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Solving Fuzzy Linear Programming Problem With Fuzzy Relational Equation Constraint

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Abstract. A linear programming problem with minimum linear objective function subject to a system of fuzzy relation equations using max-min product is considered. Some procedures to reduce the problem are established and illustrated by a numerical example.

1. Introduction
Fuzzy Relations Equations (FRE) were introduced and applied to diagnosis problems in [5,6], fuzzy relation equations on a finite set were later considered in [1] and structure of the set of solutions of such equations was studied.

The structure of solution sets of FRE with different composition operators was given in 1980s, see for instance [2,4,5]. It is well known now that the solution set of finite FRE with continuous max t-norm is determined by one maximum solution and finite number of minimal solutions. However, it is not easy to obtain all minimal solutions for a large scale problem because number of minimal solutions may increase very sharply as of the problem size increases.

In recent several years, there are still many results on developing a more effective algorithm for obtaining all minimal solutions of FRE and there has been a growing interest in a class of minimizing problems with fuzzy relational equation constraints.

We attempt to find an optimal solution to the fuzzy linear programming problem with fuzzy relational constraints which is illustrated by numerical example.

2. A necessary and sufficient condition for existence of solution definition 2.1
Consider three fuzzy binary relations P(X,Y), Q (Y,Z) and R(X,Z), defined on the sets, X = $\{x_i$ / $i \in I\}$; Y = $\{y_j$ / $j \in J\}$; Z=$\{z_k$ / $k \in K\}$ with I = N_n, J = N_m, and K=N_s. Let the membership matrices of P,Q and R be denoted by P=[p_{ij}], Q=[q_{jk}], R=[r_{ik}] respectively.

Let $p_{ij} = P(x_i, y_j)$, $q_{jk} = Q(y_j, z_k)$, $r_{ik} = R(x_i, z_k)$ This means that all entries in the matrices P,Q and R are real numbers in the unit interval [0,1].

Assume now that the three relations constrained with each other in such a way that

$$P_oQ=R \qquad (1)$$

where 'o' denotes the max-min composition. This means that

$$\max_{j\in J} \min(p_{ij}, q_{jk}) = r_{ik} \qquad (2)$$

for all and $i \in I$ and $k \in k$. That is, the matrix equation (3) encompasses n x s simultaneous equations of the form (4). these equations are referred to as fuzzy relation equations. The set of all particular matrices of the form P that satisfy (1) is called its solution and denote the set of all solutions as

$$S(Q, R) = \{P / P_oQ = R\} \qquad (3)$$

Necessary condition for existence of solutions, consider equation (2)

ie., $\max_{j\in J} \min(p_{ij}, q_{ik}) = r_{ik}$ and if $\max_{j\in J} q_{jk} < r_{ik}$

2.2 Solution Method

Consider matrix equations (1) of the simpler form $p_oQ = r$ where $p = [p_j / j \in J]$, $Q=[q_{jk} / j \in J, k \in k]$, $r = [r_k / k \in k]$ ie., p,Q and r represent a fuzzy set on y, q fuzzy relation on $Y \times Z$, and a fuzzy set on Z respectively. In our discussion the constraint equation is $x \, o \, A = b$ where $x=\{x_i / i \in I\}$, $A = \{a_{ij} / i \in I, j \in J\}$, $b = \{b_j / j \in J\}$

Denote the solution set of $p \, o \, Q = x \, o \, A = b = r$ as $X(A, B) = \{x / x \, o \, A = b\}$

When $X(A,b) \neq \phi$, the maximum solution $\bar{x} = [\bar{x}_i / i \in I]$ of (1) – (2) is determined as follows: $\bar{x}_i = \min_{j\in J} \sigma(a_{ij}, b_j)$ $\qquad (4)$

where $\sigma(a_{ij}, b_j) = \begin{cases} b_j & if \quad a_{ij} > b_j \\ i & otherwise \end{cases}$

When $\bar{x}$ determined in this way does not satisfy $x o A = b$, then $X(A,b) = \phi$.

Assume now that A and b of $xoA = b$ are given, and that we have to determine the set $\underline{X}(A,b)$ of all minimal solutions of the equation. Assume further that $\bar{x}$ has been determined by (6) and has been verified as the maximum solution. When $\bar{x}_i = 0$ for some $i \in I$, we may eliminate this component form $\bar{p}$ as well as i^{th} row from matrix A, clearly, $\bar{x}_i = 0$ implies $x_i = 0$ for each $x \in X(A,b)$. Furthermore, when $b_j = 0$ for some $j \in J$, We may eliminate this component from b and the jth column from matrix A, Since each $x \leq \bar{x}$ $(x \in \rho)$ must satisfy, in this case, the max – min equation represented by x, the jth column of Q, and $b_j = 0$.

3. A necessary condition for optimal solution

This section recalls some preliminaries of t-norms. Some properties of max-t- norm fuzzy relational equations are presented.

Definition 3.1. A triangular norm is a binary operation T on the unit interval [0.1] which

T: $[0,1]^2 \longrightarrow [0,1]$ such that for all $\alpha, \beta, \delta \in [0,1,]$

(a) $T(\alpha, \beta) = T(\beta, \alpha)$

(b) $T(\alpha, T(\beta, \delta)) = T(T(\alpha, \beta), \delta)$

(c) $T(\alpha, \beta) \leq T(\alpha, \delta)$ whenever $\beta \leq \delta$, and

(d) $T(\alpha, 1) = \alpha$.

Definition 3.2. The fuzzy linear programming problem with fuzzy relational equation constraint is defined as

$$\text{Minimize } Z(x) = \sum_{i=1}^{m} c_i x_i , \ i = 1,2....m \in I \tag{5}$$

subject to $x \in X(A,b) = \{x \in [0,1]^m / xoA = b\}$ (6)

where $c_i \in R$ is the coefficient associated with variable x_i; A=[a_{ij}] is an mxn non-negative matrix with $a_{ij} \leq 1$; $b=(b_1, b_2, ... b_n)$ is an n-dimensional vector with $0 < b_j \leq 1$, $j=1,2,...n \in J$ and the operation "o" represents the max- min composition (or the max $-$ T composition operator.

Lemma 3.3. If $x \in X(A,b)$, then for each $j \in J$ there exists $i_o \in I$ such that min (x_{io}, a_{ioj})=b_j and min (x_i, a_{ij}) $\leq$ b_j for every $i \in I$

Proof. Since xo A=b, then

$$\max_{i \in I} \{\min(x_i, a_{ij})\} = b_j \text{ for } j \in J \tag{7}$$

That means for each $j \in J$, min (x_i, a_{ij}) $\leq$ b_j. And, in order to satisfy the equality constraint, there must exist at least one $i_o \in I$ such that min (x_{io}, a_{ioj})=b_j

Definition 3.4. For a solution $x \in X(A,b)$, we call x_{io} a binding variable if min (x_{io}, a_{ioj}) = b_j for $i_o \in I$ and min (x_i,a_{ij}) $\leq$ b_j for all $i \in I$.

Lemma 3.5. Let $\bar{x} = (\bar{x}_i)_{i \in I}$ be the maximum solution, and $x = (x_i)_{i \in I}$ be a solution of (6). If x_i is binding in the jth equation, then $\bar{x}_i$ is also binding. However, if $\bar{x}_i$ is not a binding variable, x_i is also non binding for any solution x.

Proof. For any solution $x = (x_i)_{i \in I} \in X(A,b)$ we have

$$\max_{i \in I} \{\min(x_i, a_{ij}) = b_j, \forall j \in J$$

This implies min ($\bar{x}_i, a_{ij}$) $\leq b_j, \forall j \in J$. If x_i is now binding in the jth equation, then min(x_i,a_{ij})=b_j. also b_j=min(x_i,a_{ij}) $\leq$ min ($\bar{x}_i, a_{ij}$) $\leq b_j$ This implies that min ($\bar{x}_i, a_{ij}$) = b_j.

Hence $\bar{x}_i$ is also binding in the jth equation. On the other hand, if $\bar{x}_i$ is not binding in any equation then min ($\bar{x}_i, a_{ij}$) $\leq b_j$ holds for any solution x, we have min

Lemma 3.6. Let $\bar{x} = (\bar{x}_i)_{i \in I}$ be the maximum solution. If the cost co-efficient $c_i \leq 0$, $\forall i \in I$, then $\bar{x}$ is an optimal solution of problem.

Proof. For any solution x in X (A,b) we have $0 \leq x \leq \bar{x}$ Since $c_i \leq 0$, $\forall i \in I$ we have

$$\sum_{i=1}^{m} c_i x_i \geq \sum_{i=1}^{m} c_i \bar{x}_i$$

(8)

Therefore, $\bar{x}$ is an optimal solution.

4. Separation of fuzzy LPP

In this section, we study how to separate problem(5)-(6)into two sub – problems; and how to yield an optimal solution from the maximum solution and one of the minimal solutions.

4.1. Two sub–problems of model (5) – (6)

Fang et al [3] showed that an optimal solution for model (5) – (6) with max – min or max – product composition can be obtained from two sub- problems, which are formed by separating the negative and non negative coefficient in the objective function. Consider the following two problems.

$$\text{Minimize } Z^1(x) = \sum_{i=1}^{m} c_i^1 \, x_i$$

subject to $x \in X(A,b) = \left\{ x \in [0,1]^m \,/\, xoA = b \right\}$

(9)

and $\quad \text{Minimize } Z^2(x) = \sum_{i=1}^{m} c_i^2 x_i^2$

subject to $x \in X(A,b) = \left\{ x \in [0,1]^m \,/\, xoA = b \right\}$

(10)

where

$$c_i^1 = \begin{cases} c_i \text{ if } c_i < 0 \\ 0 \text{ if } c_i \geq 0 \end{cases} \quad and \quad c_i^2 = \begin{cases} o \text{ if } c_i < 0 \\ c_i \text{ if } c_i \geq 0 \end{cases} \quad \forall i \in I$$

problems (9) and (10)are subjected to the original constraint. Furthermore, $c_i = c_i^1 + c_i^2, \forall i \in I$. The maximum solution $\bar{x} = (\bar{x}_i)_{i \in I}$ is an optimal solution for problem (9) with optimal value $Z^1(\bar{x})$. Additionally, one minimal solution, say $\underline{x}^* = (\underline{x}_i^*)_{i \in I}$ is an optimal solution for problem (10) with optimal value $Z^2(\underline{x}^*)$. A new vector $x^* = (x_i^*)_{i \in I}$ is now defined by.

$$x_i^* = \begin{cases} \bar{x}_i & if \quad c_i < 0 \\ \underline{x}_i^* & if \quad c_i \geq 0 \end{cases} \quad \forall i \in I$$

(11)

It follows that $\underline{x}^* \leq x^* \leq \bar{x}$. Hence x^* is a solution of equation (6) with objective value $Z(x^*) = z^1(\bar{x}) + Z^2(\underline{x}^*)$.

The remaining task is to show that x^* as defined in (11) is an optimal solution of the original problem (5)- (6) with optimal value $Z(x^*)$

This can be seen from the following inequalities:

$$z(x) = \sum_{i=1}^{m} c_i x_i = \sum_{i=1}^{m} (c_i^1 + c_i^2) \, x_i = \sum_{i=1}^{m} c_i^1 x_i + \sum_{i=1}^{m} c_i^2 x_i$$

$$\geq \sum_{i=1}^{m} c_i^1 \bar{x}_i + \sum_{i=1}^{m} c_i^2 x_i \geq \sum_{i=1}^{m} c_i^1 \, \bar{x}_i + \sum_{i=1}^{m} c_i^2 \, \underline{x}_i^*$$

$$\geq \sum_{i=1}^{m} c_i^1 \, \bar{x}_i + \sum_{i=1}^{m} c_i^2 \, \underline{x}_i^* = z^1(\bar{x}) + Z^2(\underline{x}^*) = Z(x^*)$$

4.2. An equivalent 0-1 integer programming problem.

The following index sets are defined to find a minimal solution from X(A,b) to optimize problem (10).

$$I_j = \left(i \in I \, / \min (\bar{x}_i, a_{ij}) = b_j \right), \forall \, j \in J$$

$$J_i = \left\{ j \in J \, / \min(\bar{x}_i, a_{ij}) = b_j \right\} \forall \, i \in I$$

The index set I_j indicates the possible variables of x that may be selected as a binding variable in the jth equation. The index set J_i indicates those equations that are satisfied by $\bar{x}_i$. The following variables are defined to find minimal solution from X(A,b) to solve problem (10).

$$y_{ij} = \begin{cases} 1 & if \ i \in I_j \\ 0 & otherwise \end{cases} \qquad \forall i \in I, \forall j \in J$$

Notably, the variable $y_{ij}=1$ corresponds to a possible selection of the ith component of some minimal solutions that are binding in the jth equation. Since each solution must satisfy all equations, a minimal solution can be transformed into the selection of one variable with value 1 in each equation.

The 0-1 integer programming problem, which is equivalent to problem (10)

$$\text{Minimize } Z^2(x) = \sum_{i=1}^{m} (c_i^2 \ \max_{j \in J} \{ b_J y_{iJ} \})$$

subject to $\sum_{i=1}^{m} y_{ij} = 1, \quad \forall j \in J$ $\qquad\qquad$ (12)

$$y_{ij} = 0 \ or \ 1 , \quad \forall i \in I, j \in J$$

$$y_{ij} = 0 \qquad \forall i, j \notin I_j$$

Therefore, the objective function becomes

$Z^2(x) = \sum_{i=1}^{m} c_i^2 \max_{i \in I_j} \{ \bar{x}_i, y_{ij} \}$ Moreover, only those indices in J_i need to be considered.

$\sum_{i \in I_j} y_{ij} = 1, \quad \forall j \in J$. The 0-1 integer programming model for problem (10) is presented as follows:

$$\text{Minimize} \quad Z^2(x) = \sum_{i=1}^{m} \left(c_i^2 \max_{j \in J_i} \{ \overline{x}_i y_{ij} \} \right)$$

$$\text{subject to} \quad \sum_{i \in I_j} y_{ij} = 1, \quad \forall \; j \in J$$

$$y_{ij} = 0 \quad or \; 1, \quad \forall i \in I, \; j \in J$$

$$y_{ij} = 0 \; \forall \; i, j \; with \quad i \notin I_j$$

Any optimal solution $y^* = (y_{ij}^*)_{i \in I, j \in J}$ containing the variable $y_{ij}^* = 1$ in problem (12) corresponds to the situation where the variable x_i^* is binding in the jth equation.

5. Rules for reducing problem (10)

Consider the given matrix $A = (a_{ij})$ with $i \in I$ and $j \in J$. To develop a procedure of finding an optimal solution, the following index set are given for the value matrix $I_j(A) = \{ i \in I / \min(x_i, a_{ij}) = b_j \}$. This index set contains $i \in I$ such that x_i can be satisfied the jth equation.

Rule 1. If a singleton $I_j = \{i\}$ exists from some $j \in J$, then $\overline{x}_i$ is assigned to the ith component of any optimal solution

Proof. The index set $I_j(A) = \{i\}$ implies that the jth equation only can be satisfied by variable x_i. This implies that the ith component of any solution (hence, the variable x_i) must be binding in the jth equation, yields $x_i = \overline{x}_i$ (since $b_j > 0$)   Based on Rule 1, the jth column of M can be deleted from further consideration. The corresponding row of x_i in A can also be deleted.

Rule 2. If $Ip(A) \subseteq I_q(A)$ for some $p, q \in J$ in the value matrix A, then the qth column of M can be deleted.

Proof. Rule 2 reveals that if a singleton $I_j(A) = \{i\}$ exists for some $j \in J$ in the matrix A, and $I_j(A) \subseteq I_q(A)$ for some $q \in J$ then the qth equation can be deleted. Furthermore, the deletion can be performed when $I_j(A)$ is not a singleton.

Lemma 5.1. If $X(A,b) \neq \phi, \forall j \in J$ then $I_j \neq \phi, \forall j \in J$

Proof. From lemma 3.3, we know that there exists at least one $i_0 \in I$ that can satisfy constraint j, therefore I_j must contain at least one element.

Rule 3. If p,q,r $\in I_j$, $j \in J$ and does not belongs to any other $I_t, t \in J$ and $t \neq j$ such that $c_p \bar{x}_p > c_q \bar{x}_q > c_r \bar{x}_r$ then any optimal solution $x^* = (x_i^*)_{i \in I}$ has $x_p^* = x_q^* = 0$

Proof. Given that p,q,r $\in I_j$. This implies x_p, x_q, x_r satisfied the jth equation, they does not satisfy any other equation .Also since $c_p \bar{x}_p > c_q \bar{x}_q > c_r \bar{x}_r$ $\therefore$ In order to satisfy the jth equation, we need only one variable with minimal cost – coefficient. $\therefore$ We set $x_p^* = x_q^* = 0$.

If Rule 3 is applied in the process of finding an optimal solution then the rows of matrix A that are associated with x_p, x_q can be deleted.

Rule 4. During the process of finding an optimal solution. If x_s is an undecided decision variable such that $s \notin I_j, \forall j \in J$ then any optimal solution $x^* = (x_i^*)_{i \in I}$ has $x_s^* = 0$

Proof. Since x_s is an undedicated decision variable and $s \notin I_j, \forall j \in J$ .This implies x_s does not satisfy any equation. $\therefore$ any optimal solution $x^* = (x_i^*)_{i \in I}$ has $x_s^* = 0$ .If rule 4 is applied then the corresponding rows of A can be deleted.

5.1. An algorithm

Based on the concepts discussed before, we present an algorithm for finding an optimal solution .

Step 1. Check the necessary condition for existence of solutions.

If $\max_{j \in J} a_{ij} > b_j \forall j \in J$, continue, otherwise stop, $x(A,b) = \phi$ and problem (1) – (2) has no solution.

Step 2. Compute the vector $\bar{x} = (\bar{x}_i)_{i \in I}$ by (4).

Step 3. Check the consistency by verifying whether $\bar{x}oA = b$ stop in case of inconsistency (If consistent, then $\bar{x} = (\bar{x}_i)_{i \in I}$ is the maximum solution)

Step 4. Form two sub problems as problems (9) and (10)

Step 5. Find optimal solution for problem (10)

Step 5.1. Compute index set I_j (A) for all $j \in J$ for the given value matrix A.

Step 5.2. Apply Rules 1-4 to determine the values of decision variables as many as possible. Delete the corresponding rows and or columns in A (Thus reducing the size of the problem) Denote the reduced sub matrix by A again. If all decision variables have been set, then go to step 6.

Step 5.3. Take the (remaining) value matrix A. Employ the branch and bound method to solve for the remaining undecided decision variables.

Step 6. Generate optimal solutions for the original problem from optimal solutions of problems (9) and (10) by (11).

5.2. Numerical example

Consider the following optimization problem subject to fuzzy relations equation with

Minimize $z(x) = 5.5x_1 + 2x_2 + 5.2x_3 - 0.5x_4 + x_5 + 1.5x_6 + 2x_7 + 3x_8 + 1.25x_9$

subject to $xoA = b$

$$0 \le x_i \le 1 \qquad\qquad i = 1,2,3\ldots\ldots\ldots9$$

where $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$

	1	2	3	4	5	6	7	8	9
x_1	0.23	0.75	0.43	0.70	0.65	0.42	0.82	0.35	0.45
x_2	0.56	0.90	0.56	0.72	0.92	0.43	0.61	0.68	0.46
x_3	0.71	0.76	0.56	0.45	0.72	0.40	0.67	0.43	0.48
x_4	0.62	0.32	0.57	0.54	0.61	0.20	0.65	0.76	0.42
x_5	0.80	0.95	0.81	0.70	0.53	0.42	0.80	0.40	0.38
x_6	0.93	0.61	0.59	0.90	0.78	0.80	0.63	0.55	0.45
x_7	0.55	0.49	0.80	0.34	0.82	0.33	0.54	0.45	0.43
x_8	0.55	0.64	0.56	0.52	0.62	0.42	0.76	0.25	0.32
x_9	0.38	0.70	0.47	0.52	0.73	0.26	0.64	0.48	0.22

$A =$ (the matrix above)

$b = (0.55, 0.70, 0.56, 0.52, 0.72, 0.42, 0.64, 0.48, 0.45)$

Step 1. Since $\max\limits_{j \in J} a_{ij} > b_j \ \forall j \in J$, the necessary condition is satisfied.

Step 2. Compute the vector $\bar{x}$,

$$\bar{x} = \{0.52, 0.42, 0.45, 0.48, 0.52, 0.42, 0.56, 0.64, 0.72\}$$

Step 3. Since $\bar{x}oA = b$

i.e., Max $(\min\limits_{i \in I} (a_{ij}, x_i)) = b_j, \forall j \in J$

The problem is solvable and $\bar{x}$ is the maximum solution.

Step 4. Form two sub problems as problems (9) and (10) the following sub – problem P1, is given as problem (9) with negative co-efficient in the objective function.

P1: Minimize $Z^1(x) = -0.5x_4$

subject to $xoA = b$

The other sub problem, P2, is given as problem (10) with non negative coefficients in the objective function.

P2:

Minimize $z^2(x) = 5.5x_1 + 2x_2 + 5.2x_3 + 0x_4 + x_5 + 1.5x_6 + 2x_7 + 3x_8 + 1.25x_9$

subject to $xoA = b$

Step 5. Find optimal solution for problem (10)

Consider the given matrix.

$$A=\begin{array}{c|ccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
x_1 & 0.23 & 0.75 & 0.43 & 0.70^* & 0.65 & 0.42^* & 0.82 & 0.35 & 0.45^* \\
x_2 & 0.56 & 0.90 & 0.56 & 0.72 & 0.92 & 0.43^* & 0.61 & 0.68 & 0.46 \\
x_3 & 0.71 & 0.76 & 0.56 & 0.45 & 0.72 & 0.40 & 0.67 & 0.43 & 0.48^* \\
x_4 & 0.62 & 0.32 & 0.57 & 0.54 & 0.61 & 0.20 & 0.65 & 0.76^* & 0.42 \\
x_5 & 0.80 & 0.95 & 0.81 & 0.70^* & 0.53 & 0.42^* & 0.80^* & 0.40 & 0.38 \\
x_6 & 0.93 & 0.61 & 0.59 & 0.90 & 0.78 & 0.80 & 0.63 & 0.55 & 0.45 \\
x_7 & 0.55^* & 0.49 & 0.80^* & 0.34 & 0.82 & 0.33 & 0.54 & 0.45 & 0.43 \\
x_8 & 0.55^* & 0.64 & 0.56^* & 0.52^* & 0.62 & 0.42^* & 0.76^* & 0.25 & 0.32 \\
x_9 & 0.38 & 0.70^* & 0.47 & 0.52^* & 0.73^* & 0.26 & 0.64^* & 0.48^* & 0.22
\end{array}$$

(Since * denote $\min(x_i, a_{ij}) = b_j$

 We know that $I_j(A) = \{i \in I / \min(x_i, a_{ij}) = b_j\}$. Therefore

$I_1(A) = \{7,8\}, I_2(A) = \{9\}, I_3(A) = \{7,8\}, I_4(A) = \{1,5,8,7\}, I_5(A) = \{9\}, I_6(A) = \{1,2,5,6,8\}$
$I_7(A) = \{8,9\}, I_8(A) = \{4,9\}, I_9(A) = \{1,3\}.$

Step 5.2. Apply Rules 1-4 to determine the values of as many decision variables as possible. Delete the corresponding rows and / or columns in A.

 For the given matrix A, the index set $I_2(A) = I_5(A) = \{9\}$ indicate that the variable x_9 is the only binding variable in the 2nd and the 5th equation let $\underline{x}^* = (\underline{x}_i^*)_{i\in I}$ be any optimal solution of sub problem P2. Then $\underline{x}_9^* = \bar{x}_9$ can be assigned by rule 1. x_9 is also binding in equations 4,7 and 8 (or columns 4,7,8 of A) Hence, these columns and the corresponding row x_9, can be deleted from matrix A. After deletion the reduced matrix A becomes.

$$A=\begin{array}{c|cccc}
 & 1 & 3 & 6 & 9 \\
x_1 & 0.23 & 0.43 & 0.42^* & 0.45^* \\
x_2 & 0.56 & 0.56 & 0.43^* & 0.46 \\
x_3 & 0.71 & 0.56 & 0.40 & 0.48^* \\
x_4 & 0.62 & 0.57 & 0.20 & 0.42 \\
x_5 & 0.80 & 0.81 & 0.42^* & 0.38 \\
x_6 & 0.93 & 0.59 & 0.80^* & 0.45 \\
x_7 & 0.55^* & 0.80^* & 0.33 & 0.43 \\
x_8 & 0.55^* & 0.56^* & 0.42^* & 0.32
\end{array}$$

Since $I_1(A) = \{7,8\}, I_3(A) = \{7,8\}, I_6(A) = \{1,2,5,6,8\}, I_9(A) = \{1,3\}$.

The reduced matrix A is equivalent to 4 equations with eight variables. The index set to the reduced matrix A is such that $I_1(A) = I_3(A)$. So column 1 or 3 of A can be deleted by Rule 2. Also since $2,5,6 \in I_6$ and $2,5,6 \notin I_j, j-1,3,9$ with $c_2 \bar{x}_2 = 0.84 > c_6 \bar{x}_6 = 0.63 > c_5 \bar{x}_5 = 0.52$. we set $\underline{x}_2^* = \underline{x}_6^* = 0$ by rule 3. Also, the reduced value matrix A has x_4 with $4 \notin I_j, \forall j \in J$. We set $\underline{x}_4^* = 0$ by rule 4.

After deleting column 1 or 3 and the corresponding rows of the matrix A that are associated with x_2, x_4 and x_6 the reduced matrix A becomes,

$$A = \begin{array}{c} \\ x_1 \\ x_3 \\ x_5 \\ x_7 \\ x_8 \end{array} \begin{array}{ccc} 1 & 6 & 9 \\ \left[\begin{array}{ccc} 0.23 & 0.42^* & 0.45^* \\ 0.71 & 0.40 & 0.48^* \\ 0.80 & 0.42^* & 0.38 \\ 0.55^* & 0.33 & 0.43 \\ 0.55^* & 0.42^* & 0.32 \end{array}\right] \end{array}$$

$I_1(A) = \{7,8\}, I_6(A) = \{1,5,8\}, I_9(A) = \{1,3\}$.

Since Rules 1-4 cannot be applied to the current matrix A and five remaining variables $\{x_1, x_3, x_5, x_7, x_8\}$ are undecided goto the next step.

Step 5.3. Take the (remaining) value matrix A. Employ the branch and bound method to solve for the remaining undecided decision variables.

Since,

$$Min\ Z^2 = 5.5x_1 + 5.2x_3 + x_5 + 2x_7 + 3x_8$$

$$xoA = B, o \le x_i \le b \quad i = 1,3,5,7,8$$

where $x = (x_1, x_3, x_5, x_7, x_8)$

$$A = \begin{array}{c} \\ x_1 \\ x_3 \\ x_5 \\ x_7 \\ x_8 \end{array} \begin{array}{ccc} 1 & 6 & 9 \\ \left[\begin{array}{ccc} 0.23 & 0.42^* & 0.45^* \\ 0.71 & 0.40 & 0.48^* \\ 0.80 & 0.42^* & 0.38 \\ 0.55^* & 0.33 & 0.43 \\ 0.55^* & 0.42^* & 0.32 \end{array}\right] \end{array}$$

$I_1(A) = \{7,8\}, I_6(A) = \{1,5,8\}, I_9(A) = \{1,3\}.$

$J_1(A) = \{6,9\}, J_3(A) = \{9\}, J_5(A) = \{6\}, J_7(A) = \{1\}, J_8(A) = \{1,6\}.$
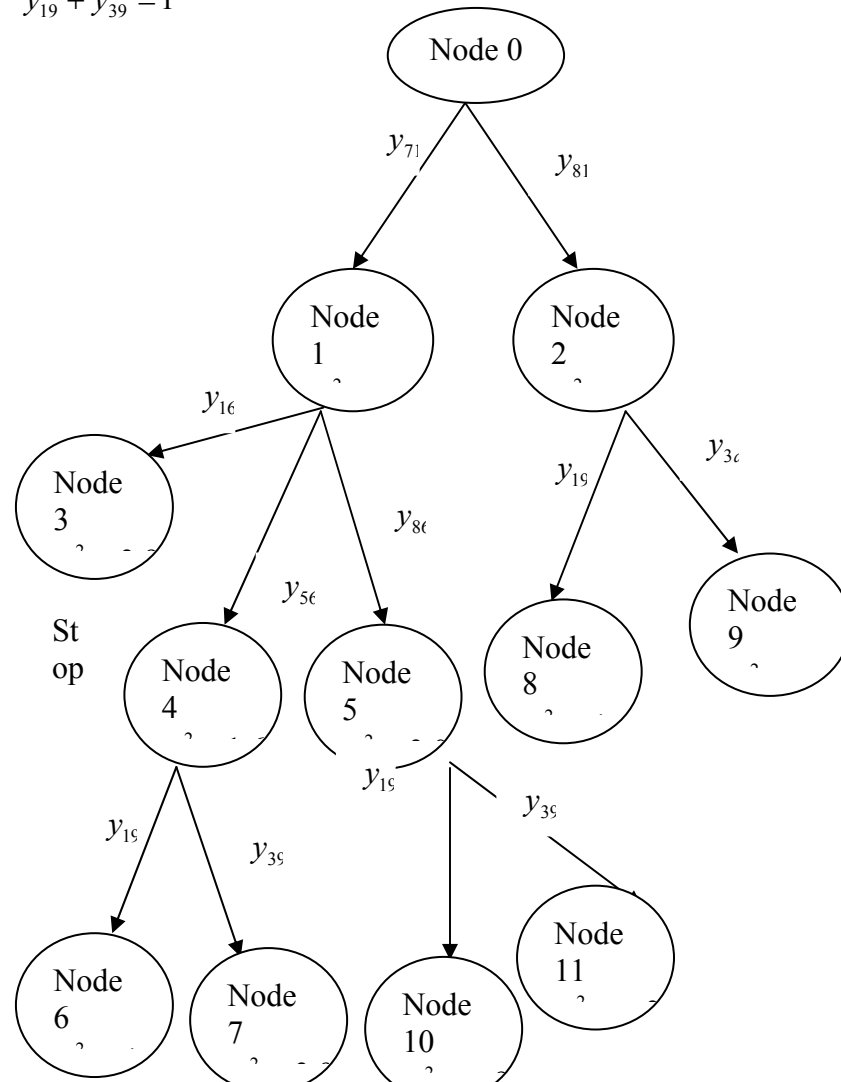
Its corresponding 0-1 integer program is

Minimize $z^2(x) = 5.5(\max_{j \in J_i} (\bar{x}_1, y_{1j})) + 5.2(\max_{j \in J_i} (\bar{x}_3 y_{3j}))$

$+ \max_{j \in J_i} (\bar{x}_5 y_{5j}) + 2(\max_{j \in J_1} (\bar{x}_7 y_{7j}) + 3(\max_{j \in J_i} (\bar{x}_8 y_{8j}))$

Subject to

$y_{71} + y_{81} = 1$

$y_{16} + y_{56} + y_{86} = 1$

$y_{19} + y_{39} = 1$

Now, consider the first constraint equation.

Either y_{71}, or y_{81} has to be 1. This yields nodes 1 and 2. If $y_{71} = 1$ then $x_7 = \bar{x}_7$ therefore the lower bound of node 1 is $2 \times 0.56 = 1.12$.

Also, if $y_{81} = 1$ then $x_8 = \bar{x}_8$. And the lower bound of node 2 is $3 \times 0.64 = 1.92$.

From node 1 we can branch further to either node 3 or node 4 or node 5 with y_{16} or y_{56} or $y_{86} = 1$ respectively if $y_{16} = 1$ then $x_1 = \bar{x}_1$ therefore the lower bound of node 3 is calculated by $1.12 + (5.5 \times 0.52) = 3.98$.

If $y_{56} = 1$ then $x_5 = \bar{x}_5$ and the lower bound of node 4 is $1.12 \times (0.52 \times 1) = 1.64$

If $y_{86} = 1$ then $x_8 = \bar{x}_8$ and the lower bound of node 5 is $1.12 + (3 \times 0.64) = 3.04$.

From node 4 we can branch further to either node 6 or node 7 with y_{19} or $y_{39} = 1$ respectively. This is equivalent to adding another constraint. Since this added constraint is the last one, we obtain the exact objective values instead of the lower bounds.

If $y_{19} = 1$ then $x_1 = \bar{x}_1$ and the objective value of node 6 is $1.64 + (5.5 \times 0.52) = 4.50$

If $y_{39} = 1$ then $x_3 = \bar{x}_3$ and the objective value of node 7 is $1.64 + (5.2 \times 0.45) = 3.98$

Since, $Z^2(x)$ at node 6 and 7 is greater than the lower bound of node 2, we can branch node 2 to either node 8 or node 9 with y_{19} or $y_{39} = 1$

If $y_{19} = 1$ then $x_1 = \bar{x}_1$ and the objective value of node 8 is $1.92 + (5.5 \times 0.52) = 4.78$

If $y_{39} = 1$ then $x_3 = \bar{x}_3$ and the objective value of node 9 is $1.92 + (5.2 \times 0.45) = 4.26$

Since $Z^2(x)$ at node 6,7,8 and 9 is greater than the lower bound of node 5 we can branch node 5 to either node 10 or node 11 with y_{19} or $y_{39} = 1$.

If $y_{19} = 1$ then $x_1 = \bar{x}_1$ and the objective value of node 10 is $3.04 + (5.5 \times 0.52) = 5.90$

If $y_{39} = 1$ then $x_3 = \bar{x}_3$ and the objective value of node 11 is $3.04 + (5.2 \times 0.45) = 5.38$

Since $Z^2(x)$ at node 7 is equal to the lower bound of node 3, we can stop branching to node 3. Moreover $Z^2(x)$ at node 7 and node 3 yields the optimal value. Figure shows the B & B of the given problem.

From the above discussion, we get the two optimal solutions.

$\underline{x}^{*1} = (0, 0, 0.45, 0.52, 0, 0.56, 0, 0.72)$ and $\underline{x}^{*2} = (0.52, 0, 0, 0, 0, 0.56, 0, 0.72)$

For the sub problem p2 with objective value $z^2(\underline{x}^{*1}) = z^2(\underline{x}^{*2}) = 4.88$.

   Now, that all the decision variables have been determined go to the next step.

Step 6. Generate optimal solutions for the original problem from optimal solutions of problem (9) and (10) by (11).

   Notably, only variable x_4 of the sub – problem P1 has a negative co-efficient in the objective function. Hence the maximum solution $\bar{x} = (\bar{x}_i)_{i \in I}$ is an optimal solution with optimal value $Z^1(\bar{x}) = c_4 \bar{x}_4 = -0.24$ for sub – problem p1. on the other hand two optimal solutions $\underline{x}^{*1}$ and $\underline{x}^{*2}$ are given for sub problem P2 with optimal value $Z^2(x^{*1}) = Z^2(\underline{x}*2) = 4.88$.

   Combining these optimal solutions derived from sub problems P1 and P2 by (11) yields two optimal solutions x^{*1} and x^{*2} as follows.

   $x^{*1} = (0, 0, 0.45, 0.48, 0.52, 0, 0.52, 0, 0, 0.56, 0, 0.72)$

and

   $x^{*2} = (0.52, 0, 0, 0.48, 0, 0, 0.56, 0, 0.72)$

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