies $(\lambda, \lambda^c) \subset (v, v^c)$.
And finally $u \cap v = \underline{0}$ implies $(u \cap v) \subset (u^c \cup v^c)$.
Therefore $(X, \tau)$ is IF-Regular(r-i).
Similarly we can prove this theorem for $k = ii, iii, iv$.

**Theorem 3.3.** Let $(X, \tau)$ be a IFTS. Then we have the following implications


**Proof:** Suppose $(X, \tau)$ is IF-Regular (r-iv). Let $x \in X$ and closed set $F = (\mu_F, \nu_F)$ with $\mu_F(x) < \nu_F(x)$, Since $(X, \tau)$ is IF-Regular (r-iv), then there exists $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $\mu_A(x) > r, \nu_A(x) = 0$ and $F \subset B$ with $(\mu_A \cap \mu_B) = \underline{0}$ and $(\nu_A \cup \nu_B)(y) > 0$ for all $y \in X$.
Now $\nu_A(x) = 0 \Rightarrow \nu_A(x) < r$, so $(X, \tau)$ is IF-Regular (r-iii). i.e., IF-Regular r-(iv) $\Rightarrow$ IF-Regular r-(iii).
Again $(\mu_A \cap \mu_B) = \underline{0}$ and $(\nu_A \cup \nu_B)(y) > 0$ for all $y \in X \Rightarrow$ $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$. so $(X, \tau)$ is IF-Regular (r-i). i.e., IF-Regular (r-iii) $\Rightarrow$ IF-Regular (r-i).
Similarly we can show that IF-Regular (r-iv) $\Rightarrow$ IF-Regular (r-ii) $\Rightarrow$ IF-Regular (r-i).

The reverse implications are not true in general which can be seen as the following examples:

**Example 3.1**. Let $X = \{x, y\}$ and $\tau$ be an intuitionistic fuzzy topology on $X$ generated by $A = \{(x, 0.6, 0.1), (y, 0.3, 0.4)\}$, $B = \{(x, 0.2, 0.3), (y, 0.5, 0.2)\}$. Consider the closed set $F = A^c = \{(x, 0.1, 0.6), (y, 0.4, 0.3)\}$, If $r = 0.5$, then clearly $(X, \tau)$ is IF-Regular(r-i) but not IF-Regular(r-ii) and IF-Regular(r-iii).

**Example 3.2**. Let $X = \{x, y\}$ and $\tau$ be an intuitionistic fuzzy topology on $X$ generated by $A = \{(x, 0.6, 0.0), (y, 0.3, 0.4)\}$, $B = \{(x, 0.2, 0.3), (y, 0.5, 0.2)\}$. Consider the closed set $F = A^c = \{(x, 0.0, 0.6), (y, 0.4, 0.3)\}$, If $r = 0.5$, then clearly $(X, \tau)$ is IF-Regular(r-ii) but not IF-Regular(r-iv).

**Example 3.3**. Let $X = \{x, y\}$ and $\tau$ be an intuitionistic fuzzy topology on $X$ generated by $A = \{(x, 0.6, 0.0), (y, 0.3, 0.4)\}$, $B = \{(x, 0.2, 0.3), (y, 0.5, 0.2)\}$, and $C = \{(x, 0.6, 0.0), (y, 0.3, 0.4)\}$. Consider the closed set $F = C^c = \{(x, 0.0, 0.6), (y, 0.4, 0.3)\}$, If $r = 0.5$, then clearly $(X, \tau)$ is IF-Regular(r-iii) but not IF-Regular(r-iv).

**Theorem 3.4.** Let $(X, \tau)$ be a IFTS and $r, s \in (0,1)$ with $r < s$, then $(X, \tau)$ is IF-Regular(s-ii) $\Rightarrow$ $(X, \tau)$ is IF-Regular(r-ii) and $(X, \tau)$ is IF-Regular(s-iv) $\Rightarrow$ $(X, \tau)$ is IF-Regular(r-iv).

**Proof:** Let $x \in X$, $F = (\mu_F, \nu_F)$ is closed in $(X, \tau)$ with $\mu_F(x) < \nu_F(x)$. Since $(X, \tau)$ is IF-Regular(s-ii), there exists $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $\mu_A(x) > s, \nu_A(x) = 0$ and $F \subset B$ with $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$. Since $r < s$, we can write $\mu_A(y) > r, \nu_A(y) = 0$. Therefore $(X, \tau)$ is IF-Regular(r-ii).

Similarly we can show that IF-Regular(s-iv) $\Rightarrow$ IF-Regular(r-iv).

The reverse implications are not true in general which can be seen as the following examples:

**Example 3.4**. Let $X = \{x, y\}$ and $\tau$ be an intuitionistic fuzzy topology on $X$ generated by $A = \{(x, 0.6, 0.0), (y, 0.3, 0.4)\}$, $B = \{(x, 0.2, 0.3), (y, 0.5, 0.2)\}$. Consider the closet set $F = A^c = \{(x, 0.0, 0.6), (y, 0.4, 0.3)\}$. If $r = 0.5$ and $s = 0.7$ then clearly $(X, \tau)$ is IF-Regular(r-ii) but not IF-Regular(s-ii).

**Example 3.5.** Let $X = \{x, y\}$ and $\tau$ be an intuitionistic fuzzy topology on $X$ generated by $A = \{(x, 0.4, 0.0), (y, 0.0, 0.3)\}$, $B = \{(x, 0.0, 0.3), (y, 0.5, 0.2)\}$. Consider the closet set $F = A^c = \{(x, 0.0, 0.4), (y, 0.3, 0.0)\}$. If $r = 0.3$ and $s = 0.5$ then clearly $(X, \tau)$ is IF-Regular(r-ivi) but not IF-Regular(s-iv).

**Theorem 3.5.** Let $(X, \tau)$ be an intuitionistic fuzzy topological space and $U$ is a non empty sub set of $X$. Then $(X, \tau)$ is IF-Regular(r-k)$\Rightarrow$ $(U, \tau_U)$ is IF-Regular(r-k) for $k \in \{i, ii, iii, iv\}$.

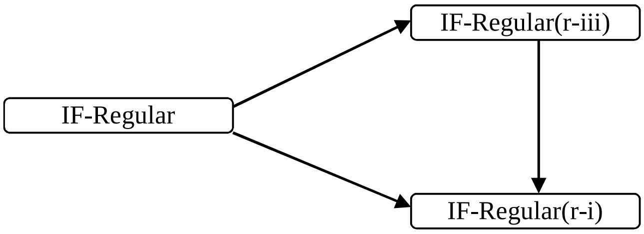
**Proof:** Suppose $(X, \tau)$ is IF-Regular(r-i).

Let $x \in U$, $F|U = (\mu_F|U, \nu_F|U)$ is closed in $(U, \tau_U)$ with $(\mu_F|U)(x) < (\nu_F|U)(x)$.

Clearly $x \in X$, $F = (\mu_F, \nu_F)$ is closed in $(X, \tau)$ with $\mu_F(x) < \nu_F(x)$ as $(U, \tau_U)$ is sub space of $(X, \tau)$.

Since $(X,\tau)$ is IF-Regular(r-i), then there exists $A=(\mu_A,\nu_A)$, $B=(\mu_B,\nu_B)\in\tau$ such that $\mu_A(x)>r,\nu_A(x)<r$, $F\subset B$ with $(\mu_A\cap\mu_B)\subset(\nu_A\cup\nu_B)$.
Since $(U,\tau_U)$ is sub space of $(X,\tau)$, then clearly $A|U=(\mu_A|U,\ \nu_A|U)\in\tau_U$, $B|U=(\mu_B|U,\ \nu_B|U)\in\tau_U$ such that $(\mu_A|U)(x)>r,(\nu_A|U)(x)<r$, $(F|U)\subset(B|U)$ with $((\mu_A|U)\cap(\mu_B|U))\subset((\nu_A|U)\cup(\nu_B|U))$.
Therefore $(U,\tau_U)$ is IF-Regular(r-i).
Similarly we can prove this theorem for $k=ii,iii,iv$.
Hence the properties IF-Regular(r-k), $k$ = $i$, $ii$, $iii$, $iv$ are hereditary.

**Theorem 3.6** . Let $(X,\tau)$ be an IFTS, then we have the following inplications where $r\in(0,1)$.




**Proof:** Suppose $(X,\tau)$ be IF-Regular. Let $x\in X$ and IFCS $F=(\mu_F,\nu_F)$ with $\mu_F(x)<\nu_F(x)$. Consider the IFP $x_{(r,r)}$. Since $\mu_F(x)<\nu_F(x)$, it is clear that $r<\mu_F(x)$ and $r>\nu_F(x)$ is impossible. So $x_{(r,r)}\notin F=(\mu_F,\nu_F)$. Since $(X,\tau)$ is IF-Regular, there exists $A=(\mu_A,\nu_A),B=(\mu_B,\nu_B)\in\tau$ such that $x_{(r,r)}\in A$, $F\subset B$ and $A\cap B=0_\sim$.
Now $x_{(r,r)}\in A\Rightarrow\ r<\mu_A(x)$ and $r>\nu_A(x)$. i.e., $\mu_A(x)>r,\nu_A(x)<r$.
Again since $A\cap B=0_\sim$, then $\mu_A\cap\mu_B=\underline{0}$ and $\nu_A\cup\nu_B=\underline{1}$. i.e., $(\mu_A\cap\mu_B)\subset(\nu_A\cup\nu_B)$.
Hence $(X,\tau)$ is IF-Regular(r-i).
Again suppose $(X,\tau)$ be IF-Regular. Let $x\in X$ and IFCS $F=(\mu_F,\nu_F)$ with $\mu_F(x)<\nu_F(x)$. Consider the IFP $x_{(r,r)}$. Since $\mu_F(x)<\nu_F(x)$, it is clear that $r<\mu_F(x)$ and $r>\nu_F(x)$ is impossible. So $x_{(r,r)}\notin F=(\mu_F,\nu_F)$. Since $(X,\tau)$ is IF-Regular, there exists $A=(\mu_A,\nu_A)$, $B=(\mu_B,\nu_B)\in\tau$ such that $x_{(r,r)}\in A$, $F\subset B$ and $A\cap B=0_\sim$.
Now $x_{(r,r)}\in A\Rightarrow\ r<\mu_A(x)$ and $r>\nu_A(x)$. i.e., $\mu_A(x)>r,\nu_A(x)<r$.
Again since $A\cap B=0_\sim$, then $\mu_A\cap\mu_B=\underline{0}$ and $\nu_A\cup\nu_B=\underline{1}$.
Now science $\nu_A\cup\nu_B=\underline{1}$, so for all $y\in X$, $(\nu_A\cup\nu_B)(y)=1$ i.e., $(\nu_A\cup\nu_B)(y)>0$
Hence $(X,\tau)$ is IF-Regular(r-iii).
The remaining implication may be proved as Theorem 3.3.

**Theorem 3.7.** Let $(X,\tau)$ and $(Y,\delta)$ be IFTSs and $f:X\to Y$ is one-one, open and continuous. Then $(Y,\delta)$ is IF-Regular(r-k)$\Rightarrow$ $(X,\tau)$ is IF- Regular (r-k) for $k\in\{i,ii,iii,iv\}$.
**Proof:** Suppose $(Y,\delta)$ is IF- Regular(r-i). Let $x\in X$, $F=(\mu_F,\nu_F)$ is closed in $(X,\tau)$ with $\mu_F(x)<\nu_F(x)$. Now $f(x)=y\in Y$ and $f(F)=(f(\mu_F),f(\nu_F))$ is closed in $(Y,\delta)$ as $f$ is open. Since $f$ is one-one then$f^{-1}(y)=\{x\}$
Now $f(\mu_F)(y)=\sup_{p\in f^{-1}(y)}\mu_F(p)=\ \mu_F(x)$ And $(\nu_F)(y)=\ inf_{p\in f^{-1}(y)}\nu_F(p)=\nu_F(x)$

Therefore $f(\mu_F)(y) < f(\nu_F)(y)$ as $\mu_F(x) < \nu_F(x)$,
Since $(Y,\delta)$ is IF- Regular(r-i). there exists $A = (\mu_A,\nu_A), B = (\mu_B,\nu_B) \in \delta$ such that $\mu_A(y) > r, \nu_A(y) < r$ and $f(F) \subset B$ with $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$.
Since $f$ is continuous, then $f^{-1}(A)$=( $f^{-1}(\mu_A), f^{-1}(\nu_A)) \in \tau$ and
$f^{-1}(B)$=( $f^{-1}(\mu_B), f^{-1}(\nu_B)) \in \tau$
Now we have $f^{-1}(\mu_A)(x) = \mu_A\big(f(x)\big) = \mu_A(y) > r$, $f^{-1}(\nu_A)(x) = \nu_A(f(x)) = \nu_A(y) < r$.
And $f(F) \subset B \Rightarrow f^{-1}(f(F)) \subset f^{-1}(B)$. But $F \subset f^{-1}(f(F))$ [5].
So $F \subset f^{-1}(B)$.
Finally for any $z \in X$ we get
$\big(f^{-1}(\mu_A) \cap f^{-1}(\mu_B)\big)(z) = min\big(f^{-1}(\mu_A)(z), f^{-1}(\mu_B)(z)\big) = min\big(\mu_A(f(z)), \mu_B(f(z)\big) = (\mu_A \cap \mu_B)f(z)$ and
$$\big(f^{-1}(\nu_A) \cup f^{-1}(\nu_B)\big)(z) = max\big(f^{-1}(\nu_A)(z), f^{-1}(\nu_B)(z)\big) = max\big(\nu_A(f(z)), \nu_B(f(z)\big) = (\nu_A \cup \nu_B)f(z)$$
But $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$, therefore $(\mu_A \cap \mu_B)f(z) < (\nu_A \cup \nu_B)f(z)$
So $\big(f^{-1}(\mu_A) \cap f^{-1}(\mu_B)\big)(z) < \big(f^{-1}(\nu_A) \cup f^{-1}(\nu_B)\big)(z)$
i.e., $\big(f^{-1}(\mu_A) \cap f^{-1}(\mu_B)\big) \subset \big(f^{-1}(\nu_A) \cup f^{-1}(\nu_B)\big)$.
Therefore $(X,\tau)$ is IF- Regular (r-i).
Similarly we can prove this theorem for $k = ii, iii, iv$.

**Theorem 3.8.** Let $(X,\tau)$ and $(Y,\delta)$ be IFTSs and $f: X \to Y$ is one-one, onto, continuous and open. Then $(X,\tau)$ is IF-Regular(r-k) $\Rightarrow$ $(Y,\delta)$ is IF-Regular(r-k) for $k \in \{i, ii, iii, iv\}$.
**Proof:** Suppose $(X,\tau)$ is IF- Regular(r-i). Let $y \in Y$ and $F = (\mu_F,\nu_F)$ is closed in $(Y,\delta)$ with $\mu_F(y) < \nu_F(y)$.
Since $f$ is one-one and onto, then there exists a unique $x \in X$ such that $f(x) = y$. i.e., $f^{-1}(y) = \{x\}$.
Clearly $f^{-1}(F) = (f^{-1}(\mu_F), f^{-1}(\nu_F))$ is closed in $(X,\tau)$ as $f$ is continuous.
Now $f^{-1}(\mu_F)(x) = \mu_F\big(f(x)\big) = \mu_F(y) < \nu_F(y) = \nu_F(f(x)) = f^{-1}(\nu_F)(x)$ ,
i.e., $f^{-1}(\mu_F)(x) < f^{-1}(\nu_F)(x)$.
Since $(X,\tau)$ is IF- Regular(r-i), then there exists s $A = (\mu_A,\nu_A), B = (\mu_B,\nu_B) \in \tau$ such that $\mu_A(x) > r, \nu_A(x) < r$ and $f^{-1}(F) \subset B$ with $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$.
Clearly $f(A) = (f(\mu_A), f(\nu_A)) \in \delta$, $f(B) = (f(\mu_B), f(\nu_B)) \in \delta$ as $f$ is open.
Now $f(\mu_A)(y) = \sup_{p\in f^{-1}(y)} \mu_A(p) = \mu_A(x) > r$ and
$(\nu_A)(y) = \inf_{p\in f^{-1}(y)} \nu_A(p) = \nu_A(x) < r$.
And $f^{-1}(F) \subset B \Rightarrow f(f^{-1}(F)) \subset f(B)$. But $f(f^{-1}(F)) = F$ as $f$ is onto [5].
so $F \subset f(B)$.
Finally, for any $z \in Y$, there exists a unique $w \in X$ such that $f(w) = z$. i.e., $f^{-1}(z) = \{w\}$,
Now $\big(f(\mu_A) \cap f(\mu_B)\big)(z)$=$min(f(\mu_A)(z), f(\mu_B)(z))$
$$= min\Big(\sup_{p\in f^{-1}(z)} \mu_A(p), \sup_{p\in f^{-1}(z)} \mu_B(p)\Big) = min\big(\mu_A(w), \mu_B(w)\big) = (\mu_A \cap \mu_B)(w)$$
And $\big(f(\nu_A) \cup f(\nu_B)\big)(z) = max(f(\nu_A)(z), f(\nu_B)(z)) =$
$max\Big(\inf_{p\in f^{-1}(z)} \nu_A(p), \inf_{p\in f^{-1}(z)} \nu_B(p)\Big) = max\big(\nu_A(w), \nu_B(w)\big) = (\nu_A \cup \nu_B)(w)$

But $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B) \Rightarrow (\mu_A \cap \mu_B(w) < (\nu_A \cup \nu_B)(w) \Rightarrow \big(f(\mu_A) \cap f(\mu_B)\big)(z) < \big(f(\nu_A) \cup f(\nu_B)\big)(z)$
i.e., $\big(f(\mu_A) \cap f(\mu_B)\big) \subset \big(f(\nu_A) \cup f(\nu_B)\big)$
Therefore $(Y, \delta)$ is IF-Regular(r-i).
Similarly we can prove this theorem for $k = ii, iii, iv$.

**Corollary 3.1.** If $(X, \tau)$ and $(Y, \delta)$ are IFTSs and $f: X \to Y$ is a homeomorphism then $(X, \tau)$ is IF-Regular(r-k) if and only if $(Y, \delta)$ is IF-Regular(r-k) for $k \in \{i, ii, iii, iv\}$.

**Remark 3.1.** IF-Regular(r-k) for $k \in \{i, ii, iii, iv\}$ are topological property.

## 4. Conclusion

In this article, we have given four new ideas of intuitionistic fuzzy regular space and established some relationships among them. We have observed from theorem 3.1 and theorem 3.2 that our notions are well defined. Also from theorem 3.5 we say that the notions are hereditary. We observed from theorem 3.3 and theorem 3.6 that our notion IF-Regular(r-i) is most general among all given notions of intuitionistic fuzzy regular topological spaces. Finally, we have observed that our notions convey topological property in the sense of intuitionistic fuzzy regular space.

## REFERENCES


1. C.L.Chang, Fuzzy topological space, *J. of Mathematical Analysis and Applications*, 24 (1968) 182-90.
2. L.A.Zadeh, Fuzzy sets, *Information and Control*, 8(3) (1965) 338-353.
3. K.T.Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20(1) (1986) 87-96.
4. D.Coker, An introduction to fuzzy subspace in intuitionistic fuzzy topological space, *Journal of Fuzzy Mathematics*, 4 (1996) 749-764.
5. D.Coker, An introduction to intuitionistic fuzzy topological spaces, *Fuzzy Sets and Systems*, 88(1) (1997) 81-89.
6. S.Bayhan and D.Coker, On fuzzy separation axioms in intuitionistic fuzzy topological space, *BUSEFAL*, 67 (1996) 77-87.
7. D.Coker and S.Bayhan, On separation axioms in intuitionistic topological space, *International Journal of Mathematics and Mathematical Sciences*, 27(10) (2001) 621-630.
8. D.Coker and S.Bayhan, On T1 and T2 separation axioms in intuitionistic fuzzy topological space, *Journal of Fuzzy Mathematics*, 11(3) (2003) 581-592.
9. A.K.Singh and R.Srivastava, Separationaxioms in intuitionistic fuzzy topological spaces, *Advances in Fuzzy Systems*, 2012 (2012) 1-7.
10. S.J.Lee and E.P.Lee, The category of intuitionistic fuzzy topological space, *Bull. Korean Math. Soc.*, 37(1) (2000) 63-76.
11. R.Saadati and J.H.Park, On the intuitionistic fuzzy topological spaces, *Chos, Solitons Fract.*, 27 (2006) 331-344.

12. E.Ahmed, M.S.Hossain and D.M.Ali, On Intuitionistic Fuzzy T0 Spaces, *Journal of Bangladesh Academy of Sciences*, 38(2) (2014) 197-207.
13. E.Ahmed, M.S.Hossain, and D.M.Ali, On intuitionistic fuzzy R0 spaces, *Annals of Pure and Applied Mathematics*, 10(1) 2015.
14. E.Ahmed, M.S.Hossain and D.M.Ali, On intuitionistic fuzzy R1 spaces, *J. Math. Comput. Sci*, 5(5) (2015) 681-693.
15. E.Ahmed, M.S.Hossain and D.M.Ali, On Intuitionistic Fuzzy T1 Spaces, *Journal of Physical Sciences*, 19 (2014) 59-66.
16. E.Ahmed, M.S.Hossain and D.M.Ali, On Intuitionistic Fuzzy T2 Spaces, *IOSR Journal of Mathematics*, 10(6) (2014) 26-30.
17. Francisco Gallego Lupianez, Separation in intiutionistic fuzzy topological spaces, *International Journal of Pure and Applied Mathematics*, 17(1) (2004) 29-34.
18. M.A.Mahbub, M.S.Hossain and M.A.Hossain, Connectedness concept in intuitionistic fuzzy topological spaces; *Notes on Intuitionistic Fuzzy Sets.*, 27(1) (2021) 72-82.
19. M.A.Mahbub, M.S.Hossain and M.A.Hossain, Separation axioms in intuitionistic fuzzy compact topological spaces; *Journal of Fuzzy Set Valued Analysis*, 2019(1) (2020) 14-23.
20. M.A.Mahbub, M.S.Hossain and M.A.Hossain, Some properties of compactness in intuitionistic fuzzy topological spaces; *Intern. J. Fuzzy Mathematical Archive*, 16(2) (2018) 39-48.
21. M.Alamgir Hossain and M.S.Hossain, Mathematical approach to identify Coronavirus disease (COVID-19) using fuzzy logic inference system; *Intern. J. Fuzzy Mathematical Archive*, 19(1) (2021) 37-47.
22. M.A.Islam and M.S.Hossain, The development of fuzzy logic controller (FLC) in air conditioning system using several types of fuzzy numbers; *Intern. J. Fuzzy Mathematical Archive*, 19(1) (2021) 1-24.
23. M.S.Islam, M.S.Hossain and M.Asaduzzaman, Level separation on intuitionistic fuzzy $T_0$ spaces, *Intern. J. Fuzzy Mathematical Archive*, 13(2) (2017) 123-133**.**
24. M.S.Islam, M.S.Hossain and M.Asaduzzaman, Level separation on intuitionistic fuzzy $T_1$ spaces, *J. Bangladesh Academy of Science*, 42(1) (2018) 73-85.
25. M.S.Islam, M.S.Hossain and M.Asaduzzaman, Level separation on Intuitionistic fuzzy $T_2$ spaces, *J. Math. Comput. Sci.*, 8(3) (2018) 353-372.