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Fuzzy and Multi-fuzzy Sequences of Fuzzy Numbers and Their Convergence at a Given Level

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Abstract. This paper explores the behavior and convergence properties of fuzzy and multi-fuzzy sequences of fuzzy numbers, with a focus on their α-level convergence, α-cut boundedness, and algebraic operations. We extend the concept to multi-fuzzy sequences of dimension r, where each component sequence consists of fuzzy numbers. Through a series of illustrative examples, we analyze convergence behavior under varying α-levels, highlighting cases where convergence occurs only at $\alpha = 1$ or fails altogether. Furthermore, we demonstrate that the sum, difference, and product of two convergent multi-fuzzy sequences also converge under the same α-level. The paper further analyzes sequences of trapezoidal fuzzy numbers—both normalized and non-normalized—identifying necessary conditions for convergence and presenting counterexamples where convergence fails, even when sufficient conditions are true. These results contribute to a deeper understanding of fuzzy sequence behavior and set a foundation for further studies in multi-fuzzy sequence analysis and multi-fuzzy calculus.

***Keywords*:** Fuzzy number; Triangular fuzzy number; Trapezoidal fuzzy number; multi-fuzzy sequence; α-level convergence; α-cut boundedness

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1. Introduction

The theory of fuzzy sets, introduced by Zadeh in 1965 [27], provides a powerful mathematical framework for modeling uncertainty and vagueness, especially in cases where classical set theory proves insufficient. Fuzzy numbers, as an extension of fuzzy set theory, provide a formal mechanism to represent and manipulate imprecise numerical quantities in various mathematical and computational contexts. While Zadeh introduced the foundational concepts, the formal definition and detailed study of fuzzy numbers as mathematical objects (with properties like normality, convexity, etc.) were further developed by later researchers Dubois, Prade etc.(late 1970s and early 1980s)[3,4], who provided rigorous mathematical frameworks for fuzzy numbers. Subsequently, many researchers worked on fuzzy numbers [1,2,10,24]. Some work related to triangular and trapezoidal fuzzy numbers is found in the papers [25,26].

Sabu Sebastian and Ramakrishnan [14,17,18,21,22,23], introduced the concept of multi-fuzzy sets, a natural extension of classical fuzzy sets, by employing multi-dimensional membership functions to represent multiple degrees of belonging simultaneously. A significant advancement in this area is the multi-fuzzy extension of crisp functions, also developed by Sabu Sebastian and Ramakrishnan [19,20]. In particular, our work explores the use of fuzzy matrices as bridge functions to construct multi-fuzzy extensions of such functions [12]. In [13], we proposed several similarity measures on multi-fuzzy sets, which further enhanced the analytical tools available for handling complex fuzzy data. Many authors have defined and explored the concept of fuzzy metric spaces and related notions in various ways [5,8,16,28]. The idea of a fuzzy sequence in a metric space was originally defined by Muthukumari et al. [11]. Building upon this, we extended the framework to define and analyze multi-fuzzy sequences, including their convergence and boundedness properties in metric spaces [6]. Furthermore, we introduced the notion of multi-fuzzy Cauchy sequences and examined their convergence behaviors within the same setting [6].

In this paper, we focus on the study of fuzzy and multi-fuzzy sequences, particularly sequences of fuzzy numbers. We explore convergence properties at different α-levels, boundedness via α-cuts, and various operations on such sequences. Special attention is given to triangular and trapezoidal fuzzy numbers due to their wide usage in practical applications and simplicity in analytical handling.

2. Preliminary

The following definitions and terminologies are used throughout this paper.

Definition 2.1. *[9] A fuzzy set A on $\mathbb{R}$ is a fuzzy number if it satisfies the following properties:*

1. A is a normal fuzzy set. That is the height $h(A) = \sup_{x\in\mathbb{R}} A(x) = 1$.
2. $\alpha -$ cut, $A_{[\alpha]} = \{x| \quad A(x) \geq \alpha\}$ is a closed interval for each $\alpha \in (0,1]$.
3. $A_{[0+]}$, the support of μ is bounded.

Definition 2.2. *[7] The $1 - cut$, A_1 is called the core of a fuzzy set A, denoted by $core(A)$, that is,*

$$\text{core}(A) = \{x \in X : A(x) = 1\}.$$

Definition 2.3. *[7] A triangular fuzzy number is a fuzzy number of the following form:*

For $a \leq b \leq c$,

$$A(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ \dfrac{c-x}{c-b}, & b \le x \le c \\ 0, & x > c \end{cases}$$

Definition 2.4. *[7] A trapezoidal fuzzy number is a fuzzy number of the following form:*
For $a \le b \le c \le d$,

$$A(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \dfrac{d-x}{d-c}, & c \le x \le d \\ 0, & x > d \end{cases}$$

Definition 2.5. *[9] Let A, B be two fuzzy numbers. The fuzzy numbers $A+B$, $A-B$ and $A.B$ are defined as follows:*

$$(A+B)(z) = \sup_{z=x+y} \min[A(x),\ B(y)]$$

$$(A-B)(z) = \sup_{z=x-y} \min[A(x),\ B(y)]$$

$$(A.B)(z) = \sup_{z=x.y} \min[A(x),\ B(y)]$$

Definition 2.6. *[18] Let r be a positive integer. A multi-fuzzy set A in a universal set X is a set of ordered $(r+1)$-tuples*

$$A = \{\langle x, \mu_1(x), \mu_2(x), \dots, \mu_r(x)\rangle : \ x \in X\},$$

where each μ_i is a function from a universal set X in to $[0,1]$. The positive integer r is called the dimension of A. The collection of all multi-fuzzy sets of dimension r is denoted by $M^r FS(X)$.

Definition 2.7. *[11] Let X be a non-empty set. A fuzzy set on $\mathbb{N} \times X$ is called a fuzzy sequence in X, that is, $A: \mathbb{N} \times X \to [0,1]$ is called a fuzzy sequence in X.*

Here, after the ordinary sequence that is a mapping from $\mathbb{N}$ to X will be named as a crisp sequence. In [11], a crisp sequence f in X is identified with a fuzzy sequence $A_f: \mathbb{N} \times X \to [0,1]$, given by,

$$A_f(n,x) = \begin{cases} 1, & \text{if } f(n) = x \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.8. *[11] Let (X,d) be a metric space, and let A be a fuzzy sequence on X.*

Let $\alpha \in (0,1]$ *and* $a \in X$. A *is said to converge to* a *at a level* α, *if*

1) for each $n \in \mathbb{N}$, there exists an element $x \in X$ such that $A(n,x) \geq \alpha$;
2) given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x,a) < \epsilon$, for all $n \geq n_0$ and for all x with $A(n,x) \geq \alpha$.

In this paper, for $\alpha, \beta \in [0,1]^r$, with $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_r]$ and $\beta = [\beta_1, \beta_2, \dots, \beta_r]$, by $\beta \geq \alpha$ we mean $\beta_i \geq \alpha_i, \forall i = 1,2,\dots,r$.

Definition 2.9. *[6] Let* X *be a non-empty set, and let* r *be a positive integer. A multi-fuzzy set* μ *of dimension r on* $\mathbb{N} \times X$ *is called a multi-fuzzy sequence of dimension* k *in* X. *That is,* $\mu: \mathbb{N} \times X \to [0,1]^r$ *given by*

$\mu(n,x) = (\mu_1(n,x), \mu_2(n,x), \dots, \mu_r(n,x))$, for $n \in N$, $x \in X$, where $\mu_i: \mathbb{N} \times X \to [0,1]$, forall $i = 1,2,\dots,r$.

Definition 2.10. *[6] Let* (X,d) *be a metric space, and let* μ *be a multi-fuzzy sequence of dimension* r *in X. Let* $\alpha \in [0,1]^r \backslash \{(0,0,\dots,0)\}$ *and* $a \in X$. *Then* μ *is said to converge to* a *at level* α, *if*

1) for each $n \in \mathbb{N}$, the set $\{x \in X: \mu(n,x) \geq \alpha\}$ is non-empty. That is $\exists x \in X$ with $\mu(n,x) \geq \alpha$;
2) given $\varepsilon > 0$, $\exists k_\varepsilon \in \mathbf{N}$ suchthat $d(x,a) < \varepsilon, \forall n \geq k_\varepsilon$ and $\forall x \in X$ with $\mu(n,x) \geq \alpha$.

Remark 2.1. *[6] We say that a multi-fuzzy sequence* μ *of dimension* r *in* X *does not converge to* $a \in X$ *at level* α *if,*

either $\exists n_0 \in \mathbb{N}$ suchthattheset $\{x \in X: \mu(n_0,x) \geq \alpha\} = \phi$. Thatis $\forall x \in X, \mu(n_0,x) < \alpha$.

or

$\exists \varepsilon_0 > 0, \forall k \in \mathbb{N}, \exists n_k \geq k$ and $x_k \in X$ with $\mu(n_k, x_k) \geq \alpha$, but $d(x_k, a) > \varepsilon$.

Definition 2.11. *[6] Let* μ *be a multi-fuzzy sequence of dimension* r *in* $\mathbb{R}$. *We say that* μ *is bounded at the level* $\alpha \in [0,1]^r \backslash \{(0,0,\dots,0)\}$, *if there exists* $M > 0$ *such that for every* $n \in \mathbb{N}$ *and for every* $x \in \mathbb{R}$, *whenever* $\mu(n,x) \geq \alpha$, *we have,* $|x| \leq M$.

Definition 2.12. *A multi-fuzzy sequence* μ *of dimension* r *in a subspace* X *of* $\mathbb{R}$ *is* α*-cut bounded multi-fuzzy sequence at* $\alpha \in [0,1]^r \backslash \{(0,0,\dots,0)\}$, *if for each* $n \in \mathbb{N}$, *the level set* $\mu^n_{[\alpha]} = \{x \in X: \ \mu(n,x) \geq \alpha\}$ *is a bounded subset of* X.

Theorem 2.1. *A convergent multi-fuzzy sequence that is* α*-cut bounded at* α *is a multi-fuzzy bounded sequence at* α.

3. Fuzzy and multi-fuzzy sequences of fuzzy numbers

In this section, we study the behavior of a multi-fuzzy sequence $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ of dimension r, where each μ_i is a fuzzy sequence of fuzzy numbers. That is $\mu_i^n: \{n\} \times \mathbb{R} \to [0,1]$ is a fuzzy number, for each $n \in \mathbb{N}$.

Example 3.1. *For* $a \in \mathbb{R}$, *define* $\mu: \mathbb{N} \times \mathbb{R} \to [0,1]$ *by,*

$$\mu(n,x) = \begin{cases} 1 - \dfrac{|x-a|}{n+1}, & if\ |x-a| \le \dfrac{1}{n} \\ 0, & otherwise \end{cases}$$

Here each μ_i^n is a triangular fuzzy number with core $\{a\}$. We can prove that μ converges to a at a level α with $\frac{1}{2} < \alpha \le 1$.

For any $n \in \mathbb{N}$, let $x = a$, then $\mu(n,x) = 1 \ge \alpha$. Now let $\varepsilon > 0$. Choose $K \in \mathbb{N}$ such that $\frac{1}{K} < \varepsilon$. Now for all $n \ge K$, $\mu(n,x) \ge \alpha$ implies $|x-a| \le \frac{1}{n} \le \frac{1}{K} < \varepsilon$.

Example 3.2. *If we define*

$$\mu(n,x) = \begin{cases} 1 - \dfrac{|x-a|}{n+1}, & if\ |x-a| \le n \\ 0, & otherwise \end{cases}$$

for $a \in \mathbb{R}$, then μ is convergent to a only at $\alpha = 1$. That is μ doesn't converge to a at a level α, where $0 < \alpha < 1$.

If $\alpha = 1$, then for any $n \in \mathbb{N}$, $\mu(n,x) \ge \alpha$ implies $x = a$, hence $|x-a| = 0$.

If $0 < \alpha < 1$, then $1 - \alpha > 0$, we can choose $K \in \mathbb{N}$ such that $0 < \frac{1}{K} < 1 - \alpha 1 - \frac{1}{K} > \alpha$. Let $\varepsilon_0 = \frac{1}{K^2}$. For $k \in \mathbb{N}$, if $k \le K$, let $n_k = K$ and $x_k = a + \frac{1}{K}$. Then $|x_k - a| = \frac{1}{K} < K$ and

$$\begin{aligned} \mu(n_k, x_k) &= 1 - \frac{|x_k - a|}{n_k + 1} \\ &= 1 - \frac{\frac{1}{K}}{K+1} \\ &= 1 - \frac{1}{K(K+1)} \\ &> 1 - \frac{1}{K} > \alpha \end{aligned}$$

But $|x_k - a| = \frac{1}{K} > \frac{1}{K^2} = \varepsilon_0$. If $k > K$, let $n_k = k$ and $x_k = a + \frac{1}{K}$. Then $|x_k - a| = \frac{1}{K}$.

In this case also $|x_k - a| > \varepsilon_0$. Hence μ doesn't converge to a at this level α.

Example 3.3.

$$\mu(n,x)=\begin{cases}1-n\left|x-\frac{1}{n}\right|, & if\left|x-\frac{1}{n}\right|\le\frac{1}{n}\\ 0, & otherwise\end{cases}$$

In this example, each μ_i^n is a triangular fuzzy number with core $\{\frac{1}{n}\}$. We can prove that μ converges to 0 at any $\alpha>0$.

If $\alpha=1$, then for any $n\in\mathbb{N}$, $\mu(n,x)\geq\alpha$ implies $x=\frac{1}{n}$, hence $|\frac{1}{n}-0|=\frac{1}{n}$.

Let $0<\alpha<1$. For any $n\in\mathbb{N}$, let $x=\frac{1}{n}$, then $\mu(n,x)=1\geq\alpha$. Now let $\varepsilon>0$. Choose $K\in\mathbb{N}$ such that $\frac{2}{K}<\varepsilon$. Now for all $n\geq K$, $\mu(n,x)\geq\alpha$ implies $|x-\frac{1}{n}|\leq\frac{1}{n}$. This implies $0<x<\frac{2}{n}-\frac{2}{n}<x<\frac{2}{n}|x-0|<\frac{2}{n}\leq\frac{2}{K}<\varepsilon$.

Example 3.4. *Let $a,b\in\mathbb{R}$. A multi-fuzzy sequence $\mu=(\mu_1,\ \mu_2)$ of fuzzy numbers of dimension 2 is given by,*

$$\mu_1(n,x)=\begin{cases}1-\frac{|x-a|}{n+1}, & if\,|x-a|\le\frac{1}{n}\\ 0, & otherwise\end{cases}$$

$$\mu_2(n,x)=\begin{cases}1-\frac{|x-b|}{n+1}, & if\,|x-b|\le\frac{1}{n}\\ 0, & otherwise\end{cases}$$

If $a\neq b$ and $\alpha=[\alpha_1,\ \alpha_2]$ with $\alpha_1,\ \alpha_2>0$, then μ is divergent at this level.

Since $a\neq b$, $\frac{|a-b|}{2}>0$, hence there exist $K\in\mathbb{N}$ such that $0<\frac{1}{K}<\frac{|a-b|}{2}$. Then for any $x\in\mathbb{R}$, either $\mu_1(K,x)=0$ or $\mu_2(K,x)=0$ or both. Therefore $\{x:\mu(K,x)\geq\alpha\}=\Phi$.

Theorem 3.1. *Every multi-fuzzy sequence of fuzzy numbers of dimension r is $\alpha-$cut bounded for a given $\alpha\in[0,1]^r\backslash\{(0,0,\ldots,0)\}$.*

Proof: Let $\mu=(\mu_1,\ \mu_2,\ldots,\mu_r)$, where each μ_i is a fuzzy sequence of fuzzy numbers. Let $\alpha=[\alpha_1,\ \alpha_2,\ldots,\alpha_r]\in[0,1]^r\backslash\{(0,0,\ldots,0)\}$. Then the $\alpha-$cut ${\mu_{i}}_{[\alpha_i]}{}^{n}$ is a closed bounded interval, for each $n\in\mathbb{N}$ and for every $i=1,2,..,r$. Also $\mu(n,x)\geq\alpha\Rightarrow\mu_i(n,x)\geq\alpha_i$, $i=1,2,\ldots,r$. Therefore the $\alpha-$cut $\mu_{[\alpha]}^n$ of μ is the intersection of a finite number of closed and bounded intervals; hence, it is a bounded subset of $\mathbb{R}$. That is μ is $\alpha-$cut bounded.

Corollary 3.2. *A convergent multi-fuzzy sequence of fuzzy numbers at a level α is multi-fuzzy bounded at that level.*

Proof: We have every multi-fuzzy sequence of fuzzy numbers of dimension r is $\alpha-$cut bounded for a given α and a convergent multi-fuzzy sequence that is $\alpha-$cut bounded at a given α is multi-fuzzy bounded at that level.

Theorem 3.3. *Let μ and η be two fuzzy sequences of fuzzy numbers. If μ and η converges to a and b respectively at a given level α. Then the fuzzy sequence of fuzzy numbers $\mu+\eta$, $\mu-\eta$ and $\mu.\eta$ converges to $a+b$, $a-b$ and $a.b$ respectively at α.*

Proof: Assume that μ and η converges to a and b respectively at a level α. Then for each $n \in \mathbb{N}$, there exist x', $y' \in \mathbb{R}$ such that $\mu(n,x') \geq \alpha$ and $\eta(n,y') \geq \alpha$. Let $z' = x' + y'$. Then

$$
\begin{aligned}
(\mu+\eta)(n,z') &= \sup_{z\prime=x+y} \min[\mu(n,x),\ \eta(n,y)] \\
&\geq \min[\mu(n,x'),\ \eta(n,y')] \\
&\geq \alpha
\end{aligned}
$$

Now for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for any $n \geq K$, $\mu(n,x) \geq \alpha$ implies $|x-a| < \frac{\varepsilon}{2}$ and $\eta(n,y) \geq \alpha$ implies $|y-b| < \frac{\varepsilon}{2}$. Therefore if $(\mu+\eta)(n,z) \geq \alpha$ for $n \geq K$ implies, $\sup_{z=x+y} \min[\mu(n,x),\ \eta(n,y)] \geq \alpha$. This implies $\min[\mu(n,x),\ \eta(n,y)] \geq \alpha$, for some $x, y \in \mathbb{R}$ with $z = x+y$. Therefore $\mu(n,x) \geq \alpha$ and $\eta(n,y) \geq \alpha$. hence $|z-(a+b)| = |x+y-(a+b)| < \varepsilon$. Similarly, we can prove that $\mu-\eta$ converges to $a-b$ at the level α.

Since a convergent multi-fuzzy sequence of fuzzy numbers at a level α is multi-fuzzy bounded at that level, we can choose $M' > 0$ such that for any $x \in X$ with $\mu(n,x) \geq \alpha$, $|x| \leq M'$. Therefore if $M = \max\{M',\ |b|\}$,

$$
\begin{aligned}
|xy-ab| &= |xy-xb+xb-ab| \\
&\leq |x||y-b| + |b||x-a| \\
&\leq M|y-b| + M|x-a|
\end{aligned}
$$

Now, for any ε>0, there exists $K \in \mathbb{N}$ such that for any $n \geq K$, $\mu(n,x) \geq \alpha$ implies $|x-a| < \frac{\varepsilon}{2M}$ and $\eta(n,y) \geq \alpha$ implies $|y-b| < \frac{\varepsilon}{2M}$. Therefore if $(\mu.\eta)(n,z) \geq \alpha$ for $n \geq K$ implies, $\sup_{z=x.y} \min[\mu(n,x),\ \eta(n,y)] \geq \alpha$. This implies $\min[\mu(n,x),\ \eta(n,y)] \geq \alpha$, for some $x, y \in \mathbb{R}$ with $z = x.y$. Therefore $\mu(n,x) \geq \alpha$ and $\eta(n,y) \geq \alpha$. Hence,

$$
\begin{aligned}
|z-ab| = |xy-ab| &= |xy-xb+xb-ab| \\
&\leq M|y-b| + M|x-a| \\
&< \varepsilon
\end{aligned}
$$

4. Sequences of trapezoidal fuzzy numbers

Now we consider sequences of trapezoidal fuzzy numbers.

Example 4.1. *Let* μ *be a sequence of trapezoidal fuzzy numbers defined by*

$$\mu^n(x) = \mu(n,x) = \begin{cases} nx+2, & \text{if } \frac{-2}{n} \leq x \leq \frac{-1}{n} \\ 1, & \text{if } \frac{-1}{n} \leq x \leq \frac{1}{n} \\ 2-nx, & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \text{otherwise} \end{cases}$$

This sequence converges to 0 at any level $0 < \alpha \leq 1$.

For each $n \in \mathbb{N}$, let $x = \frac{1}{n}$, for this x, $\mu(n,x) = 1 \geq \alpha$. Now for $\varepsilon > 0$, choose $K \in \mathbb{N}$ such that $\frac{2}{K} < \varepsilon$. If $n \geq K$, $\mu(n,x) \geq \alpha$ implies $\frac{-2}{n} \leq x \leq \frac{-1}{n}$ or $\frac{-1}{n} \leq x \leq \frac{1}{n}$ or $\frac{1}{n} \leq x \leq \frac{2}{n}$. In any case, $|x| \leq \frac{2}{n} < \varepsilon$.

Theorem 4.1. *A sequence of trapezoidal fuzzy numbers, with core* $[a_n, b_n]$, *for each* $n \in \mathbb{N}$, *is converge to* $a \in \mathbb{R}$ *at a level* $0 < \alpha \leq 1$. *Then* $\bigcap_{n=1}^{\infty} [a_n, b_n] = \{a\}$.

Proof: Assume that μ is a sequence of trapezoidal fuzzy numbers and it converges to a at a level $0 < \alpha \leq 1$. If $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \{a\}$. That is there exist $b \in \mathbb{R}$ such that $b \in \bigcap_{n=1}^{\infty} [a_n, b_n]$, with $a < b$. Then, we have $[a,b] \subset \bigcap_{n=1}^{\infty} [a_n, b_n]$. Therefor $\mu(n,x) = 1$, if, $a \leq x \leq b$. Implies μ does not converge to a. For let $\varepsilon_0 = \frac{b-a}{2} > 0$. For each $k \in \mathbb{N}$, we have $\mu(k,b) = 1 \geq \alpha$, but $|a-b| = b-a > \varepsilon_0$.

Remark 4.1. *The converse of the above result is not true. That is if* μ *is a sequence of trapezoidal fuzzy numbers with core* $[a_n, b_n]$, *for each* $n \in \mathbb{N}$, *and* $\bigcap_{n=1}^{\infty} [a_n, b_n] = \{a\}$. *Then* μ *need not converge to* a *at a given level* α. *Consider the following example. For* $n > 1$, *let,*

$$\mu^n(x) = \mu(n,x) = \begin{cases} \dfrac{n+x}{n-\frac{1}{n}}, & \text{if } -n \leq x \leq \frac{-1}{n} \\ 1, & \text{if } \frac{-1}{n} \leq x \leq \frac{1}{n} \\ \dfrac{n-x}{n-\frac{1}{n}} & \text{if } \frac{1}{n} \leq x \leq n \\ 0, & \text{otherwise} \end{cases}$$

$$\mu^1(x) = \mu(1,x) = \begin{cases} x+2, & if\ -2 \le x \le -1 \\ 1, & if\ -1 \le x \le 1 \\ 2-x, & if\ 1 \le x \le 2 \\ 0, & otherwise \end{cases}$$

Here $\bigcap_{n=1}^{\infty} [a_n, b_n] = \{0\}$. But μ does not converge to 0 at a level $0 < \alpha < 1$.

Let $K \in \mathbb{N}$ such that $0 < \frac{1}{K} < 1 - \alpha 1 - \frac{1}{K} > \alpha$. Let $\varepsilon_0 = \frac{1}{K}$. For each $k > 1$, let let $n_k = k$ and $x_k = \frac{1}{k} + \frac{1}{K}$. Then $\frac{1}{k} < x_k < k$ and

$$\begin{aligned} \mu(n_k, x_k) &= \frac{k - (\frac{1}{k} + \frac{1}{K})}{k - \frac{1}{k}} \\ &= 1 - \frac{k}{K(k^2-1)} \\ &> 1 - \frac{1}{K} > \alpha \end{aligned}$$

But $|x_k - 0| = \frac{1}{k} + \frac{1}{K} > \frac{1}{K} = \varepsilon_0$.

Example 4.2. *Let $a, b \in \mathbb{R}$ with $a < b$. Define a sequence of trapezoidal fuzzy numbers as follows:*

Here $\bigcap_{n=1}^{\infty} [a_n, b_n] = [a, b]$. Therefore, by the above theorem, it is divergent at any level $\alpha > 0$.

$$\mu^n(x) = \mu(n,x) = \begin{cases} nx - na + 1, & if\ a - \frac{1}{n} \le x \le a \\ 1, & if\ a \le x \le b \\ nb - nx + 1, & if\ b \le x \le b + \frac{1}{n} \\ 0, & otherwise \end{cases}$$

4.1. Sequence of non-normalized trapezoidal fuzzy numbers

In this session, we first consider sequences of non-normalized trapezoidal fuzzy numbers with height a fixed real number γ, where $0 < \gamma < 1$, for each $n \in \mathbb{N}$. Here if $\gamma < \alpha \le 1$, then the sequence is divergent at this level. Because $\{x : \mu(n,x) \ge \alpha\} = \phi$. But it may converge if $0 < \alpha \le \gamma$.

Example 4.3. *Consider a sequence of non-normalized trapezoidal fuzzy numbers with height 0.5 defined as follows:*

$$\mu^n(x) = \mu(n,x) = \begin{cases} 0.5(nx+2), & if\ \frac{-2}{n} \le x \le \frac{-1}{n} \\ 0.5, & if\ \frac{-1}{n} \le x \le \frac{1}{n} \\ 0.5(2-nx), & if\ \frac{1}{n} \le x \le \frac{2}{n} \\ 0, & otherwise \end{cases}$$

This sequence converges to 0 at a level α with $0 < \alpha \le 0.5$.

Now, we move to sequences of non-normalized trapezoidal fuzzy numbers with varying heights. That is for each $n \in \mathbb{N}$, the height of n^{th} term is γ_n, where $0 < \gamma_n < 1$.

Example 4.4.

$$\mu^n(x) = \mu(n,x) = \begin{cases} \frac{1}{n+1}(nx+2), & if\ \frac{-2}{n} \le x \le \frac{-1}{n} \\ \frac{1}{n+1}, & if\ \frac{-1}{n} \le x \le \frac{1}{n} \\ \frac{1}{n+1}(2-nx), & if\ \frac{1}{n} \le x \le \frac{2}{n} \\ 0, & otherwise \end{cases}$$

Here, the height of μ^n is $\frac{1}{n+1}$ for each $n \in \mathbb{N}$. We can prove that this sequence is divergent at any level. For let $0 < \alpha \le 1$. Then there exists $K \in \mathbb{N}$ such that $\frac{1}{K} < \alpha$. For this K, $\mu(K,x) \le \frac{1}{K+1} < \alpha$, for every $x \in \mathbb{R}$. Therefore $\{x: \mu(n,x) \ge \alpha\} = \phi$.

Example 4.5.

$$\mu^n(x) = \mu(n,x) = \begin{cases} \frac{n}{n+1}(nx+2), & if\ \frac{-2}{n} \le x \le \frac{-1}{n} \\ \frac{n}{n+1}, & if\ \frac{-1}{n} \le x \le \frac{1}{n} \\ \frac{n}{n+1}(2-nx), & if\ \frac{1}{n} \le x \le \frac{2}{n} \\ 0, & otherwise \end{cases}$$

This sequence is divergent if $\frac{1}{2} < \alpha < 1$. Because for $n = 1$, $\mu(1,x) \le \frac{1}{2} < \alpha$, for every $x \in \mathbb{R}$. But if $0 < \alpha \le \frac{1}{2}$, the sequence converges to 0 at this level.

Theorem 4.2. *If μ is a sequence of non-normalized trapezoidal fuzzy numbers with varying heights γ_n and if $\lim_{n\to\infty} \gamma_n = 0$, the sequence is divergent at any level α.*

Proof: Let $0 < \alpha \le 1$. Since $\lim_{n\to\infty} \gamma_n = 0$, there exist $K \in \mathbb{N}$ such that $|\gamma_n - 0| = \gamma_n < \alpha$, for all $n \ge K$. Therefore, for all $n \ge K$, $\mu(n,x) \le \gamma_n < \alpha$. Hence for such n, $\{x: \mu(n,x) \ge \alpha\} = \phi$.

5. Conclusion

In this paper, we have investigated the behavior and convergence properties of fuzzy and multi-fuzzy sequences of fuzzy numbers, with a particular emphasis on α-level analysis. Through detailed examples, we demonstrated how the convergence of such sequences depends on the structure and nature of the underlying fuzzy numbers, whether triangular, trapezoidal, or non-normalized trapezoidal.

We established that a convergent multi-fuzzy sequence of fuzzy numbers at a given level is necessarily bounded, and further demonstrated that standard arithmetic operations—such as addition, subtraction, and multiplication—preserve convergence under appropriate conditions. Our investigation into sequences of trapezoidal fuzzy numbers revealed that convergence is closely linked to the behavior of their core intervals across the sequence. Additionally, the study of non-normalized trapezoidal fuzzy numbers highlighted that convergence depends critically on the sequential variation of core values.

These findings not only enhance the theoretical understanding of multi-fuzzy sequences but also on more sophisticated applications in areas involving imprecise data and dynamic uncertainty modeling.

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