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New Algebraic Operations of Picture Fuzzy Multisets

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Abstract. Picture fuzzy multiset is an extension of picture fuzzy set. Here in this paper, two new algebraic operations of picture fuzzy multisets were proposed and some vital properties based on them were obtained. Finally, example was given to validate the new algebraic operations. It was discovered that the algebraic operations in picture fuzzy sets were also hold in picture fuzzy multisets.

Keywords: Multiset, fuzzy multiset, picture fuzzy set, picture fuzzy multiset, algebraic laws

AMS Mathematics Subject Classification (2010): 03E72, 08A72

1. Introduction

Cuong and Kreinovich [2] generalised two theories, theory of fuzzy sets (FSs) proposed by Zadeh [20] and theory of intuitionistic fuzzy sets (IFSs) introduced by Atanassov [1] to form an interesting concept, theory of picture fuzzy sets (PFSs). This theory was proposed to incorporate a vital concept called neutrality degree that was not taken into account in IFSs. This degree of neutrality can be seen in a voting environment where voters have four different options; voting for, abstain from voting, voting against and refuse voting. Thus, PFSs constitute membership degrees; positive, neutral and negative membership degrees together with refusal margin. Many researchers have studied and applied the concept to medical diagnosis, Covid-19 medicine selection, building material and minerals field recognitions [see [3, 4, 8, 13, 14, 12, 17, 18] for details].

Yagar [19], introduced fuzzy multisets (FMs) as an extension of FSs allowing membership function to occur more than once. Shinoj and Sunil [16] initiated intuitionistic fuzzy multisets (IFMs) from the work of Yagar and Atanassov in order to allow both membership function and non-membership function occuring more that once. Cao et al [7], introduced the concept of picture fuzzy multisets (PFMSs) as an extension of PFSs. The PFMSs play a vital role in decision-making process by providing a robust framework for dealing with uncertainty and imprecision by representing positive, neutral and negative membership degrees for elements with multiple instances. Few researchers have studied the PFMSs and applied it see [7, 9] for details.

In this paper, we introduced two new algebraic operations of picture fuzzy multisets and some vital properties based on the new algebraic operations were established, and example was given to validate the new algebraic operations. It was discovered that the algebraic laws in picture fuzzy sets were also hold in picture fuzzy multisets.

2. Preliminaries

In this section, we recall some basic definitions.

Definition 2.1. [20] Given a nonempty set $\mathcal{C}$. A fuzzy set (FS) $\mathcal{D}$ of $\mathcal{C}$ is written as

$$\mathcal{D} = \{\langle \frac{\sigma_{\mathcal{D}}(r)}{r} \rangle | r \in \mathcal{C}\},$$

with a membership function

$$\sigma_{\mathcal{D}}: \mathcal{C} \longrightarrow [0,1]$$

where the function $\sigma_{\mathcal{D}}(r)$ denotes the degree of membership of $r \in \mathcal{C}$.

Definition 2.2. [19] A fuzzy multiset (FMS) $\mathcal{D}$ drawn from $\mathcal{C}$ is characterised by a count membership function $cm_{\mathcal{D}}$ such that $cm_{\mathcal{D}}: \mathcal{C} \to \mathcal{N}$, where $\mathcal{N}$ is the set of all crisp multisets drawn from [0,1]. Then, for any r $\in \mathcal{C}$, the value $cm_{\mathcal{D}}(r)$ is a crisp multiset drawn from [0,1]. For any r $\in \mathcal{C}$, the membership sequence is defined as the decreasingly ordered sequence of elements in $cm_{\mathcal{D}}(r)$. It is denoted by $(\sigma_{\mathcal{D}}^1(r), \sigma_{\mathcal{D}}^2(r), \cdots, \sigma_{\mathcal{D}}^k(r))$ where $\sigma_{\mathcal{D}}^1(r) \geq \sigma_{\mathcal{D}}^2(r) \geq \cdots \geq \sigma_{\mathcal{D}}^k(r)$.

Definition 2.3. [1] Given a nonempty set $\mathcal{C}$. An intuitionistic fuzzy set (IFS) $\mathcal{D}$ of $\mathcal{C}$ is written as

$$\mathcal{D} = \{\langle \frac{\sigma_{\mathcal{D}}(r), \tau_{\mathcal{D}}(r)}{r} \rangle | r \in \mathcal{C}\},$$

where the functions $\sigma_{\mathcal{D}}: \mathcal{C} \to [0,1]$ and $\tau_{\mathcal{D}}: \mathcal{C} \to [0,1]$ are called the membership and non-membership degrees of $r \in \mathcal{C}$, respectively, and for every $r \in \mathcal{C}$,

$$0 \leq \sigma_{\mathcal{D}}(r) + \tau_{\mathcal{D}}(r) \leq 1.$$

Definition 2.4. [16] An intuitionistic fuzzy multiset (IFMS) $\mathcal{D}$ drawn from $\mathcal{C}$ is characterised by count membership function $cm_{\mathcal{D}}$ and count nonmembership function $cn_{\mathcal{D}}$ such that $cn_{\mathcal{D}}: \mathcal{C} \to \mathcal{N}$ *and* $cn_{\mathcal{D}}: \mathcal{C} \to \mathcal{N}$, respectively, where $\mathcal{N}$ is the set of all crisp multisets drawn from [0,1], such that for any r $\in \mathcal{C}$, the membership sequence is defined as the decreasingly ordered sequence of elements in $cm_{\mathcal{D}}(r)$, denoted by $(\sigma_{\mathcal{D}}^1(r), \sigma_{\mathcal{D}}^2(r), \cdots, \sigma_{\mathcal{D}}^k(r))$ where $\sigma_{\mathcal{P}}^1(r) \geq \sigma_{\mathcal{D}}^2(r) \geq \cdots \sigma_{\mathcal{D}}^k(r)$ and the nonmembership sequence is given as $(\tau_{\mathcal{D}}^1(r), \tau_{\mathcal{D}}^2(r) \cdots, \tau_{\mathcal{D}}^k(r))$ such that $0 \leq \sigma_{\mathcal{D}}^k(r) + \tau_{\mathcal{D}}^k(r) \leq 1$ *for any* $r \in \mathcal{C}$.

Thus, an IFMS is given as

$$\mathcal{D} = \{\langle r, (\sigma_{\mathcal{D}}^1(r), \sigma_{\mathcal{D}}^2(r) \cdots, \sigma_{\mathcal{D}}^k(r)), (\tau_{\mathcal{D}}^1(r), \tau_{\mathcal{D}}^2(r), \cdots, \tau_{\mathcal{D}}^k(r)) \rangle \mid r \in \mathcal{C}\}.$$

This can be rewritten as

$$\mathcal{D} = \{\langle \frac{(\sigma_{\mathcal{D}}^1(r), \sigma_{\mathcal{D}}^2(r) \cdots, \sigma_{\mathcal{D}}^k(r)), (\tau_{\mathcal{D}}^1(r), \tau_{\mathcal{D}}^2(r), \cdots, \tau_{\mathcal{D}}^k(r))}{r} \rangle \mid r \in \mathcal{C}\}.$$

Definition 2.5. [2] Given a nonempty set $\mathcal{C}$. A PFS $\mathcal{D}$ of $\mathcal{C}$ is written as

$$\mathcal{D} = \{\langle \frac{\sigma_{\mathcal{D}}(r), \tau_{\mathcal{D}}(r), \gamma_{\mathcal{D}}(r)}{r} \rangle \mid r \in \mathcal{C}\},$$

where the functions

$$\sigma_{\mathcal{D}}(r), \tau_{\mathcal{D}}(r), \gamma_{\mathcal{D}}(r): \mathcal{C} \to [0,1],$$

are called the positive, neutral and negative membership degrees of $r \in \mathcal{C}$ to $\mathcal{D}$, and for all element $r \in \mathcal{C}$,

$$0 \leq \sigma_{\mathcal{D}}(r) + \tau_{\mathcal{D}}(r) + \gamma_{\mathcal{D}}(r) \leq 1.$$

For each PFS $\mathcal{D}$ of $\mathcal{C}$,

$$\pi_{\mathcal{D}}(r) = 1 - (\sigma_{\mathcal{D}}(r) + \tau_{\mathcal{D}}(r) + \gamma_{\mathcal{D}}(r))$$

is the refusal membership degree of $r \in \mathcal{C}$.

Definition 2.6. [7] Given a nonempty set $\mathcal{C}$. The PFMS $\mathcal{D}$ in $\mathcal{C}$ is characterised by three functions namely positive membership count function pmc, neutral membership count function $n_e mc$ and negative membership count function nmc such that $pmc, n_e mc, nmc: \mathcal{C} \to \mathcal{W}$, where $\mathcal{W}$, is refer to collection of crisp multisets taken from [0,1]. Thus, every element $r \in \mathcal{C}$, pmc is the crisp multiset from [0,1] whose positive membership sequence is defined by $(\sigma_{\mathcal{D}}^1(r), \sigma_{\mathcal{D}}^2(r), \cdots, \sigma_{\mathcal{D}}^k(r))$ such that $\sigma_{\mathcal{D}}^1(r) \geq \sigma_{\mathcal{D}}^2(r) \geq \cdots \geq \sigma_{\mathcal{D}}^k(r)$, $n_e mc$ is the crisp multiset from [0,1] whose neutral membership sequence is defined by $(\tau_{\mathcal{D}}^1(r), \tau_{\mathcal{D}}^2(r), \cdots, \tau_{\mathcal{D}}^k(r))$ and nmc is the crisp multiset from [0,1] whose negative membership sequence is defined by $(\eta_{\mathcal{D}}^1(r), \eta_{\mathcal{D}}^2(r), \cdots, \eta_{\mathcal{D}}^k(r))$, these can be either decreasing or increasing functions satisfying $0 \leq \sigma_{\mathcal{D}}^k(r) + \tau_{\mathcal{D}}^k(r) + \eta_{\mathcal{D}}^k(r) \leq 1$ $\forall\, r \in \mathcal{C}, k = 1,2, \cdots, n$.
Thus, $\mathcal{D}$ is represented by

$$\mathcal{D} = \{\langle \frac{\sigma_{\mathcal{D}}^d(r), \tau_{\mathcal{D}}^d(r), \eta_{\mathcal{D}}^k(r)}{r} \rangle \mid r \in \mathcal{C}\}.$$

The set of all picture fuzzy multisets over $\mathcal{C}$, is denoted as PFMS($\mathcal{C}$).

Definition 2.7. [7] Let $\mathcal{C}$ be nonempty set and $\mathcal{D}$ and $\mathcal{E}$ be PFMSs drawn from $\mathcal{C}$. Then,

1. The support of $\mathcal{D}$ is given as
$$supp(\mathcal{D}) = \{\sigma_{\mathcal{D}}^k(r) > 0 \; or \; \tau_{\mathcal{D}}^k(r) > 0 \; or \; \eta_{\mathcal{D}}^k(r) > 0 \mid r \in \mathcal{C}\}, k = 1,2, \cdots, n.$$
2. The height of $\mathcal{D}$ is given as
$$ht = max\{supp(\sigma_{\mathcal{D}}^k(r), \tau_{\mathcal{D}}^k(r), \eta_{\mathcal{D}}^k(r))\}, r \in \mathcal{C}, k = 1,2, \cdots, n.$$
3. The PFMSs $\mathcal{D}$ and $\mathcal{E}$ are similar if
$$\sigma_{\mathcal{D}}^k(r) = \sigma_{\mathcal{E}}^k(r) \text{ or } \tau_{\mathcal{D}}^k(r) = \tau_{\mathcal{E}}^k(r) \text{ or } \eta_{\mathcal{D}}^k(r) = \eta_{\mathcal{E}}^k(r),$$
for at least one k and $r \in \mathcal{C}, k = 1,2, \cdots, n$.
4. The PFMSs $\mathcal{D}$ and $\mathcal{E}$ are comparable or equal denoted as, $\mathcal{D} = \mathcal{E}$, if $\sigma_{\mathcal{D}}^k(r) \leq \sigma_{\mathcal{E}}^k(r)$ or $\tau_{\mathcal{D}}^k(r) \leq \tau_{\mathcal{E}}^k(r)$ or $\eta_{\mathcal{D}}^k(r) \leq \eta_{\mathcal{E}}^k(r)$ for all $r \in \mathcal{C}, k = 1,2, \cdots, n$.
5. The PFMS $\mathcal{D}$ is a sub-PFMS of $\mathcal{E}$ denoted as, $\mathcal{D} \subseteq \mathcal{E}$, if $\sigma_{\mathcal{D}}^k(r) \leq \sigma_{\mathcal{E}}^k(r)$, $\tau_{\mathcal{D}}^k(r) \leq \tau_{\mathcal{E}}^k(r)$ and $\eta_{\mathcal{D}}^k(r) \geq \eta_{\mathcal{E}}^k(r)$ for $r \in \mathcal{C}, k = 1,2, \cdots, n$.
6. The PFMS $\mathcal{D}$ is a proper sub-PFMS of $\mathcal{E}$ denoted as, $\mathcal{D} \subset \mathcal{E}$, if $\sigma_{\mathcal{D}}^k(r) < \sigma_{\mathcal{E}}^k(r)$, $\tau_{\mathcal{D}}^k(r) < \tau_{\mathcal{E}}^k(r)$ and $\eta_{\mathcal{D}}^k(r) > \eta_{\mathcal{E}}^k(r)$ for $r \in \mathcal{C}, k = 1,2, \cdots, n$.

Definition 2.8. [7] Let $\mathcal{D}, \mathcal{E} \in$ PFMS($\mathcal{C}$). That is;

$$\mathcal{D} = \{\langle r, \sigma_{\mathcal{D}}^k(r), \tau_{\mathcal{D}}^k(r), \eta_{\mathcal{D}}^k(r)\rangle \mid r \in \mathcal{C}\}$$

and

$$\mathcal{E} = \{\langle r, \sigma_{\mathcal{E}}^k(r), \tau_{\mathcal{E}}^k(r), \eta_{\mathcal{E}}^i(r)\rangle \mid r \in \mathcal{C}\},$$

where $k = 1,2, \cdots, n$. Then, the following basic operations hold:

- Union

$$\mathcal{D} \cup \mathcal{E} = \{(r, (\sigma_{\mathcal{D}}^k(r) \vee \sigma_{\mathcal{E}}^k(r)), (\tau_{\mathcal{D}}^k(r) \wedge \tau_{\mathcal{E}}^k(r)), (\eta_{\mathcal{D}}^k(r) \wedge \eta_{\mathcal{E}}^k(r))) \mid r \in \mathcal{C}\},$$

- Intersectuon

  $\mathcal{D} \cap \mathcal{E} = \{(r, (\sigma_{\mathcal{D}}^k(r) \wedge \sigma_{\mathcal{E}}^k(r)), (\tau_{\mathcal{D}}^k(r) \vee \tau_{\mathcal{E}}^k(r)), (\eta_{\mathcal{D}}^k(r) \vee \eta_{\mathcal{E}}^k(r)))| \, r \in \mathcal{C}\},$
- Complement

  $\mathcal{D}^c = \{(r, \eta_{\mathcal{D}}^k(r), \tau_{\mathcal{D}}^k(r), \sigma_{\mathcal{D}}^k(z))| \, r \in \mathcal{C}\}.$
- The Cartesian product

  $\mathcal{D} \times \mathcal{E} = \{\langle (r_1, r_2), \sigma_{\mathcal{D}}^k(r_1) \wedge \sigma_{\mathcal{E}}^k(r_2), \tau_{\mathcal{D}}^k(r_1) \wedge \tau_{\mathcal{E}}^k(r_2), \eta_{\mathcal{D}}^k(r_1) \vee \eta_{\mathcal{E}}^k(r_2)\rangle | \, r_1, r_2 \in \mathcal{C}\}.$

Theorem 2.1. [7] *For every PFMS P, Q, R in PFMS($\mathcal{C}$)*

1. **Involution** $\overline{\overline{P}} = P$.

2. **Commutative Rule** $P \cap Q = Q \cap P$, $P \cup Q = Q \cup P$, $P \times Q = Q \times P$.

3. **Associative Rule** $P \cap (Q \cap R) = (P \cap Q) \cap R$,
   $P \cup (Q \cup R) = (P \cup Q) \cup R$, $(P \times Q) \times R = P \times (Q \times R)$.

4. **Distributive Rule** $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$,
   $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$,
   $P \times (Q \cup R) = (P \times Q) \cup (P \times R)$, $P \times (Q \cap R) = (P \times Q) \cap (P \times R)$.

5. **Idempotent Rule** $P \cap P = P$, $P \cup P = P$.

6. **De Morgan's Rule** $\overline{P \cap Q} = \overline{P} \cup \overline{Q}$, $\overline{P \cup Q} = \overline{P} \cap \overline{Q}$.

3. New algebraic operations of picture fuzzy multisets

Let $\mathcal{D}, \mathcal{E}, \mathcal{F}$ be PFMSs. That is

$$\mathcal{D} = \{\langle r, \sigma_{\mathcal{D}}^k(r), \tau_{\mathcal{D}}^k(r), \eta_{\mathcal{D}}^k(r)\rangle | \, r \in \mathcal{C}\},$$

$$\mathcal{E} = \{\langle r, \sigma_{\mathcal{E}}^k(r), \tau_{\mathcal{E}}^k(r), \eta_{\mathcal{E}}^i(r)\rangle | \, r \in \mathcal{C}\},$$

and

$$\mathcal{F} = \{\langle r, \sigma_{\mathcal{F}}^k(r), \tau_{\mathcal{F}}^k(r), \eta_{\mathcal{F}}^k(r)\rangle | \, r \in \mathcal{C}\}$$

where $k = 1, 2, \cdots, n$.

Then, define addition and multiplication on PFMSs as

$$\mathcal{D} \oplus \mathcal{E} = \{(r, \sigma_{\mathcal{D}\oplus\mathcal{E}}^k(r), \tau_{\mathcal{D}\oplus\mathcal{E}}^k(r), \eta_{\mathcal{D}\oplus\mathcal{E}}^k(r)) \mid r \in \mathcal{C}\}$$

where

$$\sigma_{\mathcal{D}\oplus\mathcal{E}}^k(r) = \sigma_{\mathcal{D}}^k(r) + \sigma_{\mathcal{E}}^k(r) - \sigma_{\mathcal{D}}^k(r)\sigma_{\mathcal{E}}^k(r),$$

$$\tau_{\mathcal{D}\oplus\mathcal{E}}^k(r) = \tau_{\mathcal{D}}^k(r)\tau_{\mathcal{E}}^k(r)$$

and

$$\eta_{\mathcal{D}\oplus\mathcal{E}}^k(r) = \eta_{\mathcal{D}}^k(r)\eta_{\mathcal{E}}^k(r).$$

$$\mathcal{D} \otimes \mathcal{E} = \{(r, \sigma_{\mathcal{D}\otimes\mathcal{E}}^k(r), \tau_{\mathcal{D}\otimes\mathcal{E}}^k(r), \eta_{\mathcal{D}\otimes\mathcal{E}}^k(r)) \mid r \in \mathcal{C}\}$$

where

$$\sigma_{\mathcal{D}\otimes\mathcal{E}}^k(r) = \sigma_{\mathcal{D}}^k(r)\sigma_{\mathcal{E}}^k(r),$$

$$\tau^k_{\mathcal{D}\otimes\mathcal{E}}(r) = \tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r) - \tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r)$$

and

$$\eta^k_{\mathcal{D}\otimes\mathcal{E}}(r) = \eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r) - \eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r).$$

Theorem 3.1. *Let $\mathcal{D}, \mathcal{E}, \mathcal{F}$ be in PFMS($\mathcal{C}$. Then,*

- $\mathcal{D} \oplus \mathcal{E} = \mathcal{E} \oplus \mathcal{D}$.
- $\mathcal{D} \otimes \mathcal{E} = \mathcal{E} \otimes \mathcal{D}$.
- $\mathcal{D} \oplus (\mathcal{E} \oplus \mathcal{F}) = (\mathcal{D} \oplus \mathcal{E}) \oplus \mathcal{F}$.
- $\mathcal{D} \otimes (\mathcal{E} \otimes \mathcal{F}) = (\mathcal{D} \otimes \mathcal{E}) \otimes \mathcal{F}$.
- $\mathcal{D} \oplus (\mathcal{E} \cup \mathcal{F}) = (\mathcal{D} \oplus \mathcal{E}) \cup (\mathcal{D} \oplus \mathcal{F})$.
- $\mathcal{D} \oplus (\mathcal{E} \cap \mathcal{F}) = (\mathcal{D} \oplus \mathcal{E}) \cap (\mathcal{D} \oplus \mathcal{F})$.
- $\mathcal{D} \otimes (\mathcal{E} \cup \mathcal{F}) = (\mathcal{D} \otimes \mathcal{E}) \cup (\mathcal{D} \otimes \mathcal{F})$.
- $\mathcal{D} \otimes (\mathcal{E} \cap \mathcal{F}) = (\mathcal{D} \otimes \mathcal{E}) \cap (\mathcal{D} \otimes \mathcal{F})$.

Proof:

- $\mathcal{D} \oplus \mathcal{E} = \{(r, \sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r) - \sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r), \tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r), \eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r)) \mid r \in \mathcal{C}\}$
$= \{(r, \sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{D}}(r) - \sigma^k_{\mathcal{E}}(r)\sigma^k_{\mathcal{D}}(r), \tau^k_{\mathcal{E}}(r)\tau^k_{\mathcal{D}}(r), \eta^k_{\mathcal{E}}(r)\eta^k_{\mathcal{D}}(r)) \mid r \in \mathcal{C}\}$
$= \mathcal{E} \oplus \mathcal{D}$.

- $\mathcal{D} \otimes \mathcal{E} = \{(r, \sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r), \tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r) - \tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r), \eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r) -$
$\eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r)) \mid r \in \mathcal{C}\}$
$= \{(r, \sigma^k_{\mathcal{E}}(r)\sigma^k_{\mathcal{D}}(r), \tau^k_{\mathcal{E}}(r) + \tau^k_{\mathcal{D}}(r) - \tau^k_{\mathcal{E}}(r)\tau^k_{\mathcal{D}}(r), \eta^k_{\mathcal{E}}(r) + \eta^k_{\mathcal{D}}(r) -$
$\eta^k_{\mathcal{E}}(r)\eta^k_{\mathcal{D}}(r))$
$= \mathcal{E} \otimes \mathcal{D}$

- $\mathcal{D} \oplus (\mathcal{E} \oplus \mathcal{F}) = \mathcal{D} \oplus \{(r, \sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{F}}(r) -$
$\sigma^k_{\mathcal{E}}(r)\sigma^k_{\mathcal{F}}(r), \tau^k_{\mathcal{E}}(r)\tau^k_{\mathcal{F}}(r), \eta^k_{\mathcal{E}}(r)\eta^k_{\mathcal{F}}(r)) \mid r \in \mathcal{C}\}$
$= \{(r, \sigma^k_{\mathcal{D}}(r) + (\sigma^k_{\mathcal{E}}(r) + \sigma^k_{\mathcal{F}}(r)) -$
$\sigma^k_{\mathcal{D}}(r)(\sigma^k_{\mathcal{E}}(r)\sigma^k_{\mathcal{F}}(r)), \tau^k_{\mathcal{D}}(r)(\tau^k_{\mathcal{E}}(r)\tau^k_{\mathcal{F}}(r)),$
$\eta^k_{\mathcal{D}}(r)(\eta^k_{\mathcal{E}}(r)\eta^k_{\mathcal{F}}(r))) \mid r \in \mathcal{C}\}$
$= \{(r, (\sigma^k_{\mathcal{D}}(r) + \sigma^k_{\mathcal{E}}(r)) + \sigma^k_{\mathcal{F}}(r) -$
$(\sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r))\sigma^k_{\mathcal{F}}(r), (\tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r))\tau^k_{\mathcal{F}}(r),$
$(\eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r))\eta^k_{\mathcal{F}}(r))) \mid r \in \mathcal{C}\}$
$= (\mathcal{D} \oplus \mathcal{E}) \oplus \mathcal{F}$.

- $\mathcal{D} \otimes (\mathcal{E} \otimes \mathcal{F}) = \mathcal{D} \otimes \{(r, \sigma^k_{\mathcal{E}}(r)\sigma^k_{\mathcal{F}}(r), \tau^k_{\mathcal{E}}(r) + \tau^k_{\mathcal{F}}(r) - \tau^k_{\mathcal{E}}(r)\tau^k_{\mathcal{F}}(r),$
$\eta^k_{\mathcal{E}}(r) + \eta^k_{\mathcal{F}}(r) - \eta^k_{\mathcal{E}}(r)\eta^k_{\mathcal{F}}(r)) \mid r \in \mathcal{C}\}$
$= \{(r, \sigma^k_{\mathcal{D}}(r)(\sigma^k_{\mathcal{E}}(r)\sigma^k_{\mathcal{F}}(r)), \tau^k_{\mathcal{D}}(r) + (\tau^k_{\mathcal{E}}(r)\tau^k_{\mathcal{F}}(r)) -$
$\tau^k_{\mathcal{D}}(r)(\tau^k_{\mathcal{E}}(r)\tau^k_{\mathcal{F}}(r)),$
$\eta^k_{\mathcal{D}}(r) + (\eta^k_{\mathcal{E}}(r)\eta^k_{\mathcal{F}}(r)) - \eta^k_{\mathcal{D}}(r) + (\eta^k_{\mathcal{E}}(r)\eta^k_{\mathcal{F}}(r))) \mid r \in \mathcal{C} \mid r \in \mathcal{C}\}$
$= \{(r, (\sigma^k_{\mathcal{D}}(r)\sigma^k_{\mathcal{E}}(r))\sigma^k_{\mathcal{F}}(r), (\tau^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{E}}(r)) + \tau^k_{\mathcal{F}}(r) -$
$(\tau^k_{\mathcal{D}}(r)\tau^k_{\mathcal{E}}(r))\tau^k_{\mathcal{F}}(r),$
$(\eta^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{E}}(r)) + \eta^k_{\mathcal{F}}(r) - (\eta^k_{\mathcal{D}}(r)\eta^k_{\mathcal{E}}(r))\eta^k_{\mathcal{F}}(r)) \mid r \in \mathcal{C} \mid r \in \mathcal{C}\}$

$= (\mathcal{D} \otimes \mathcal{E}) \otimes \mathcal{F}.$

• $\mathcal{D} \oplus (\mathcal{E} \cup \mathcal{F}) = \mathcal{D} \oplus \{(r, \sigma_{\mathcal{E}}^k(r) \vee \sigma_{\mathcal{F}}^k(r), \tau_{\mathcal{E}}^k(r) \wedge \tau_{\mathcal{F}}^k(r), \eta_{\mathcal{E}}^k(r) \wedge \eta_{\mathcal{F}}^k(r)) \mid r \in \mathcal{C}\}$
$= \{(r, \sigma_{\mathcal{D}}^k(r) + (\sigma_{\mathcal{E}}^k(r) \vee \sigma_{\mathcal{F}}^k(r)) - \sigma_{\mathcal{D}}^k(r) + (\sigma_{\mathcal{E}}^k(r) \vee \sigma_{\mathcal{F}}^k(r)),$
$\tau_{\mathcal{D}}^k(r)(\tau_{\mathcal{E}}^k(r)\tau_{\mathcal{F}}^k(r)), \eta_{\mathcal{D}}^k(r)(\eta_{\mathcal{E}}^k(r)\eta_{\mathcal{F}}^k(r))) \mid r \in \mathcal{C}\}$
$= \{(r, \sigma_{\mathcal{D}}^k(r) + \sigma_{\mathcal{E}}^k(r) \vee \sigma_{\mathcal{D}}^k(r) + (\sigma_{\mathcal{F}}^k(r) - (\sigma_{\mathcal{D}}^k(r)\sigma_{\mathcal{E}}^k(r)) \vee$
$(\sigma_{\mathcal{D}}^k(r)(\sigma_{\mathcal{F}}^k(r)),$
$(\tau_{\mathcal{D}}^k(r)\tau_{\mathcal{E}}^k(r)) \wedge (\tau_{\mathcal{D}}^k(r)(\tau_{\mathcal{E}}^k(r))), (\eta_{\mathcal{D}}^k(r)\eta_{\mathcal{E}}^k(r)) \wedge (\eta_{\mathcal{D}}^k(r)(\eta_{\mathcal{E}}^k(r)))) \mid r \in \mathcal{C}\}$
$= \{(r, \sigma_{\mathcal{D}}^k(r) + \sigma_{\mathcal{E}}^k(r) - \sigma_{\mathcal{D}}^k(r)\sigma_{\mathcal{E}}^k(r), \tau_{\mathcal{D}}^k(r)\tau_{\mathcal{E}}^k(r), \eta_{\mathcal{D}}^k(r)\eta_{\mathcal{E}}^k(r)) \mid r \in \mathcal{C}\}$
$\cup \{(r, \sigma_{\mathcal{D}}^k(r) + \sigma_{\mathcal{F}}^k(r) - \sigma_{\mathcal{D}}^k(r)\sigma_{\mathcal{F}}^k(r), \tau_{\mathcal{D}}^k(r)\tau_{\mathcal{F}}^k(r), \eta_{\mathcal{D}}^k(r)\eta_{\mathcal{F}}^k(r)) \mid r \in \mathcal{C}\}$
$= (\mathcal{D} \oplus \mathcal{E}) \cup (\mathcal{D} \oplus \mathcal{F}).$

Similarly, we can prove (6), $\mathcal{D} \oplus (\mathcal{E} \cap \mathcal{F}) = (\mathcal{D} \oplus \mathcal{E}) \cap (\mathcal{D} \oplus \mathcal{F})$.

• $\mathcal{D} \otimes (\mathcal{E} \cup \mathcal{F}) = \mathcal{D} \otimes \{(r, \sigma_{\mathcal{E}}^k(r) \vee \sigma_{\mathcal{F}}^k(r), \tau_{\mathcal{E}}^k(r) \wedge \tau_{\mathcal{F}}^k(r), \eta_{\mathcal{E}}^k(r) \wedge$
$\eta_{\mathcal{F}}^k(r)) \mid r \in \mathcal{C}\}$
$= \{(r, \sigma_{\mathcal{D}}^k(r)(\sigma_{\mathcal{E}}^k(r) \vee \sigma_{\mathcal{F}}^k(r)), \tau_{\mathcal{D}}^k(r) + (\tau_{\mathcal{E}}^k(r)\tau_{\mathcal{F}}^k(r)) -$
$\tau_{\mathcal{D}}^k(r)(\tau_{\mathcal{E}}^k(r)\tau_{\mathcal{F}}^k(r)),$
$\eta_{\mathcal{D}}^k(r) + (\eta_{\mathcal{E}}^k(r)\eta_{\mathcal{F}}^k(r)) - \eta_{\mathcal{D}}^k(r)(\eta_{\mathcal{E}}^k(r)\eta_{\mathcal{F}}^k(r))) \mid r \in \mathcal{C}\}$
$= \{(r, \sigma_{\mathcal{D}}^k(r)\sigma_{\mathcal{E}}^k(r) \vee \sigma_{\mathcal{D}}^k(r)\sigma_{\mathcal{F}}^k(r), (\tau_{\mathcal{D}}^k(r) + \tau_{\mathcal{E}}^k(r)) \vee (\tau_{\mathcal{D}}^k(r) + \tau_{\mathcal{F}}^k(r))$
$-(\tau_{\mathcal{D}}^k(r)\tau_{\mathcal{E}}^k(r)) \vee (\tau_{\mathcal{D}}^k(r)\tau_{\mathcal{F}}^k(r)), (\eta_{\mathcal{D}}^k(r) + \eta_{\mathcal{E}}^k(r)) \vee (\eta_{\mathcal{D}}^k(r) + \eta_{\mathcal{F}}^k(r))$
$-(\eta_{\mathcal{D}}^k(r)\eta_{\mathcal{E}}^k(r)) \vee (\eta_{\mathcal{D}}^k(r)(\eta_{\mathcal{F}}^k(r)))) \mid r \in \mathcal{C}\}$
$= \{(r, \sigma_{\mathcal{D}}^k(r)\sigma_{\mathcal{E}}^k(r), (\tau_{\mathcal{D}}^k(r) + \tau_{\mathcal{E}}^k(r)) - (\tau_{\mathcal{D}}^k(r)\tau_{\mathcal{E}}^k(r)),$
$(\eta_{\mathcal{D}}^k(r) + \eta_{\mathcal{E}}^k(r)) - (\eta_{\mathcal{D}}^k(r)\eta_{\mathcal{E}}^k(r))) \mid r \in \mathcal{C}\}$
$\cup \{(r, \sigma_{\mathcal{D}}^k(r)\sigma_{\mathcal{F}}^k(r), (\tau_{\mathcal{D}}^k(r) + \tau_{\mathcal{F}}^k(r)) - (\tau_{\mathcal{D}}^k(r)\tau_{\mathcal{F}}^k(r)),$
$(\eta_{\mathcal{D}}^k(r) + \eta_{\mathcal{F}}^k(r)) - (\eta_{\mathcal{D}}^k(r)\eta_{\mathcal{F}}^k(r))) \mid r \in \mathcal{C}\}$
$= (\mathcal{D} \otimes \mathcal{E}) \cup (\mathcal{D} \otimes \mathcal{F}).$

Similarly, we can prove (8), $\mathcal{D} \otimes (\mathcal{E} \cap \mathcal{F}) = (\mathcal{D} \otimes \mathcal{E}) \cap (\mathcal{D} \otimes \mathcal{F})$.

4. Verification of the algebraic laws

Let $\mathcal{C} = \{a, b, c\}$

$$\mathcal{D} = \{(a, 0.7,0.2,0.1), (b, 0.6,0.2,0.2), (c, 0.4,0.4,0.2)\},$$

$$\mathcal{E} = \{(a, 0.8,0.1,0.1), (b, 0.5,0.3,0.2), (c, 0.3,0.4,0.4)\}$$

and

$$\mathcal{F} = \{(a, 0.5,0.4,0.1), (b, 0.6,0.3,0.1), (c, 0.7,0.15,0.15)\}.$$

Then,

• Commutativity:

$$\mathcal{D} \cup \mathcal{E} = \{(a, 0.8,0.1,0.1), (b, 0.6,0.2,0.2), (c, 0.4,0.4,0.2)\}$$
$$\mathcal{E} \cup \mathcal{D} = \{(a, 0.8,0.1,0.1), (b, 0.6,0.2,0.2), (c, 0.4,0.4,0.2)\}$$
$$\Rightarrow \mathcal{D} \cup \mathcal{E} = \mathcal{E} \cup \mathcal{D}.$$

$$\mathcal{D} \cap \mathcal{E} = \{(a, 0.7,0.2,0.1), (b, 0.5,0.3,0.2), (c, 0.3,0.4,0.4)\}$$
$$\mathcal{E} \cap \mathcal{D} = \{(a, 0.7,0.2,0.1), (b, 0.5,0.3,0.2), (c, 0.3,0.4,0.4)\} \Rightarrow \mathcal{D} \cap \mathcal{E} = \mathcal{E} \cap \mathcal{D}.$$

$$\mathcal{D} \oplus \mathcal{E} = \{(a, 0.94,0.02,0.01), (b, 0.80,0.06,0.04), (c, 0.58,0.16,0.08)\}$$

$$\mathcal{E} \oplus \mathcal{D} = \{(a, 0.94,0.02,0.01), (b, 0.80,0.06,0.04), (c, 0.58,0.16,0.08)\}$$
$\Rightarrow \mathcal{D} \oplus \mathcal{E} = \mathcal{E} \oplus \mathcal{D}.$
$$\mathcal{D} \otimes \mathcal{E} = \{(a, 0.56,0.28,0.19), (b, 0.30,0.44,0.36), (c, 0.12,0.64,0.52)\}$$
$$\mathcal{E} \oplus \mathcal{D} = \{(a, 0.56,0.28,0.19), (b, 0.30,0.44,0.36), (c, 0.12,0.64,0.52)\}$$
$$\Rightarrow \mathcal{D} \otimes \mathcal{E} = \mathcal{E} \otimes \mathcal{D}.$$

• Associativity:

$\mathcal{D} \cup (\mathcal{E} \cup \mathcal{F}) = (\mathcal{D} \cup \mathcal{E}) \cup \mathcal{F}$
$$\mathcal{E} \cup \mathcal{F} = \{(a, 0.80,0.100.10), (b, 0.60,0.30,0.10), (c, 0.70,0.15,0.15)\}$$
$$\mathcal{D} \cup (\mathcal{E} \cup \mathcal{F}) = \{(a, 0.80,0.10,0.10), (b, 0.62,0.20,0.10), (c, 0.70,0.15,0.015)\}$$
$$(\mathcal{D} \cup \mathcal{E}) \cup \mathcal{F} = \{(a, 0.80,0.10,0.10), (b, 0.62,0.20,0.10), (c, 0.70,0.15,0.015)\}$$
$\Rightarrow \mathcal{D} \cup (\mathcal{E} \cup \mathcal{F}) = (\mathcal{D} \cup \mathcal{E}) \cup \mathcal{F}.$

$\mathcal{D} \cap (\mathcal{E} \cap \mathcal{F}) = (\mathcal{D} \cap \mathcal{E}) \cap \mathcal{F}$
$$\mathcal{E} \cap \mathcal{F} = \{(a, 0.50,0.400.10), (b, 0.50,0.30,0.20), (c, 0.30,0.40,0.40)\}$$
$$\mathcal{D} \cap (\mathcal{E} \cap \mathcal{F}) = \{(a, 0.50,0.400.10), (b, 0.50,0.30,0.20), (c, 0.30,0.40,0.40)\}$$
$$(\mathcal{D} \cap \mathcal{E}) \cap \mathcal{F} = \{(a, 0.50,0.400.10), (b, 0.50,0.30,0.20), (c, 0.30,0.40,0.40)\}$$
$\Rightarrow \mathcal{D} \cap (\mathcal{E} \cap \mathcal{F}) = (\mathcal{D} \cap \mathcal{E}) \cap \mathcal{F}.$

$$\mathcal{D} \oplus (\mathcal{E} \oplus \mathcal{F}) = (\mathcal{D} \oplus \mathcal{E}) \oplus \mathcal{F}$$
$$\mathcal{E} \oplus \mathcal{F} = \{(a, 0.90,0.04,0.01), (b, 0.80,0.09,0.02), (c, 0.79,0.06,0.06)\}$$
$$\mathcal{D} \oplus (\mathcal{E} \oplus \mathcal{F}) = \{(a, 0.97,0.008,0.001), (b, 0.92,0.018,0.004), (c, 0.874,0.024,0.012)\}$$

$$(\mathcal{D} \oplus \mathcal{E}) \oplus \mathcal{F} = \{(a, 0.97,0.008,0.001), (b, 0.92,0.018,0.004), (c, 0.874,0.024,0.012)\}$$
$\Rightarrow \mathcal{D} \oplus (\mathcal{E} \oplus \mathcal{F}) = (\mathcal{D} \oplus \mathcal{E}) \oplus \mathcal{F}.$

$\mathcal{D} \otimes (\mathcal{E} \otimes \mathcal{F}) = (\mathcal{D} \otimes \mathcal{E}) \otimes \mathcal{F}$
$$\mathcal{D} \otimes (\mathcal{E} \otimes \mathcal{F}) = \{(a, 0.28,0.568,0.271), (b, 0.18,0.608,0.424), (c, 0.84,0.694,0.592)\}.$$

$$(\mathcal{D} \otimes \mathcal{E}) \otimes \mathcal{F} = \{(a, 0.28,0.568,0.271), (b, 0.18,0.608,0.424), (c, 0.84,0.694,0.592)\}.$$
$\Rightarrow \mathcal{D} \otimes (\mathcal{E} \otimes \mathcal{F}) = (\mathcal{D} \otimes \mathcal{E}) \otimes \mathcal{F}.$

• Distributivity:
$$\mathcal{D} \oplus (\mathcal{E} \cup \mathcal{F}) = \{(a, 0.94,0.02,0.01), (b, 0.84,0.06,0.02), (c, 0.82,0.06,0.03)\}$$

$$(\mathcal{D} \oplus \mathcal{E}) \cup (\mathcal{D} \oplus \mathcal{F}) = \{(a, 0.94,0.02,0.01), (b, 0.84,0.06,0.02), (c, 0.82,0.06,0.03)\}$$
$$\Rightarrow \mathcal{D} \oplus (\mathcal{E} \cup \mathcal{F}) = (\mathcal{D} \oplus \mathcal{E}) \cup (\mathcal{D} \oplus \mathcal{F}).$$

$$\mathcal{D} \oplus (\mathcal{E} \cap \mathcal{F}) = \{(a, 0.85,0.08,0.01), (b, 0.80,0.06,0.04), (c, 0.58,0.16,0.08)\}$$
$$(\mathcal{D} \oplus \mathcal{E}) \cap (\mathcal{D} \oplus \mathcal{F}) = \{(a, 0.85,0.08,0.01), (b, 0.80,0.06,0.04), (c, 0.58,0.16,0.08)\}$$

$$\Rightarrow \mathcal{D} \oplus (\mathcal{E} \cup \mathcal{F}) = (\mathcal{D} \oplus \mathcal{E}) \cap (\mathcal{D} \oplus \mathcal{F}).$$

$$\mathcal{D} \otimes (\mathcal{E} \cup \mathcal{F}) = \{(a, 0.56,0.28,0.19), (b, 0.36,0.44,0.28), (c, 0.28,0.49,0.32)\}$$

$$(\mathcal{D} \otimes \mathcal{E}) \cup (\mathcal{D} \otimes \mathcal{F}) = \{(a, 0.56,0.28,0.19), (b, 0.36,0.44,0.28), (c, 0.28,0.49,0.32)\}$$

$$\Rightarrow \mathcal{D} \otimes (\mathcal{E} \cup \mathcal{F}) = (\mathcal{D} \otimes \mathcal{E}) \cup (\mathcal{D} \otimes \mathcal{F}).$$

$$\mathcal{D} \otimes (\mathcal{E} \cap \mathcal{F}) = \{(a, 0.35,0.52,0.19), (b, 0.30,0.44,0.36), (c, 0.12,0.64,0.52)\}$$

$$(\mathcal{D} \otimes \mathcal{E}) \cap (\mathcal{D} \otimes \mathcal{F}) = \{(a, 0.35,0.52,0.19), (b, 0.30,0.44,0.36), (c, 0.12,0.64,0.52)\}$$

$$\Rightarrow \mathcal{D} \otimes (\mathcal{E} \cap \mathcal{F}) = (\mathcal{D} \otimes \mathcal{E}) \cap (\mathcal{D} \otimes \mathcal{F}).$$

5. Conclusion

In this paper, two new algebraic operations were defined and some properties related to them were obtained. Example was given to validate the two new algebraic operations. It was established that the algebraic laws in PFSs are also hold in PFMSs because PFMSs are extension of PFSs. In our future work, we investigate some algebraic structures of picture fuzzy multisets and its applications to decision-making problems in medicine, appointment procedure and so on.

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