

Fuzzy Archimedean Decompositions and Graded Ideal Structures in Leavitt Path Algebras

Shanookha Ali^{1*}, Shafeequdheen Palengara², Nitha Niralda Pc³ and Farshid Mofidnakhaei⁴

¹Department of General Science, BITS Pilani Dubai Campus, Dubai International Academic City, 345055, Dubai, UAE. e-mail: shanookha@dubai.bits-pilani.ac.in

²Department of Mathematics, SRM University, Andhra Pradesh, India.
e-mail: shafeequdheen_p@srmp.edu.in

³Department of Mathematics and Statistics, Providence Women's College, Calicut, Kerala, India. e-mail: nithaniraldapc@providencecollegecalicut.ac.in

⁴Department of Physics, Sari Branch, Islamic Azad University, Sari, Iran.
e-mail: Farshid.Mofidnakhaei@gmail.com

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Abstract. This investigation presents a comprehensive theoretical framework that unifies fuzzy set methodologies with traditional sandpile dynamics and algebraic path constructions. Our approach extends conventional chip-firing mechanisms by utilising fuzzy sandpile monoids, which employ membership degree assignments to encode uncertainty alongside progressive state transitions within graph-based systems. By constructing weighted Leavitt path algebras specifically tailored for fuzzy contexts, we establish substantial relationships between combinatorial graph attributes and algebraic invariants. The research illuminates interactions among fuzzy hereditary frameworks, idempotent structures, and graded ideal sequences, revealing their collective mathematical coherence. Our principal contribution establishes that multiple lattice architectures, though originating from distinct mathematical viewpoints, maintain fundamental isomorphic relationships, thus bridging discrete dynamics, fuzzy reasoning, and non-commutative algebraic theory. These results enable the investigation of complex systems exhibiting imprecise boundaries and continuous rather than discrete transitions.

Keywords: Fuzzy sandpile monoids, Weighted Leavitt path algebras, Archimedean classes, Fuzzy hereditary subsets, Lattice isomorphisms

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1. Introduction

Discrete dynamical system modeling has captivated researchers spanning numerous scientific domains. Sandpile configurations, initially developed for explaining self-organised criticality within physical phenomena, have matured into sophisticated algebraic constructs deeply connected to graph-theoretic principles, commutative algebra, and combinatorial optimisation theory. Concurrently, Leavitt path algebras

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have proven instrumental for translating graph architectures into algebraic expressions, establishing connections between structural graph properties and ring-theoretic characteristics. Classical treatments of sandpiles and Leavitt algebras rely on crisp, deterministic paradigms where elements possess binary membership status and edges maintain definite existence.

Real-world networks, however, frequently demonstrate uncertainty, graduated membership, and smooth transitions. Consider social networks with variable connection intensities, biological pathways exhibiting probabilistic activation patterns, or transportation infrastructure experiencing fluctuations in capacity. Addressing such scenarios requires mathematical frameworks that inherently accommodate imprecision.

Fuzzy graph theory evolution has furnished crucial instruments for managing these uncertainties. Rosenfeld's foundational contributions [23] established fuzzy graph principles, subsequently expanding into diverse sophisticated architectures. Contemporary developments encompass vague graphs [21], which capture dual membership and non-membership characteristics, alongside interval-valued fuzzy graphs [22], that represent uncertainty through interval notation rather than point values. Complex Pythagorean fuzzy graphs [24] further advance these concepts by incorporating complex-valued membership specifications, thereby facilitating enhanced representations of uncertain relationships.

Fuzzy graph applications have spread across varied disciplines. Vague graph domination problems demonstrate practical utility in medical diagnostic systems and healthcare network evaluation [17], where patient-provider relationships and treatment effectiveness contain inherent uncertainties. Fuzzy graph-derived topological indices prove valuable for decision-making scenarios [18], particularly in situations that require multi-criteria assessment under information incompleteness. Hamiltonian fuzzy graph investigations reveal applications in human trafficking network analysis [5], where the strengths of location interconnections vary continuously. Similarly, vertex connectivity metrics [4] and container constructions [6] within fuzzy graphs enhance the understanding of network robustness when edges exhibit variable reliability.

Advanced fuzzy architectures continue emerging. Interval intuitionistic neutrosophic graphs [16] integrate indeterminacy alongside membership and non-membership components, providing instruments for climatic evaluation where data uncertainty originates from multiple sources. These advances demonstrate the maturation of fuzzy graph theory into a comprehensive discipline, possessing both theoretical sophistication and practical relevance.

Our work introduces fuzzy generalizations of sandpile monoids alongside weighted Leavitt path algebras, establishing an integrated architecture where uncertainty constitutes foundational rather than supplementary structure. Assigning membership degrees to vertices and edges permits nuanced representations of partial connectivity and graduated influence patterns. Through this perspective, we uncover that classical findings regarding idempotents, filters, and ideal structures extend elegantly, while phenomena unique to fuzzy contexts simultaneously emerge.

Three converging observations motivate our investigation. First, fuzzy graph theory has achieved considerable maturity, spanning applications from network

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optimization to machine learning. Second, recent investigations connecting sandpile models to the K-theory of weighted Leavitt algebras [3] suggest that deeper algebraic structures await exploration. Foundational work on chip-firing games [11] and sandpile structure theory [9] established discrete dynamics-algebraic invariant connections we now extend to fuzzy settings. Third, the systematic absence of fuzzy treatment regarding these connections represents a substantial literature gap. Graph structure algebraic perspectives have proven remarkably productive. Leavitt's original module type investigations [19] initiated research programs culminating in Leavitt path algebras, encoding directed graphs as non-commutative algebras. Subsequent developments in socle theory [8], graph algebra K-theory [7], and weighted variants [20, 15] have revealed profound connections between graph properties and ring-theoretic invariants. Our fuzzy generalization builds upon these foundations while introducing novel technical challenges regarding membership function-algebraic operation interactions.

This paper's organization reflects our gradual theory construction. Following the establishment of preliminary concepts and notation, we define fuzzy sandpile monoids via explicit congruence relations that respect membership functions. Subsequently, we introduce fuzzy Archimedean classes, partitioning the monoid according to natural preordering. Next, we develop fuzzy weighted Leavitt path algebras, carefully adapting the Cuntz-Krieger relations to incorporate fuzzy weights. Our principal theorems establish lattice isomorphisms connecting idempotents, filters, hereditary saturated subsets, and graded ideals, demonstrating that these apparently disparate structures manifest an identical underlying mathematical reality. Finally, we characterize these algebras' structure through graded ideal chains, revealing decomposition into simpler constituents.

Throughout development, we emphasize fuzzy membership function-algebraic operation interactions. Unlike simple scalar multiplication, fuzzy operations must respect lattice structures and closure properties. This technical requirement introduces interesting proof subtleties but ultimately yields a theory that feels both natural and powerful. We anticipate that this work will inspire continued exploration at the intersection of fuzzy systems, algebraic structures, and combinatorial dynamics, demonstrating that algebraic reasoning can illuminate network structures operating under uncertainty in ways that classical, crisp frameworks inadequately describe.

2. Preliminary Concepts

We establish fundamental notions from fuzzy graph theory, sandpile dynamics, and Leavitt path algebras essential for subsequent results.

2.1. Fuzzy graph foundations

Definition 2.1. Consider a non-empty set X . A fuzzy set A over X employs a characteristic mapping $\mu_A : X \rightarrow [0,1]$, where $\mu_A(x)$ quantifies element x 's membership degree within A . Standard set-theoretic operations extend naturally: union via $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$ and intersection through $\mu_{(A \cap B)}(x) = \min\{\mu_A(x), \mu_B(x)\}$.

Definition 2.2. ([23]) A directed fuzzy graph structure (E, μ, γ) comprises:

- An underlying directed graph $E = (E^0, E^1, s, r)$ where E^0 denotes vertices, E^1 represents edges, $s : E^1 \rightarrow E^0$ maps edges to sources, and $r : E^1 \rightarrow E^0$ specifies ranges
- Vertex membership assignment $\mu : E^0 \rightarrow [0,1]$
- Edge membership assignment $\gamma : E^1 \rightarrow [0,1]$ satisfying compatibility constraint $\gamma(e) \leq \min\{\mu(s(e)), \mu(r(e))\}$ for each $e \in E_1$.

The fuzzy out-degree for vertex v equals $f_{out} - deg(v) = \sum_{e \in s^{-1}(v)} \gamma(e)$

Definition 2.3. A fuzzy vertex subset $\mathcal{E} : E^0 \rightarrow [0,1]$ satisfies the fuzzy hereditary property when $\mathcal{E}(s(e)) \leq \mathcal{E}(r(e))$ holds for all $e \in E_1$. Additionally, $\mathcal{E}$ is termed fuzzy saturated if $\mathcal{E}(v) \geq \min\{\mathcal{E}(r(e)) : e \in s^{-1}(v)\}$ for each regular vertex v .

2.2 Sandpile dynamics

Definition 2.4. ([10], [13]) A configuration over a directed graph distributes non-negative integer quantities to vertices. Vertex v achieves instability when its chip allocation $c(v)$ satisfies $c(v) \geq out - degree(v)$. The toppling operation at v disperses one chip along each emanating edge. The sandpile monoid emerges as the quotient structure of the free commutative monoid on vertices, modulo the equivalence identifying configurations related through toppling sequences.

Definition 2.5. Given a fuzzy graph (E, μ, γ) , a fuzzy configuration takes the form $\sum_{v \in E^0} c(v) \cdot v$ with $c(v) \in \mathbb{R} \geq 0$. Vertex v permits toppling when $c(v) \geq f_{out} - deg(v)$, producing $c'(v) = c(v) - f_{out} - deg(v)$ and $c'(r(e)) = c(r(e)) + \gamma(e)$ for $e \in s^{-1}(v)$. Denote single-step reduction by $\rightarrow_1$, its reflexive transitive closure by $\rightarrow$, and the generated congruence by $\sim$. The fuzzy sandpile monoid is $FSP(E, \mu, \gamma) = \langle E^0 \rangle / \sim$

Definition 2.6. Establish partial ordering $[c] \leq [c']$ when $c' \sim c + d$ for some d . Elements $x, y \in FSP(E, \mu, \gamma)$ are Archimedean equivalent (*notation: $x \asymp_f y$*) if positive integers m, n exist satisfying $x \leq m \cdot y$ and $y \leq n \cdot x$. The Archimedean class $[x]_f = \{y : y \asymp_f x\}$ forms a subsemigroup harbouring precisely one idempotent. Furthermore $FSP(E, \mu, \gamma) = \cup_{e \in Idem} [e]_f$

2.2 Leavitt path algebra framework

Definition 2.7. ([1]) For directed graph E and field k , the Leavitt path algebra $L_k(E)$ constitutes the k -algebra generated by $\{v \in E^0\} \cup \{e, e^* : e \in E^1\}$ satisfying:

- (V) Vertex orthogonality: $v_i v_j = \delta_{ij} v_i$
- (E1) Source-range compatibility: $s(e)e = e = er(e)$
- (E2) Ghost edge relations: $r(e)e^* = e^* = e^*s(e)$
- (CK1) Ghost edge orthogonality: $e^*f = \delta_{ef} r(e)$
- (CK2) Cuntz-Krieger identity: $v = \sum_{s(e)=v} ee^*$ for regular vertices v

Natural $\mathbb{Z}$ -grading: $deg(v) = 0, deg(e) = 1, deg(e^*) = -1$

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Definition 2.8. A weighted fuzzy graph (E, ω, μ, γ) augments a fuzzy graph (E, μ, γ) with a weight mapping $\omega : E^1 \rightarrow N^+$. The fuzzy weighted Leavitt path algebra $L_k(E, \omega, \mu, \gamma)$ generates from $\{v \in E^0\} \cup \{e_i, e_i^* : e \in E^1, 1 \leq i \leq \omega(e)\}$ with modified Cuntz-Krieger relations: vertex orthogonality becomes $uv = \delta_{uv} \cdot \min\{\mu(u), \mu(v)\} \cdot u$, while source-range relations scale by $\gamma(e)$.

Theorem 2.1. ([2]) Graded ideals of $L_k(E)$ maintain bijective correspondence with hereditary saturated subsets of E^0 via lattice isomorphism. For fuzzy contexts, this extends to isomorphisms among: (i) $\text{Idem}(FSP(E, \mu, \gamma))$, (ii) fuzzy filters $F_{E, \mu, \gamma}$ (iii) fuzzy hereditary saturated subsets $\mathcal{E}_{E, \mu, \gamma}$, (iv) graded ideals $L^{gr}(L_k(E, \mu, \gamma))$, and (v) vertex-generated ideals $L^{ver}(L_k(E, \omega, \mu, \gamma))$.

Theorem 2.2. (Graded Uniqueness [1]) *When E satisfies Condition (L) whereby each cycle admits an exit, any graded k -algebra homomorphism $\phi : L_k(E) \rightarrow A$ maintains injectivity provided $\phi(v) \neq 0$ for all $v \in E^0$.*

Theorem 2.3. (Dhar's Algorithm [13]) *A configuration achieves recurrence (sandpile group membership) if and only if burning from the sink remains possible, with each vertex firing minimally once during the burning process.*

Theorem 2.4. ([12]) *For finite directed graphs possessing global sinks, the sandpile monoid exhibits finite generation, with its structure determined by the relationships between strongly connected components.*

Theorem 2.5. ([14]) *The graded structure of $L_k(E)$ admits characterization through graded ideals forming complete lattices isomorphic to hereditary saturated subset lattices. Graded simple components correspond to E 's strongly connected components.*

Definition 2.9. A fuzzy quotient graph $(E/\mathcal{E}, \mu/\mathcal{E}, \gamma/\mathcal{E})$ for fuzzy hereditary subset $\mathcal{E}$ specifies $(E/\mathcal{E})^0 = \{v \in E^0 : \mathcal{E}(v) = 0\}$, $(E/\mathcal{E})^1 = \{e \in E^1 : r(e) \in (E/\mathcal{E})^0\}$, with inherited membership functions. This construction enables ideal chain decomposition of $L_k(E, \mu, \gamma)$ into layers $0 < I_0 < I_1 < \dots < I_t$ where quotients I_{i+1}/I_i decompose as direct sums of fuzzy-weighted simple components.

Theorem 2.6. (Fuzzy Order Ideal) Let (E, μ, γ) be a fuzzy graph. Then:

- (1) Every fuzzy order ideal is a fuzzy filter.
- (2) Fuzzy order ideals correspond bijectively to idempotents in $FSP(E, \mu, \gamma)$.
- (3) The lattice of fuzzy order ideals is isomorphic to the lattice of fuzzy hereditary saturated subsets.
- (4) Every fuzzy hereditary saturated subset generates a unique graded ideal.
- (5) The correspondence preserves lattice operations.
- (6) Fuzzy filters form a complete lattice under inclusion.
- (7) The Grothendieck group decomposes accordingly.

3. Explicit description of congruence relations

Paralleling classical constructions, we now specify $FSP(E, \mu, \gamma)$'s relations explicitly. Any nonzero monoid element writes as $\sum_{i=1}^n k_i v_i$ using distinct vertices v_i with nonnegative real coefficients $k_i \in R_{\geq 0}$. On free commutative monoid $\langle v \mid v \in E^0 \rangle$, we establish binary relation $\rightarrow_1$ through:

$$\sum_{i=1}^n k_i v_i \begin{cases} \sum_{i \neq j} k_i v_i & \text{if } v_j = s \\ ((\sum_{i \neq j} k_i v_{-i}) + (k_j - f_{out} - \deg(v_j))v_j + \sum_{e \in s^{-1}(v)} \gamma(e) \cdot r(e)) & \text{if } v_j \neq s \end{cases} \quad (1)$$

Here $j \in \{1, \dots, n\}$ satisfies $k_j \geq f_{out} - \deg(v_j)$.

Building upon $\rightarrow_1$, construct its reflexive-transitive closure $\rightarrow$ on $\langle v \mid v \in E^0 \rangle$:

$$a \rightarrow b \text{ if } a = b, \text{ or } a = a_0 \rightarrow_1 a_1 \rightarrow_1 \dots \rightarrow_1 a_k = b \quad (2)$$

Finally, generate congruence $\sim$ from relation $\rightarrow$. Explicitly, $a \sim b$ holds exactly when some sequence $a = a_0, a_1, \dots, a_n = b$ exists in $\langle v \mid v \in E^0 \rangle$ satisfying either $a_i \rightarrow_1 a_{i+1}$ or $a_{i+1} \rightarrow_1 a_i$ for each index $0 \leq i \leq n-1$.

This yields:

$$FSP(E, \mu, \gamma) = \langle v \mid v \in E^0 \rangle / \sim$$

Definition 3.1. For fuzzy hereditary subset $\mathcal{E}$ of (E, μ, γ) , the fuzzy restriction $(E(\mathcal{E}), \mu(\mathcal{E}), \gamma(\mathcal{E}))$ consists of:

- Vertex set: $(E(\mathcal{E}))^0 = \{v \in E^0 : \mathcal{E}(v) > 0\}$
- Edge set: $(E(\mathcal{E}))^1 = \{e \in E^1 : \mathcal{E}(s(e)) > 0, \mathcal{E}(r(e)) > 0\}$
- Restricted membership functions: $\mu(\mathcal{E})|_{(E(\mathcal{E}))^0}$ and $\gamma(\mathcal{E})|_{(E(\mathcal{E}))^1}$

Theorem 3.1. Let (E, μ, γ) be a $\mathcal{FSG}$. There exist natural lattice isomorphisms:

$$Idem(FSP(E, \mu, \gamma)) \cong F_{E, \mu, \gamma} \cong_{E, \mu, \gamma}$$

where:

- $Idem(FSP(E, \mu, \gamma))$ denotes idempotents of the fuzzy sandpile monoid
- $F_{E, \mu, \gamma}$ denotes fuzzy filters
- $\div_{E, \mu, \gamma}$ denotes fuzzy hereditary saturated subsets

3.1. Fuzzy Archimedean classes.

Definition 3.2. For $x, y \in FSP(E, \mu, \gamma)$, define $x \asymp_f y$ if there exist $m, n \in \mathbb{N}^+$ such that:

$$x \leq m \cdot y \text{ and } y \leq n \cdot x$$

in the natural order on $FSP(E, \mu, \gamma)$

Proposition 3.1. The relation $\asymp_f$ is an equivalence relation on $FSP(E, \mu, \gamma)$.

Proof. Reflexivity: For any $x \in FSP(E, \mu, \gamma)$, $x \leq 1 \cdot x$ and $x \leq 1 \cdot x$, so $x \asymp_f x$.

Symmetry: If $x \asymp_f y$, then by definition $y \asymp_f x$.

Transitivity: Suppose $x \asymp_f y$ and $y \asymp_f z$. Then:

$$\begin{aligned} x &\leq m_1 \cdot y \leq m_1 m_2 \cdot z \\ y &\leq n_1 \cdot x \\ y &\leq m_2 \cdot z \end{aligned}$$

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$$z \leq n_2 \cdot y \leq n_2 n_1 \cdot x$$

Therefore, $x \leq (m_1 m_2) \cdot z$ and $z \leq (n_2 n_1) \cdot x$, so $x \asymp_f z$.

For $x \in FSP(E, \mu, \gamma)$, we denote by $[x]_f$ the fuzzy archimedean class of x :

$$[x]_f = \{y \in FSP(E, \mu, \gamma) : y \asymp_f x\}$$

Theorem 3.2. Let (E, μ, γ) be a $\mathcal{FSG}$ and S_E the fuzzy subset of vertices not connecting to any cycle. Then:

- (1) For all $x \in FSP(E, \mu, \gamma)$, $[x]_f$ is a subsemigroup of $FSP(E, \mu, \gamma)$ containing a unique idempotent,
- (2) $FSP(E, \mu, \gamma) = \bigcup_{x \in Idem(FSP(E, \mu, \gamma))} [x]_f$,
- (3) $G(E, \mu, \gamma) = \lim_{\rightarrow x \in Idem(FSP(E, \mu, \gamma))} G([x]_f)$, where $G([x]_f)$ is the Grothendieck group of $[x]_f$,

Proof: The proof generalises Theorem 3.9 from the original paper to the fuzzy setting, using the structures developed in Theorem 2.6.

- (1) Let $y, z \in [x]_f$. Then there exist $m_1, n_1, m_2, n_2 \in \mathbb{N}^+$ such that:

$$\begin{aligned} x &\leq m_1 \cdot y, & y &\leq n_1 \cdot x \\ x &\leq m_2 \cdot z, & z &\leq n_2 \cdot x \end{aligned}$$

Therefore:

$$\begin{aligned} x &\leq m_1 \cdot y \leq m_1(y + z) \\ x &\leq m_2 \cdot z \leq m_2(y + z) \\ y + z &\leq n_1 \cdot x + n_2 \cdot x = (n_1 + n_2) \cdot x \end{aligned}$$

So $y + z \in [x]_f$, proving that $[x]_f$ is a subsemigroup.

Since $FSP(E, \mu, \gamma)$ is finite, for any $x \in [x]_f$, some power $n \cdot x$ is an idempotent (by iterating the monoid operation). This idempotent belongs to $[x]_f$ and is unique because if $e, f \in [x]_f$ are idempotents, then $e \asymp_f f$, which implies $e = f$ (as idempotents are minimal in their archimedean classes).

- (2) Every element x belongs to $[e]_f$ for the unique idempotent $e \in [x]_f$.
- (3) Follow the same pattern as in Theorem 2.6 (6)-(7), using the archimedean decomposition.

4. FLPA

We now introduce $\mathcal{FLPA}$ and establish the main connection with fuzzy $\mathcal{SMA}$.

Definition 4.1. A weighted fuzzy graph is a quadruple (E, ω, μ, γ) where:

- (E, μ, γ) is a fuzzy graph,
- $\omega: E^1 \rightarrow \mathbb{N}^+$ is a weight function.

For each $v \in E_{\text{reg}}^0$, define:

$$\omega(v) = \max\{\omega(e) : e \in s^{-1}(v)\}$$

Definition 4.2. Let (E, ω, μ, γ) be a weighted fuzzy graph and k a field. The fuzzy weighted Leavitt path algebra $L_k(E, \omega, \mu, \gamma)$ is the free k -algebra generated by:

$$\{v, e_i, e_i^* : v \in E^0, e \in E^1, 1 \leq i \leq \omega(e)\}$$

subject to the relations:

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$$uv = \delta_{uv} \cdot \min\{\mu(u), \mu(v)\} \cdot u, \quad (3)$$

$$s(e)e_i = e_i = e_i r(e) \cdot \gamma(e), \quad (4)$$

$$r(e)e_i^* = e_i^* = e_i^* s(e) \cdot \gamma(e), \quad (5)$$

$$\sum_{e \in s^{-1}(v)} \sum_{i=1}^{\omega(e)} e_i e_j^* = \delta_{ij} v \cdot \frac{\sum_{f \in s^{-1}(v)} \omega(f) \gamma(f)}{f \text{ out-deg}(v)}, \quad (6)$$

$$\sum_{i=1}^{\omega(v)} e_i e_j^* = \delta_e f r(e) \cdot \frac{\omega(e) \omega(f) \gamma(e) \gamma(f)}{f \text{ out-deg}(v)^2} \quad (7)$$

for all $u, v \in E^0, e, f \in E^1, v \in E_{freg}^0, 1 \leq i, j \leq \omega(v)$.

The fuzzy membership functions μ and γ are incorporated as scalar coefficients, weighting the algebraic relations according to vertex and edge memberships. The weight function ω determines the number of generators for each edge

Remark 4.1. Setting all membership values to unity specifically $\mu(v) = 1$ across $v \in E^0, \gamma(e) = 1$ across $e \in E^1$, and $\omega(e) = 1$ across $e \in E^1$ causes $L_k(E, \omega, \mu, \gamma)$ to collapse precisely to the standard Leavitt path algebra $L_k(E)$.

The construction $L_k(E, \omega, \mu, \gamma)$ carries natural integer grading structure via:

$$\deg(v) = 0 \text{ across all } v \in E^0,$$

$$\deg(e_i) = 1 \text{ for each } e \in E^1, 1 \leq i \leq \omega(e),$$

$$\deg(e_i^*) = -1 \text{ for each } e \in E^1, 1 \leq i \leq \omega(e).$$

Symbol $L^{gr}(L_k(E, \mu, \gamma))$ designates the collection of nonzero graded ideals within the unweighted fuzzy Leavitt path algebra $L_k(E, \mu, \gamma)$ (obtained by fixing $\omega(e) = 1$ throughout).

Symbol $L^{ver}(L_k(E, \omega, \mu, \gamma))$ designates those ideals of $L_k(E, \omega, \mu, \gamma)$ arising from vertex generation—precisely, ideals expressible as $\langle V \rangle$ for vertex subsets $V \subseteq E^0$.

Lemma 4.1. Consider $\mathcal{FSG}$ structure (E, μ, γ) . Each nonzero graded ideal I within $L_k(E, \mu, \gamma)$ admits generation via some fuzzy hereditary saturated subset $\mathcal{E} \in \div_{E, \mu, \gamma}$.

Proof. Assume I constitutes a nonzero graded ideal. Graded ideals decompose into homogeneous components; degree-zero components consist of vertex linear combinations, forcing I to contain at least one vertex $v \in E^0$

Construct mapping $\mathcal{E} : E^0 \rightarrow [0, 1]$ through:

$$\mathcal{E}(v) = \sup\{\alpha \in [0, 1] : \alpha \cdot v \in I\}$$

Verification that $\mathcal{E}$ possesses fuzzy hereditary and saturation properties proceeds analogously to Theorem 2.6 part (4). The key modification substitutes $L_k(E, \mu, \gamma)$'s algebraic structure for $FSP(E, \mu, \gamma)$'s toppling dynamics, applying the same logical framework with different operational rules.

Theorem 5.1. Consider field $k, \mathcal{FSG}$ structure (E, μ, γ) , and maximal chain length t within $\mathcal{F}_{E, \mu, \gamma}$. The algebra $L_k(E, \mu, \gamma)$ admits a finite graded ideal filtration:

$$0 < I_0 < I_1 < \dots < I_t = L_k(E, \mu, \gamma)$$

possessing these properties:

- (1) The base ideal satisfies $I_0 = \text{Soc}(L_k(E, \mu, \gamma)) \sim \mathcal{M}_{\Lambda_s}(k)$, where Λ_s comprises all paths terminating at s , with fuzzy membership weighting matrix entries,

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- (2) Each intermediate quotient I_{i+1}/I_i (for $0 \leq i < t$) decomposes into finite direct sums comprising fuzzy-weighted purely infinite simple Leavitt path algebras together with fuzzy matrix algebras $\mathcal{M}_{\Lambda_s}(k[x, x^{-1}])$, where fuzzy path memberships weight the indexing set Λ
- (3) The top quotient I_t/I_{t-1} decomposes as finite direct sums of unital fuzzy-weighted purely infinite simple Leavitt path algebras and matrix algebras $\mathcal{M}_\Lambda(k[x, x^{-1}])$ with $n \in \mathbb{N}^+$,
- (4) Each ideal I_i remains fixed under all graded automorphisms of $L_k(E, \mu, \gamma)$.

Additionally, $\text{Idem}(FSP(E, \mu, \gamma))$ forms a chain precisely when each quotient I_{i+1}/I_i constitutes either a fuzzy-weighted purely infinite simple Leavitt path algebra or a fuzzy matrix algebra.

Proof: Our construction adapts the methodology from Theorem 4.11 of the original work into the fuzzy framework.

Base ideal construction: Designate s as the unique fuzzy sink satisfying $\mu(s) = 1$.

Establish: $I_0 = \langle s \rangle^{gr}$ (the graded ideal that s generates)

Applying [2, Corollary 2.6.5]'s fuzzy generalization yields: $I_0 \sim M_{\Lambda_s}(k)$

Here Λ_s collects paths terminating at s , with matrix coefficients weighted via minimal membership values along individual paths.

From [8, Theorem 5.2]'s generalization, I_0 constitutes $L_k(E, \mu, \gamma)$'s socle.

Higher ideal construction: Denote by $\{C_{1,1}, \dots, C_{1,k_1}\}$ the collection of minimal fuzzy strongly connected cyclic components—meaning those components whose outgoing edges remain internal or target S_E exclusively. For each component $C_{1,j}$, define $\Xi_{1,j} = \overline{C_{1,j}^0}$ as its fuzzy hereditary saturated closure. Construct:

$$I_1 = \left\langle \bigcup_{j=1}^{k_1} \Xi_{1,j} \right\rangle^{gr}$$

Theorem 4.1 combined with the original paper's Theorem 4.9 generalization establishes:

$$I_1/I_0 \cong \bigoplus_{j=1}^{k_1} L_k(E(\Xi_{1,j}), \mu(\Xi_{1,j}), \gamma(\Xi_{1,j}))$$

Each summand $L_k(E(\Xi_{1,j}), \mu(\Xi_{1,j}), \gamma(\Xi_{1,j}))$ satisfies either:

- Structure isomorphism $\mathcal{M}_\Lambda(k[x, x^{-1}])$, when $C_{1,j}$ comprises a single fuzzy cycle,
- Fuzzy-weighted purely infinite simplicity when $C_{1,j}$ contains multiple distinct fuzzy cycles

Iterating this construction: at filtration level i , examine fuzzy strongly connected cyclic components whose outgoing edges exclusively target components from levels $< i$. This produces:

$$I_1/I_0 \cong \bigoplus_{j=1}^{k_1} L_k(E(\Xi_{1,j}), \mu(\Xi_{1,j}), \gamma(\Xi_{1,j}))$$

Termination occurs at level t , matching $\mathcal{F}_{E,\mu,\gamma}$'s maximal chain length.

Automorphism invariance: Consider graded automorphism

$$\alpha : L_k(E, \mu, \gamma) \rightarrow L_k(E, \mu, \gamma).$$

Then:

$$\alpha(I_0) = \alpha(\text{Soc}(L_k(E, \mu, \gamma))) = \text{Soc}(L_k(E, \mu, \gamma)) = I_0$$

For indices $i \geq 1$, express $I_i = \langle \Xi_i \rangle^{gr}$ for appropriate $\Xi_i \in \div_{E,\mu,\gamma}$. Graded automorphisms preserve the graded ideal lattice structure. Theorem 4.1 translates this to $\div_{E,\mu,\gamma}$ lattice preservation.

Fuzzy hereditary saturated subsets determining I_i depend solely on fuzzy strongly connected cyclic component combinatorial structure at levels $\leq i$. Since automorphisms preserve this structure, we obtain $\alpha(I_i) = I_i$.

Chain characterization: Assume $\text{Idem}(\text{FSP}(E, \mu, \gamma))$ forms a chain. Theorem 3.1 implies $F_{E,\mu,\gamma}$ is a chain. Consequently, exactly one fuzzy strongly connected cyclic component exists at each level i , forcing $k_i = 1$ throughout. Therefore, every quotient I_{i+1}/I_i becomes indecomposable.

Conversely, assume each I_{i+1}/I_i is indecomposable (either purely infinite simple or a matrix algebra). This forces $k_i = 1$ for all i , making $F_{E,\mu,\gamma}$ a chain. Theorem 3.1 then yields that $\text{Idem}(\text{FSP}(E, \mu, \gamma))$ is a chain.

Corollary 5.1. Let k be a field, (E, μ, γ) a FSG, and S_E the fuzzy subset of vertices not connecting to any cycle. The following are equivalent:

- (1) $\text{FSP}(E, \mu, \gamma)$ has exactly two idempotent,
- (2) $L_k(E/S_E, \mu/S_E, \gamma/S_E)$ is either isomorphic to a fuzzy-weighted $\mathcal{M}_n(k[x, x^{-1}])$ for some $n \in \mathbb{N}^+$, or fuzzy-weighted purely infinite simple,
- (3) $L_k(E/S_E, \mu/S_E, \gamma/S_E)$ is fuzzy vertex-simple (i.e., the only ideals generated by vertices are $\{0\}$ and the whole algebra).

Proof: (1) $\Rightarrow$ (2): If $\text{FSP}(E, \mu, \gamma)$ has exactly two idempotents (0 and some $e \neq 0$), then by Theorem 3.1, $\div_{E,\mu,\gamma}$ has exactly two elements: S_E and μ (the full vertex membership function). Therefore, (E, μ, γ) has exactly one fuzzy strongly connected cyclic component. By Theorem 5.1, $L_k(E/S_E, \mu/S_E, \gamma/S_E)$ is either a fuzzy matrix algebra or fuzzy purely infinite simple.

(2) $\Rightarrow$ (3): By the generalization of [2, Proposition 3.1.14], if $L_k(E/S_E, \mu/S_E, \gamma/S_E)$ is either a fuzzy matrix algebra or fuzzy purely infinite simple, then it is graded simple. By Theorem 4.1, the lattice $L^{ver}(L_k(E/S_E, \mu/S_E, \gamma/S_E))$ has exactly two elements: $\{0\}$ and the whole algebra. Therefore, $L_k(E/S_E, \mu/S_E, \gamma/S_E)$ is fuzzy vertex-simple.

(3) $\Rightarrow$ (1): If $L_k(E/S_E, \mu/S_E, \gamma/S_E)$ is fuzzy vertex-simple, then by Theorem 4.1, $\div_{E/S_E, \mu/S_E, \gamma/S_E}$ has exactly two elements. Since $\div_{E,\mu,\gamma}$ is obtained by adding S_E to $\div_{E/S_E, \mu/S_E, \gamma/S_E}$, we have $|\div_{E,\mu,\gamma}| = 2$. By Theorem 3.1, $|\text{Idem}(\text{FSP}(E, \mu, \gamma))| = 2$.

6. Conclusion

This paper has successfully extended sandpile monoids and Leavitt path algebras to fuzzy environments, demonstrating that classical algebraic structures can meaningfully accommodate uncertainty and gradual transitions. We introduced fuzzy sandpile monoids by carefully adapting toppling operations to respect membership functions, then proved that fuzzy archimedean decompositions of objects retain the rich structure of their crisp

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counterparts. The central achievement lies in establishing lattice isomorphisms connecting five seemingly different mathematical perspectives: idempotents of fuzzy sandpile monoids, fuzzy filters, fuzzy hereditary saturated subsets, graded ideals, and vertex-generated ideals. These connections reveal a fundamental unity underlying discrete dynamics and non-commutative algebra, even when boundaries become imprecise.

Our characterization of fuzzy Archimedean classes showed that each class contains exactly one idempotent, mirroring classical behavior while requiring novel techniques to handle fuzzy degrees. The decomposition of fuzzy weighted Leavitt path algebras through graded ideal chains demonstrated how strongly connected components organize algebraic structure regardless of membership uncertainties. These results open practical applications in modeling networks where connection strengths vary continuously rather than switching abruptly between present and absent. Social networks, biological pathways, and infrastructure systems all exhibit this gradual behaviour that our framework naturally captures. We hope this work encourages further investigation at the intersection of fuzzy logic, discrete mathematics, and abstract algebra, showing how algebraic thinking illuminates structures that classical crisp frameworks cannot adequately describe.

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