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The Power Graph as a Hypergraph and Iterated-Powerset Power Graphs as n-SuperHyperGraphs

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Abstract. Graph theory provides a fundamental language for modeling relational structure and connectivity via vertices and edges [1]. For a group G, the (undirected) *power graph* has vertex set G and joins two distinct elements whenever one is a positive integral power of the other, while the *directed power graph* orients an arc $x \to y$ precisely when $y = x^m$ for some $m \in \mathbb{N}_{\geq 1}$. Hypergraphs extend ordinary graphs by allowing an edge to connect an arbitrary number of vertices [2], and SuperHyperGraphs further generalize hypergraphs via iterated powerset constructions to capture hierarchical, set-valued incidence patterns. In this paper, we show that the power graph can be viewed as a 2-uniform hypergraph and that the directed power graph admits a natural representation as a directed hypergraph. Motivated by the powerset-based hierarchy of SuperHyperGraphs, we then introduce the n*-level power graph* and the *directed* n*-level power graph*, whose vertices lie in iterated nonempty powersets of G and whose adjacency is induced by level-wise exponentiation maps. We prove that these constructions fit naturally within the SuperHyperGraph and directed SuperHyperGraph paradigms, respectively.

Keywords: Superhypergraph, Hypergraph, Power Graph, Directed Power Graph, Group Theory

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1. Introduction

Classical graphs represent *binary* relationships: every edge links exactly two vertices [1]. In many settings, however, interactions are inherently *higher–order*. Examples include group conversations, collaborative tasks involving multiple participants, and scenarios where several resources are consumed jointly. Such multiway relations are modeled naturally by *hypergraphs*, where a single *hyperedge* may connect any nonempty subset of vertices [3], [4], [5]. Related concepts such as Fuzzy HyperGraphs[6], [7], [8], Directed HyperGraphs [9], HyperGraph Partitioning[10], [11], HyperGraph Neural Networks[12], [13], and Neutrosophic HyperGraphs [14], [15], [16] are also known.

Higher arity alone is not the whole story. Contemporary data often come with *hierarchical* or *multi-scale* organization—for instance, communities containing subcommunities, systems assembled from subsystems, or service stacks layered across multiple levels. To encode this kind of nested incidence, *SuperHyperGraphs* enrich the vertex domain by iterating the powerset operation on a finite base set. The resulting models allow vertices to be set-valued objects (sets of sets, and so forth), enabling a principled representation of multi-level structure [17], [18], [19]. Table 1 highlights the main differences among graphs, hypergraphs, and n-SuperHyperGraphs.

Table 1: *Key distinctions among graphs, hypergraphs, and n-SuperHyperGraphs.*

Concept	*Notation*	*Edge family*	*Core extension principle*
Graph [20]	$G = (V, E)$	$E \subseteq \binom{V}{2}$	Edges encode *pairwise* relations between vertices.
Hypergraph [3]	$H = (V, \mathcal{E})$	$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Hyperedges may join *any* nonempty subset of vertices, capturing higher-order interactions.
n-SuperHyperGraph [17]	$\mathrm{SuHyG}^{(n)} = (V, \mathcal{E})$ (on a base set V_0)	$V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$, $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Vertices lie in an n-fold powerset hierarchy and may be *set-valued*; edges are subsets of the supervertex set, allowing *nested* and *multi-level* incidence patterns.

Notation. $\mathcal{P}(X) = \{A \mid A \subseteq X\}$, $\binom{V}{2} = \{\{u, v\} \subseteq V \mid u \neq v\}$, and $\mathcal{P}^0(X) = X$, $\mathcal{P}^{k+1}(X) = \mathcal{P}(\mathcal{P}^k(X))$.

Let G be a group. The (undirected) *power graph* has vertex set G and joins two distinct elements whenever one is a (positive) integral power of the other. Its directed variant, the *directed power graph*, places an arc $x \to y$ precisely when $y = x^m$ for some $m \in \mathbb{N}_{\geq 1}$. In this work, we observe that the power graph is equivalently a 2-uniform hypergraph (on the same vertex set), and that the directed power graph can be viewed as a directed hypergraph whose hyperarcs have singleton tail and singleton head. To avoid confusion with the standard *nth power of a graph* G^n (defined via graph distances), we use the term *n-level power graph* for our powerset-lifted construction.

Motivated by the powerset-based hierarchy underlying SuperHyperGraphs, we further introduce an *n-level power graph* and a *directed n-level power graph* whose vertices live in an iterated (nonempty) powerset domain and whose adjacency is induced by the corresponding level-wise exponentiation maps. Concretely, write $\mathcal{P}_*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$ and $\mathcal{P}_*^0(G) := G$, $\mathcal{P}_*^{n+1}(G) := \mathcal{P}_*(\mathcal{P}_*^n(G))$. For each $m \in \mathbb{N}_{\geq 1}$, define maps $\varphi_m^{(n)}$ recursively by

$$\varphi_m^{(0)}(x) := x^m \quad (x \in G), \qquad \varphi_m^{(k+1)}(A) := \{\varphi_m^{(k)}(a) \mid a \in A\} \quad (A \in \mathcal{P}_*^{k+1}(G)).$$

We prove that the resulting graphs form subclasses of n-SuperHyperGraphs and directed n-SuperHyperGraphs, respectively.

2. Preliminaries

We begin by reviewing the basic terminology and notation used throughout this paper.

2.1. Power graph and directed power graph

The power graph of a group records exponentiation relations among elements: two distinct vertices are adjacent whenever one is a (positive) integer power of the other. Its directed version orients an arc $x \to y$ exactly when y is a positive power of x.

Definition 1. (Power Graph). [21], [22], [23] Let G be a group and let $\mathbb{N} := \{1,2,3,\dots\}$. The *power graph* of G is the simple undirected graph

$$P(G) = (V, E), \qquad V := G,$$

with edge set

$$E := \{\{x, y\} \subseteq G \mid x \neq y,\ \exists m \in \mathbb{N}:\ y = x^m \text{ or } \exists m \in \mathbb{N}:\ x = y^m\}.$$

Equivalently, $\{x, y\} \in E$ if and only if there is an arc $x \to y$ or $y \to x$ in the directed power graph $\vec{P}(G)$ (Definition 3).

Example 2. (Power Graph of the cyclic group of order 4). Let $G = \langle a \mid a^4 = e\rangle = \{e, a, a^2, a^3\}$. Then $P(G) = (V, E)$ has

$$V = \{e, a, a^2, a^3\}, \qquad E = \{\{e, a\}, \{e, a^2\}, \{e, a^3\}, \{a, a^2\}, \{a, a^3\}, \{a^2, a^3\}\}.$$

Definition 3. (Directed Power Graph). [21], [23] Let G be a group and let $\mathbb{N} := \{1,2,3,\dots\}$. The *directed power graph* of G is the digraph

$$\vec{P}(G) = (V, A), \qquad V := G,$$

whose arc set is

$$A := \{(x, y) \in G \times G \mid x \neq y,\ \exists m \in \mathbb{N}:\ y = x^m\}.$$

Example 4. (Directed power graph of the cyclic group of order 4). On the same group $G = \langle a \mid a^4 = e\rangle$, the digraph $\vec{P}(G) = (V, A)$ has

$$V = \{e, a, a^2, a^3\},$$

and the nontrivial arcs are

$$A = \{(a, a^2),\ (a, a^3),\ (a, e),\ (a^2, e),\ (a^3, a),\ (a^3, a^2),\ (a^3, e)\}.$$

Indeed, each listed arc (x, y) satisfies $y = x^m$ for some $m \in \mathbb{N}$ and $x \neq y$.

1.2. SuperHyperGraphs

A *hypergraph* extends an ordinary (simple) graph by permitting an edge to connect more than two vertices simultaneously: a *hyperedge* is an arbitrary nonempty subset of the vertex set [2], [13]. In many applications, however, one encounters not only higher-order interactions but also *hierarchical* or *multi-level* organization (for example, components built from subcomponents or communities within communities). Motivated by such nested incidence patterns, *SuperHyperGraphs* enrich the vertex domain by iterating the powerset construction on a base set, so that vertices themselves may be set-valued objects (sets of sets, and so on) [24], [25], [26], [27], [28], [29]. Reported applications include molecular modeling, network analysis, and signal processing (e.g., [30], [31]). Throughout, the parameter n in $\mathcal{P}^n(\cdot)$ and in an n-SuperHyperGraph denotes a *nonnegative* integer.

Definition 5. (Powerset). [32] Let S be a set. The *powerset* of S, denoted $\mathcal{P}(S)$, is the set of all subsets of S:

$$\mathcal{P}(S) := \{A \mid A \subseteq S\}.$$

Definition 6. (Iterated powersets). For any set S and integer $n \geq 0$, define recursively

$$\mathcal{P}^0(S) := S, \qquad \mathcal{P}^{n+1}(S) := \mathcal{P}(\mathcal{P}^n(S)).$$

When needed, we also write

$$\mathcal{P}_*(S) := \mathcal{P}(S) \setminus \{\emptyset\},$$

and one may define iterated nonempty powersets analogously by $\mathcal{P}_*^0(S) := S$ and $\mathcal{P}_*^{n+1}(S) := \mathcal{P}_*(\mathcal{P}_*^n(S))$.

Definition 7. (Hypergraph). [2], [3] A *hypergraph* is a pair $H = (V(H), E(H))$ consisting of

a (typically finite) vertex set $V(H)$, and

a collection $E(H) \subseteq \mathcal{P}(V(H)) \setminus \{\emptyset\}$ whose elements are called *hyperedges*.

If every hyperedge has cardinality r, then H is called r*-uniform*; in particular, 2-uniform hypergraphs coincide with simple graphs.

Definition 8. (2-section (shadow graph)). Let $H = (V, E)$ be a hypergraph. The 2*-section* (or *shadow graph*) of H is the simple graph $[H]_2 = (V, E_2)$ defined by

$$E_2 := \{\{u, v\} \subseteq V \mid u \neq v, \exists e \in E: \{u, v\} \subseteq e\}.$$

Theorem 9. *Let G be a group and let $P(G) = (V, E)$ be its power graph as in Definition 1, i.e., $V = G$, $E = \{\{x, y\} \subseteq G \mid x \neq y, \exists m \in \mathbb{N}: y = x^m$ or $x = y^m\}$. Define a hypergraph $H := (V, E) = (G, E)$. Then H is 2-uniform, and its 2-section $[H]_2$ (Definition 8) coincides with $P(G)$.*

Proof: Each hyperedge $e \in E$ is, by definition, a 2-element subset of G. Hence

$$E \subseteq \binom{G}{2} \subseteq \mathcal{P}(G) \setminus \{\emptyset\},$$

so $H = (G, E)$ is a 2-uniform hypergraph.

For the 2-section $[H]_2 = (G, E_2)$, we have $\{x, y\} \in E_2$ if and only if there exists $e \in E$ with $\{x, y\} \subseteq e$. Since $|e| = 2$ for all $e \in E$, this is equivalent to $\{x, y\} \in E$. Therefore $E_2 = E$, and hence $[H]_2 = P(G)$. □

Definition 10. (n-SuperHyperGraph). [17], [33], [34] Let V_0 be a nonempty (typically finite) base set and fix $n \geq 0$. An n*-SuperHyperGraph* on V_0 is a hypergraph $\mathrm{SuHyG}^{(n)} = (V, \mathcal{E})$ such that

$$V \subseteq \mathcal{P}^n(V_0), \qquad \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Elements of V are called n*-supervertices*, and elements of $\mathcal{E}$ are called n*-superedges*. In particular, each superedge is a nonempty set of n-supervertices, while each n-supervertex itself may be a set-valued object in the iterated powerset hierarchy $\mathcal{P}^n(V_0)$.

Example 11. (3-SuperHyperGraph: global project organization). We illustrate Definition 10 for $n = 3$ by modeling a simple hierarchy within a multinational project structure.

Base set (employees). Let

$$V_0 = \{\text{Alice, Bob, Carol, Dave}\}.$$

Level 1 (teams). Choose three teams in $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$:

$$T_1 = \{\text{Alice, Bob}\}, \quad T_2 = \{\text{Bob, Carol}\}, \quad T_3 = \{\text{Carol, Dave}\}.$$

Level 2 (departments). Choose two departments in $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$:

$$D_1 = \{T_1, T_2\}, \quad D_2 = \{T_2, T_3\}.$$

Level 3 (divisions / supervertices). Define two 3-supervertices in $\mathcal{P}^3(V_0)$:

$$v_1 = \{D_1, D_2\}, \qquad v_2 = \{D_1, \{T_1, T_3\}\}.$$

Set

$$V := \{v_1, v_2\} \subseteq \mathcal{P}^3(V_0), \qquad \mathcal{E} := \{\{v_1, v_2\}\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Interpretation. The 3-supervertices v_1, v_2 represent high-level divisions built from departments and (indirectly) from teams and employees. The unique superedge $\{v_1, v_2\}$ encodes a strategic collaboration between these two divisions on a company-wide initiative. Hence $\mathrm{SuHyG}^{(3)} = (V, \mathcal{E})$ is a concrete 3-SuperHyperGraph on the base set V_0.

1.3. Directed SuperHyperGraphs

Directed SuperHyperGraphs extend SuperHyperGraphs in the same spirit that directed graphs extend undirected graphs: hyperedges carry an orientation from a *source* part to a *target* part. We begin with directed hypergraphs and then introduce their powerset-based n-level analogues.

Definition 12. (Directed hypergraph). A *directed hypergraph* is a pair $H = (V, E)$, where V is a (typically finite) set of vertices and E is a set of *directed hyperedges*. Each directed hyperedge $e \in E$ is an ordered pair

$$e = (T_e, H_e),$$

where $T_e \subseteq V$ is the *tail* (source set) and $H_e \subseteq V$ is the *head* (target set), and one assumes

$$T_e \neq \emptyset, \qquad H_e \neq \emptyset.$$

When $|T_e| = |H_e| = 1$ for every $e \in E$, the structure reduces to an ordinary directed graph.

Definition 13. (Underlying digraph). Let $H = (V, E)$ be a directed hypergraph. Its *underlying digraph* is the digraph $U(H) = (V, A)$ with

$$A := \{(u, v) \in V \times V \mid \exists e = (T_e, H_e) \in E\colon u \in T_e,\ v \in H_e\}.$$

Thus $u \to v$ is an arc in $U(H)$ whenever some directed hyperedge sends flow from a tail containing u to a head containing v.

Theorem 14. *Let G be a group and let $\vec{P}(G) = (G, A)$ be its directed power graph (Definition 3), where $A = \{(x, y) \in G \times G \mid x \neq y,\ y = x^m$ for some $m \in \mathbb{N}\}$. Define a directed hypergraph $H = (V, E_H)$ by $V := G, \qquad E_H := \{(\{x\}, \{y\}) \mid (x, y) \in A\}$. Then H is a directed hypergraph (Definition 12), and its underlying digraph $U(H)$ Definition 13 coincides with $\vec{P}(G)$. In particular, $\vec{P}(G)$ is realized as a directed hypergraph whose directed hyperedges all have singleton tails and heads.*

Proof: For each $(x, y) \in A$, the pair $(\{x\}, \{y\})$ is a directed hyperedge with nonempty tail and head, hence $H = (V, E_H)$ satisfies Definition 12.

Now consider the underlying digraph $U(H) = (V, A')$. By Definition 13, an arc $x \to y$ belongs to A' if and only if there exists some hyperedge $(T, H) \in E_H$ such that $x \in T$ and

$y \in H$. Since every hyperedge in E_H is of the form $(\{u\},\{v\})$, this is equivalent to $(\{x\},\{y\}) \in E_H$, i.e., $(x,y) \in A$. Therefore $A' = A$, and hence $U(H) = \vec{P}(G)$. □

Definition 15. (Directed n-SuperHyperGraph). (cf. [17], [26]) Let S be a nonempty base set and let $n \geq 0$. Consider the iterated powersets $\mathcal{P}^n(S)$ defined by $\mathcal{P}^0(S) = S$ and $\mathcal{P}^{k+1}(S) = \mathcal{P}(\mathcal{P}^k(S))$. A *directed n-SuperHyperGraph* is a pair

$$\mathrm{DSuHG}^{(n)} = (V, \mathcal{E}),$$

where $V \subseteq \mathcal{P}^n(S)$ is the set of n-*supervertices*, and

$$\mathcal{E} \subseteq (\mathcal{P}(V)\backslash\{\emptyset\}) \times (\mathcal{P}(V)\backslash\{\emptyset\})$$

is the set of *directed* n-*superedges*. Each $e \in \mathcal{E}$ is written as

$$e = (\mathrm{Tail}(e), \mathrm{Head}(e)),$$

where $\mathrm{Tail}(e)$ and $\mathrm{Head}(e)$ are nonempty subsets of V, interpreted as the source and target collections of n-supervertices, respectively.

Example 16. (Directed 2-SuperHyperGraph: corporate workflow). We model a simplified approval workflow involving two teams and a central review committee.

Base set (employees).

$$S = \{\text{Alice, Bob, Carol, Dave}\}.$$

Level 1 (teams).

$$T_1 = \{\text{Alice, Bob}\}, \qquad T_2 = \{\text{Carol, Dave}\}, \qquad T_1, T_2 \in \mathcal{P}(S).$$

Level 2 (departments / supervertices).

$$D_1 = \{T_1\}, \qquad D_2 = \{T_2\}, \qquad D_{\mathrm{all}} = \{T_1, T_2\}, \qquad D_1, D_2, D_{\mathrm{all}} \in \mathcal{P}^2(S).$$

Vertex set.

$$V = \{D_1, D_2, D_{\mathrm{all}}\} \subseteq \mathcal{P}^2(S).$$

Directed superedges (flow of requests). Define two directed 2-superedges by

$$\mathcal{E} := \{(\{D_1\}, \{D_{\mathrm{all}}\}),\ (\{D_2\}, \{D_{\mathrm{all}}\})\}.$$

Then $\mathrm{DSuHG}^{(2)} = (V, \mathcal{E})$ is a directed 2-SuperHyperGraph. Each superedge $(\{D_i\}, \{D_{\mathrm{all}}\})$ indicates that department D_i forwards its deliverables (or approval requests) to the central committee D_{all}.

Theorem 17. *Every directed hypergraph can be represented as a directed* 1*-SuperHyperGraph. More precisely, let $H = (V_0, E)$ be a directed hypergraph with $E \subseteq (\mathcal{P}(V_0)\backslash\{\emptyset\}) \times (\mathcal{P}(V_0)\backslash\{\emptyset\})$. Set $S := V_0$ and $V := \{\{v\} \mid v \in V_0\} \subseteq \mathcal{P}^1(S) = \mathcal{P}(S)$. For each hyperedge $e = (T_e, H_e) \in E$, define* $\mathrm{Tail}'(e) := \{\{v\} \mid v \in T_e\} \subseteq V$, $\mathrm{Head}'(e) := \{\{v\} \mid v \in H_e\} \subseteq V$, *and set* $\mathcal{E}' := \{(\mathrm{Tail}'(e), \mathrm{Head}'(e)) \mid e \in E\}$. *Then* $\mathrm{DSuHG}^{(1)} := (V, \mathcal{E}')$ *is a directed* 1*-SuperHyperGraph and encodes the same directed incidence as* H.

Proof: By construction, $V \subseteq \mathcal{P}^1(S)$. Moreover, for each $e = (T_e, H_e) \in E$, the sets $\mathrm{Tail}'(e)$ and $\mathrm{Head}'(e)$ are nonempty subsets of V, since $T_e \neq \emptyset$ and $H_e \neq \emptyset$. Hence

$$(\mathrm{Tail}'(e), \mathrm{Head}'(e)) \in (\mathcal{P}(V)\backslash\{\emptyset\}) \times (\mathcal{P}(V)\backslash\{\emptyset\}),$$

so $\mathcal{E}'$ satisfies Definition 15 for $n = 1$. Finally, the correspondence $e \mapsto (\mathrm{Tail}'(e), \mathrm{Head}'(e))$ is injective and preserves the intended source/target incidence at the level of vertices, so the directed hypergraph H is faithfully represented by $\mathrm{DSuHG}^{(1)}$. □

Corollary 18. *For every integer $n \geq 1$, every directed hypergraph admits a faithful representation as a directed n-SuperHyperGraph.*

Proof: Let $\mathrm{DSuHG}^{(1)} = (V, \mathcal{E}')$ be the directed 1-SuperHyperGraph obtained in Theorem 17. For each $n \geq 1$, define

$$\iota_n : \mathcal{P}(S) \to \mathcal{P}^n(S)$$

recursively by

$$\iota_1(A) := A, \qquad \iota_{k+1}(A) := \{\iota_k(A)\} \quad (k \geq 1).$$

Each ι_n is injective, and therefore $\iota_n(V) \subseteq \mathcal{P}^n(S)$. Applying ι_n to every supervertex and transporting every tail and head along this injection yields a directed n-SuperHyperGraph isomorphic, as a directed incidence structure, to $\mathrm{DSuHG}^{(1)}$. Hence it faithfully represents the original directed hypergraph. □

3. Main results

As the main contributions of this paper, we show that:

- Every n-level power graph $P^{(n)}(G)$ can be viewed as a special case of an n-SuperHyperGraph (in particular, as a 2-uniform n-SuperHyperGraph whose vertices are n-supervertices).
- Every directed n-level power graph $\vec{P}^{(n)}(G)$ admits a natural realization as a directed n-SuperHyperGraph in which each hyperarc has a singleton tail and (in general) a multi-vertex head encoding all one-step power images.

3.1. n-level power graphs

The classical power graph of a group is built from the exponentiation relation $y = x^m$ ($m \in \mathbb{N}_{\geq 1}$) on individual elements. To obtain a hierarchical variant, we lift exponentiation through iterated *nonempty* powersets, so that vertices may themselves be set-valued objects (sets of sets, and so on). Throughout this subsection, G denotes a group (written multiplicatively) and $\mathbb{N} := \{1,2,3,\ldots\}$.

Definition 19. (Iterated exponentiation on iterated nonempty powersets). Let G be a group and let $m \in \mathbb{N}$. Define the base map

$$\varphi_m^{(0)} : G \to G, \qquad \varphi_m^{(0)}(x) := x^m.$$

Write $\mathcal{P}_*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$ and define the iterated nonempty powersets by

$$\mathcal{P}_*{}^0(G) := G, \qquad \mathcal{P}_*{}^{k+1}(G) := \mathcal{P}_*\big(\mathcal{P}_*{}^k(G)\big) \quad (k \geq 0).$$

Recursively, for each $k \geq 0$ define

$$\varphi_m^{(k+1)} : \mathcal{P}_*{}^{k+1}(G) \to \mathcal{P}_*{}^{k+1}(G), \qquad \varphi_m^{(k+1)}(A) := \left\{\varphi_m^{(k)}(a) \mid a \in A\right\}.$$

Equivalently, $\varphi_m^{(k)}$ is obtained by iterating the direct-image extension of $x \mapsto x^m$ through k levels of nonempty powersets.

Definition 20. (Undirected n-level power graph). Fix an integer $n \geq 0$. Let

$$V_n := \mathcal{P}_*{}^n(G),$$

and define an edge set $E_n \subseteq \binom{V_n}{2}$ by

$$E_n := \left\{\{A, B\} \subseteq V_n \;\mid\; A \neq B,\ \exists m \in \mathbb{N} \text{ such that } B = \varphi_m^{(n)}(A) \text{ or } A = \varphi_m^{(n)}(B)\right\}.$$

The n-*level power graph* of G is the simple graph

$$P^{(n)}(G) := (V_n, E_n).$$

Remark 21. In the 0-level case, the vertex domain reduces to the underlying group:

$$V_0 = \mathcal{P}_*^0(G) = G.$$

Moreover, by definition of the level-wise exponentiation maps,

$$\varphi_m^{(0)}(x) = x^m \qquad (x \in G,\ m \in \mathbb{N}_{\geq 1}).$$

Hence the adjacency rule for $P^{(0)}(G)$ agrees exactly with that of the classical power graph $P(G)$. In particular, we may identify

$$P^{(0)}(G) = P(G).$$

Example 22. (A small second-level power graph). Let $G = \langle a \mid a^4 = 1 \rangle = \{1, a, a^2, a^3\}$. Consider the level $n = 2$ vertex

$$A := \{\{a, a^3\}, \{a^2\}\} \in \mathcal{P}_*{}^2(G).$$

For $m = 2$, we compute at level $k = 1$:

$$\varphi_2^{(1)}(\{a, a^3\}) = \{a^2, (a^3)^2\} = \{a^2\}, \qquad \varphi_2^{(1)}(\{a^2\}) = \{(a^2)^2\} = \{1\}.$$

Hence, at level $k = 2$,

$$\varphi_2^{(2)}(A) = \left\{\varphi_2^{(1)}(\{a, a^3\}),\ \varphi_2^{(1)}(\{a^2\})\right\} = \{\{a^2\}, \{1\}\} \in \mathcal{P}_*{}^2(G).$$

Since $\varphi_2^{(2)}(A) \neq A$, the pair $\left\{A, \varphi_2^{(2)}(A)\right\}$ is an edge of $P^{(2)}(G)$.

Theorem 23. *Let G be a group and let $n \geq 0$. Then the n-level power graph $P^{(n)}(G) = (V_n, E_n)$ is the 2-section of a 2-uniform n-SuperHyperGraph whose n-supervertex domain is $V_n = \mathcal{P}_*{}^n(G)$.*

More concretely, set $\mathrm{SuHyG}^{(n)} := (V_n, \mathcal{E}_n)$, $\qquad \mathcal{E}_n := E_n \subseteq \binom{V_n}{2}$. *Then* $\mathrm{SuHyG}^{(n)}$ *is a 2-uniform n-SuperHyperGraph (in the sense that $V_n \subseteq \mathcal{P}^n(G)$), and its 2-section coincides with $P^{(n)}(G)$.*

Proof: By induction on n, the recursive construction of nonempty powersets gives

$$\mathcal{P}_*^n(G) \subseteq \mathcal{P}^n(G).$$

Hence $V_n = \mathcal{P}_*^n(G)$ is a valid set of n-supervertices over the base set G. Moreover, every element of $\mathcal{E}_n = E_n$ is a 2-element subset of V_n; therefore

$$\mathcal{E}_n \subseteq \binom{V_n}{2} \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\},$$

so $\mathrm{SuHyG}^{(n)}$ is a 2-uniform n-SuperHyperGraph.

Finally, the 2-section of a 2-uniform hypergraph joins two distinct vertices precisely when they form one of its 2-element hyperedges. Since the superedges are exactly the pairs $\{A, B\} \in E_n$, the resulting adjacency relation is exactly that of $P^{(n)}(G)$. Therefore the 2-section of $\mathrm{SuHyG}^{(n)}$ coincides with $P^{(n)}(G)$. □

Theorem 24. (Invariance under group isomorphisms). *Let $\psi: G \to H$ be a group isomorphism. For each $n \geq 0$, define maps $\Psi^{(n)}: \mathcal{P}_*{}^n(G) \to \mathcal{P}_*{}^n(H)$ recursively by $\Psi^{(0)} := \psi$, $\qquad \Psi^{(k+1)}(A) := \{\Psi^{(k)}(a) \mid a \in A\} \quad (k \geq 0)$. Then $\Psi^{(n)}$ is a graph isomorphism $P^{(n)}(G) \cong P^{(n)}(H)$.*

Proof: Each $\Psi^{(k)}$ is bijective because ψ is bijective and the direct-image extension of a bijection to nonempty powersets is again bijective. To distinguish the domains, write $\varphi_{m,G}^{(k)}$ and $\varphi_{m,H}^{(k)}$ for the iterated exponentiation maps on G and H, respectively. We claim that

$$\Psi^{(k)} \circ \varphi_{m,G}^{(k)} = \varphi_{m,H}^{(k)} \circ \Psi^{(k)} \qquad (k \geq 0).$$

For $k = 0$, this follows from the homomorphism identity $\psi(x^m) = \psi(x)^m$. Assume the identity holds at level k. For any $A \in \mathcal{P}_*^{k+1}(G)$,

$$\Psi^{(k+1)}\left(\varphi_{m,G}^{(k+1)}(A)\right) = \Psi^{(k+1)}\left(\{\varphi_{m,G}^{(k)}(a) \mid a \in A\}\right)$$
$$= \{\Psi^{(k)}(\varphi_{m,G}^{(k)}(a)) \mid a \in A\} = \{\varphi_{m,H}^{(k)}(\Psi^{(k)}(a)) \mid a \in A\}$$
$$= \varphi_{m,H}^{(k+1)}\left(\Psi^{(k+1)}(A)\right).$$

Thus the commutation identity holds for every $k \geq 0$.

Now suppose $\{A, B\} \in E_n(G)$. Then, for some $m \in \mathbb{N}$, either $B = \varphi_{m,G}^{(n)}(A)$ or $A = \varphi_{m,G}^{(n)}(B)$. By the commutation identity, the corresponding power relation holds between $\Psi^{(n)}(A)$ and $\Psi^{(n)}(B)$ in H; hence

$$\{\Psi^{(n)}(A), \Psi^{(n)}(B)\} \in E_n(H).$$

Applying the same argument to the inverse group isomorphism $\psi^{-1}: H \to G$ gives the converse implication. Therefore adjacency is preserved in both directions, and $\Psi^{(n)}$ is a graph isomorphism $P^{(n)}(G) \cong P^{(n)}(H)$. □

Proposition 25. (Automorphism action). *For each $n \geq 0$, the automorphism group* $\mathrm{Aut}(G)$ *acts on $P^{(n)}(G)$ by graph automorphisms via the construction in Theorem 24. In particular, every inner automorphism of G induces an automorphism of $P^{(n)}(G)$.*

Theorem 26. (Diameter bound for finite groups). *Let G be a finite group of exponent e (i.e., $g^e = 1$ for all $g \in G$). For any $n \geq 0$, the graph $P^{(n)}(G)$ has diameter at most* 2. *More explicitly, define the iterated identity vertex $\mathbf{1}_n \in V_n$ by $\mathbf{1}_0 := 1 \in G$, $\quad \mathbf{1}_{k+1} := \{\mathbf{1}_k\} \quad (k \geq 0)$. Then every vertex $A \in V_n$ is adjacent to $\mathbf{1}_n$, unless $A = \mathbf{1}_n$.*

Proof: We show that $\varphi_e^{(n)}(A) = \mathbf{1}_n$ for every $A \in V_n = \mathcal{P}_*{}^n(G)$. For $n = 0$, $\varphi_e^{(0)}(g) = g^e = 1 = \mathbf{1}_0$ for all $g \in G$. Assume $\varphi_e^{(k)}(a) = \mathbf{1}_k$ holds for all $a \in \mathcal{P}_*{}^k(G)$. If $A \in \mathcal{P}_*{}^{k+1}(G)$, then A is a nonempty set of elements $a \in \mathcal{P}_*{}^k(G)$, hence

$$\varphi_e^{(k+1)}(A) = \{\varphi_e^{(k)}(a) \mid a \in A\} = \{\mathbf{1}_k\} = \mathbf{1}_{k+1}.$$

By induction, $\varphi_e^{(n)}(A) = \mathbf{1}_n$ for all $A \in V_n$. Therefore, if $A \neq \mathbf{1}_n$, then $\{A, \mathbf{1}_n\} \in E_n$, so every vertex is at distance at most 1 from $\mathbf{1}_n$. Consequently, any two vertices are connected by a path of length at most 2, and $\mathrm{diam}(P^{(n)}(G)) \leq 2$. □

3.2. Directed n-level power graphs

The directed power graph $\vec{P}(G)$ orients the power relation on the elements of a group. We extend this construction hierarchically by transporting the exponentiation maps through iterated nonempty powersets. Throughout this subsection, G denotes a group written multiplicatively and $\mathbb{N} := \{1,2,3, \dots\}$.

Definition 27. (Directed n-level power graph). Let $n \geq 0$ and set

$$V_n := \mathcal{P}_*^n(G),$$

where $\mathcal{P}_*^n(G)$ and the iterated exponentiation maps $\varphi_m^{(n)}: V_n \to V_n$ are as in Definition 19. The *directed n-level power graph* of G is the digraph

$$\vec{P}^{(n)}(G) := (V_n, A_n),$$

with arc set

$$A_n := \left\{(A, B) \in V_n \times V_n \mid A \neq B,\ \exists m \in \mathbb{N}: B = \varphi_m^{(n)}(A)\right\}.$$

Remark 28. In the 0-level case, the vertex domain is the group itself:

$$V_0 = \mathcal{P}_*^0(G) = G.$$

Furthermore, the level-wise exponentiation maps satisfy

$$\varphi_m^{(0)}(x) = x^m \qquad (x \in G,\ m \in \mathbb{N}_{\geq 1}).$$

Therefore, the arc condition in $\vec{P}^{(0)}(G)$, namely "$x \to y$ iff $y = \varphi_m^{(0)}(x)$ for some $m \in \mathbb{N}_{\geq 1}$," is exactly the defining condition of the classical directed power graph $\vec{P}(G)$. Consequently,

$$\vec{P}^{(0)}(G) = \vec{P}(G).$$

Example 29. (A directed second-level power graph). Let $G = \langle a \mid a^7 = 1 \rangle = \{1, a, a^2, \dots, a^6\}$ be the cyclic group of order 7. Consider the two nonempty subsets

$$A_1 := \{1, a, a^2\}, \qquad A_2 := \{a^3, a^4\} \subseteq G,$$

and the level-2 vertex $P := \{A_1, A_2\} \in \mathcal{P}_*{}^2(G)$. For $m = 2$, we compute at level 1:

$$\varphi_2^{(1)}(A_1) = \{1, a^2, a^4\}, \qquad \varphi_2^{(1)}(A_2) = \{a^6, a\}.$$

Therefore, at level 2,

$$\varphi_2^{(2)}(P) = \left\{\varphi_2^{(1)}(A_1),\ \varphi_2^{(1)}(A_2)\right\} =: Q \in \mathcal{P}_*{}^2(G),$$

and $P \neq Q$. Hence $(P, Q) \in A_2$, i.e., $P \to Q$ is an arc in $\vec{P}^{(2)}(G)$.

Interpretation. The map $x \mapsto x^2$ permutes the elements of G (since $\gcd(2,7) = 1$), and $\varphi_2^{(2)}$ applies this relabeling coherently at two nested levels: first inside each block A_i, and then on the collection of blocks.

Proposition 30. (Canonical embeddings of vertex sets). *For each $n \geq 0$, the singleton map $\iota_n: V_n \to V_{n+1}$, $\qquad \iota_n(A) := \{A\}$, is injective. Thus the vertex sets form a nested family up to canonical identification via $V_n \cong \iota_n(V_n) \subseteq V_{n+1}$.*

Proof: If $\{A\} = \{B\}$, then $A = B$. Hence ι_n is injective. □

Theorem 31. *For every group G and integer $n \geq 0$, the directed n-level power graph $\vec{P}^{(n)}(G) = (V_n, A_n)$ can be realized as a directed hypergraph (indeed, as a directed n-SuperHyperGraph) in which each directed hyperedge has a singleton tail but, in general, a multi-vertex head. In this realization, a single hyperedge emanating from A encodes all one-step power images of A at once.*

More precisely, for each $A \in V_n$ set $H_n(A) := \left\{\varphi_m^{(n)}(A) \mid m \in \mathbb{N}_{\geq 1},\ \varphi_m^{(n)}(A) \neq A\right\} = \{B \in V_n \setminus \{A\} \mid (A, B) \in A_n\}$, and define the family of directed hyperedges $\mathcal{E}_n$

$:= \{\, (\{A\}, H_n(A)) \mid A \in V_n,\ H_n(A) \neq \emptyset \,\} \subseteq (\mathcal{P}(V_n)\setminus\{\emptyset\}) \times (\mathcal{P}(V_n)\setminus\{\emptyset\})$. *Then* $\mathrm{DSuHG}^{(n)} := (V_n, \mathcal{E}_n)$ *is a directed* n*-SuperHyperGraph (Definition 15), and its underlying digraph is exactly* $\vec{P}^{(n)}(G)$.

Proof: By construction, every element of $\mathcal{E}_n$ has the form $(\{A\}, H_n(A))$ with $\{A\} \neq \emptyset$ and $H_n(A) \neq \emptyset$; hence both tail and head are nonempty subsets of V_n. Therefore $(V_n, \mathcal{E}_n)$ satisfies Definition 15 and is a directed n-SuperHyperGraph.

Let $D(\mathrm{DSuHG}^{(n)})$ denote the underlying digraph of $(V_n, \mathcal{E}_n)$, so that an arc $A \to B$ exists in $D(\mathrm{DSuHG}^{(n)})$ if and only if there is a directed hyperedge $(T, H) \in \mathcal{E}_n$ with $A \in T$ and $B \in H$. Since each tail is a singleton, $A \in T$ forces $T = \{A\}$, and thus

$$A \to B \text{ in } D(\mathrm{DSuHG}^{(n)}) \quad \Leftrightarrow \quad B \in H_n(A).$$

By the definition of $H_n(A)$, the latter holds if and only if $B = \varphi_m^{(n)}(A)$ for some $m \in \mathbb{N}_{\geq 1}$ with $B \neq A$, i.e., if and only if $(A, B) \in A_n$. Hence the arc sets coincide, and therefore $D(\mathrm{DSuHG}^{(n)}) = \vec{P}^{(n)}(G)$. □

Theorem 32. (Composition and transitivity of the power relation). *Let* G *be a group and let* $n \geq 0$ *. On* $V_n = \mathcal{P}_*^n(G)$ *define the (power) relation* $A \preccurlyeq_n B \ :\Leftrightarrow\ \exists m \in \mathbb{N}_{\geq 1}$ such that $B = \varphi_m^{(n)}(A)$. *Then* $\preccurlyeq_n$ *is transitive. Equivalently, the exponentiation maps satisfy the composition rule* $\varphi_{m_2}^{(n)} \circ \varphi_{m_1}^{(n)} = \varphi_{m_1 m_2}^{(n)} \qquad (m_1, m_2 \in \mathbb{N}_{\geq 1})$. *Consequently, in the directed* n*-level power graph* $\vec{P}^{(n)}(G)$ *(with loops deleted), whenever* $(A, B) \in A_n$ *and* $(B, C) \in A_n$ *and* $A \neq C$*, we also have* $(A, C) \in A_n$.

Proof: Assume $A \preccurlyeq_n B$ and $B \preccurlyeq_n C$. Then there exist $m_1, m_2 \in \mathbb{N}_{\geq 1}$ such that

$$B = \varphi_{m_1}^{(n)}(A), \qquad C = \varphi_{m_2}^{(n)}(B).$$

On the base level $n = 0$, we have $\varphi_m^{(0)}(x) = x^m$, hence

$$\varphi_{m_2}^{(0)}\left(\varphi_{m_1}^{(0)}(x)\right) = (x^{m_1})^{m_2} = x^{m_1 m_2} = \varphi_{m_1 m_2}^{(0)}(x).$$

For $n \geq 1$, each $\varphi_m^{(n)}$ is obtained by iterated direct images, so applying the preceding identity elementwise yields

$$\varphi_{m_2}^{(n)}\left(\varphi_{m_1}^{(n)}(A)\right) = \varphi_{m_1 m_2}^{(n)}(A) \qquad (A \in V_n).$$

Therefore

$$C = \varphi_{m_2}^{(n)}(B) = \varphi_{m_2}^{(n)}\left(\varphi_{m_1}^{(n)}(A)\right) = \varphi_{m_1 m_2}^{(n)}(A),$$

which shows $A \preccurlyeq_n C$ and hence transitivity of $\preccurlyeq_n$.

For the final claim, if $(A, B) \in A_n$ and $(B, C) \in A_n$ then $A \preccurlyeq_n B$ and $B \preccurlyeq_n C$, so $A \preccurlyeq_n C$ and thus $C = \varphi_m^{(n)}(A)$ for some $m \in \mathbb{N}_{\geq 1}$. If moreover $A \neq C$, this exactly means $(A, C) \in A_n$. (If $A = C$, transitivity may fail in $\vec{P}^{(n)}(G)$ because loops are excluded by definition.) □

Theorem 33. (One-step reachability to the identity supervertex). *Let* G *be a finite group of exponent* e*, i.e.,* $g^e = 1 \qquad$ for all $g \in G$. *Define the iterated identity supervertex* $\mathbf{1}_n \in V_n$ *recursively by* $\mathbf{1}_0 := 1 \in G, \qquad \mathbf{1}_{k+1} := \{\mathbf{1}_k\} \qquad (k \geq 0)$. *Then, for every* $n \geq 0$

and every vertex $A \in V_n$ with $A \neq \mathbf{1}_n$, we have $(A, \mathbf{1}_n) \in A_n$. Equivalently, every vertex $A \in V_n \setminus \{\mathbf{1}_n\}$ has a directed path of length one to $\mathbf{1}_n$.

Proof: We first show that

$$\varphi_e^{(n)}(A) = \mathbf{1}_n$$

for every $n \geq 0$ and every

$$A \in V_n = \mathcal{P}_*{}^n(G).$$

For $n = 0$, we have $V_0 = G$, and hence, for every $g \in G$,

$$\varphi_e^{(0)}(g) = g^e = 1 = \mathbf{1}_0.$$

Assume that, for some $k \geq 0$,

$$\varphi_e^{(k)}(a) = \mathbf{1}_k \qquad \text{for every } a \in V_k.$$

Let $A \in V_{k+1} = \mathcal{P}_*(V_k)$. Since A is a nonempty subset of V_k, the recursive definition of $\varphi_e^{(k+1)}$ gives

$$\begin{aligned}\varphi_e^{(k+1)}(A) &= \left\{\varphi_e^{(k)}(a) \mid a \in A\right\} \\ &= \{\mathbf{1}_k\} \\ &= \mathbf{1}_{k+1}.\end{aligned}$$

Therefore, by induction,

$$\varphi_e^{(n)}(A) = \mathbf{1}_n$$

for every $n \geq 0$ and every $A \in V_n$.

Now let $A \in V_n$ satisfy $A \neq \mathbf{1}_n$. Since

$$\mathbf{1}_n = \varphi_e^{(n)}(A),$$

the definition of the directed n-level power graph implies

$$(A, \mathbf{1}_n) \in A_n.$$

Thus every nonidentity vertex has a directed path of length one to $\mathbf{1}_n$. □

4.Conclusion

In this paper, we investigated power graphs and directed power graphs from the viewpoint of hypergraphs and SuperHyperGraphs. We first showed that the ordinary power graph of a group can be regarded as a 2-uniform hypergraph whose 2-section recovers the original graph, while the directed power graph can be represented naturally as a directed hypergraph with singleton tails and heads. Building on the iterated powerset structure of SuperHyperGraphs, we then introduced the n-level power graph and the directed n-level power graph by extending group exponentiation through successive levels of nonempty powersets. We proved that these structures can be realized as an n-SuperHyperGraph and a directed n-SuperHyperGraph, respectively. We also established several basic properties, including invariance under group isomorphisms, a diameter bound for finite groups, the composition property of the induced power maps, and one-step reachability to the iterated identity supervertex in the directed setting.

As future work, it would be worthwhile to investigate further algebraic and graph-theoretic properties of these hierarchical power structures, including connectivity, automorphisms, cliques, cycles, domination, and computational aspects. We also plan to develop uncertainty-aware extensions based on Fuzzy Sets [35], [36], Intuitionistic Fuzzy Sets [37], Neutrosophic Sets [38], [39], [40], and Plithogenic Sets [41], [42], [43]. Such extensions may provide more flexible models for hierarchical power relations when the underlying algebraic or relational information is incomplete, vague, or uncertain. Furthermore, as a direction for future research, it would be worthwhile to investigate the properties of the power graph and related concepts introduced in this paper in the settings of HyperGroups[44], [45] and SuperHyperGroups[46].

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Use of Generative AI and AI-Assisted Tools
I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

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