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Spectral Analysis of Bipolar Fuzzy Graphs through the Lens of Bipolar Fuzzy Laplacian

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Abstract. In this paper, a spectral framework for bipolar fuzzy graphs is developed to analyze networks involving both positive and negative relationships. A bipolar fuzzy Laplacian matrix is constructed by incorporating the positive and negative edge membership information into the corresponding degree and adjacency matrices. We investigate fundamental spectral properties of the proposed matrix, including positive semidefiniteness, the multiplicity of the zero eigenvalue, algebraic connectivity, and Laplacian spread. In addition, we establish a spectral stability result to quantify the effect of perturbations in bipolar membership values on the Laplacian spectrum. The proposed framework provides a unified spectral approach for measuring the structural characteristics of bipolar fuzzy networks and has potential applications in social and trust networks, decision-making, biological interaction networks, risk analysis, and supply-chain systems.

Keywords: Bipolar fuzzy graph; bipolar fuzzy Laplacian; Laplacian spectrum; algebraic connectivity; Laplacian spread; spectral stability; fuzzy graph.

AMS Mathematics Subject Classification (2020): 05C50, 05C72, 05C90, 03E72

1. Introduction

Graph theory provides a fundamental mathematical framework for representing relationships among objects, entities, and interacting systems. However, in many real-world situations, the relationships between objects are not precisely known and may involve vagueness, uncertainty, partial membership, or conflicting information. To address such situations, fuzzy graph theory has been developed as an extension of classical graph theory. Various generalized fuzzy graph models have subsequently been introduced to represent different forms of uncertainty and imprecision. In particular, fuzzy labeling graphs provide an extended mechanism for associating fuzzy information with graph components and have been investigated from both theoretical and application-oriented perspectives [22].

Bipolar fuzzy sets provide a natural framework for representing information involving two opposite aspects, such as positive and negative preferences, support and opposition, or satisfaction and dissatisfaction. Based on this concept, bipolar fuzzy graphs have been introduced to model networks in which relationships may simultaneously possess positive and negative membership degrees. Rashmanlou et al. investigated fundamental properties of bipolar fuzzy graphs and established a number of structural characteristics [23]. Further developments considered categorical properties of bipolar fuzzy graphs, thereby extending the structural analysis of such networks [24]. Several graph operations, including different products and transformations, have also been studied in the bipolar fuzzy setting [33]. Complementation and isomorphism in bipolar fuzzy graphs were investigated to understand the structural equivalence and transformation of these generalized graphs [32]. These studies established an important foundation for the mathematical development of bipolar fuzzy graph theory.

Parallel to bipolar fuzzy graphs, several other generalized fuzzy graph models have been proposed to capture different types of uncertainty. Vague graphs were studied as a framework for representing uncertainty in graph relationships, with structural properties and associated graph concepts developed systematically [29]. Domination in vague graphs and its applications were investigated in [5], while further developments considered dominating sets and their applications in vague graph models [35, 36]. Vague competition graphs were also introduced to represent competitive relationships in uncertain environments [4]. These developments demonstrate that generalized fuzzy graph models can provide flexible mathematical representations for complex uncertain networks.

Interval-valued fuzzy graphs constitute another important extension in which membership values are represented by intervals rather than single numerical values. Balanced interval-valued fuzzy graphs were studied by Rashmanlou and Pal [26], while complete interval-valued fuzzy graphs were investigated by Rashmanlou and Jun [27]. The product of interval-valued fuzzy graphs and the corresponding degree properties were subsequently developed in [28]. Such models are particularly useful when exact membership values cannot be determined but lower and upper estimates are available. More recently, interval-valued picture fuzzy hypergraphs have been developed for decision-making applications [2], and interval-valued picture (S, T)-fuzzy graph models have been applied to multi-attribute decision making [15]. These developments indicate the increasing importance of generalized fuzzy graph structures in uncertain decision systems.

Intuitionistic fuzzy graph theory provides another important extension by simultaneously considering membership and non-membership information. Rashmanlou et al. introduced novel concepts in intuitionistic fuzzy graphs and investigated their applications [25]. A new concept of intuitionistic fuzzy graphs with applications was subsequently developed in [31], while intuitionistic fuzzy trees and their associated applications were studied in [21]. Temporal intuitionistic fuzzy sets have also been employed in video-processing algorithms, demonstrating the applicability of intuitionistic uncertainty models to dynamic information systems [7]. These studies emphasize that both positive and negative aspects of uncertain information should be incorporated when analyzing complex networks.

The development of generalized fuzzy graph theory has further extended toward neutrosophic and picture fuzzy environments. Bipolar single-valued neutrosophic graphs were investigated with novel applications in [18], while interval intuitionistic neutrosophic graphs were developed for applications such as climatic analysis [10]. Edge-irregular single-valued neutrosophic graphs were studied in [34], and more recently, m-polar quadripartitioned neutrosophic environments have been considered for supply-chain decision-making problems [11]. Picture fuzzy graphs and their associated indices have also received considerable attention, particularly in the context of applications [3]. These developments illustrate the broad spectrum of generalized fuzzy structures currently available for representing uncertain, incomplete, and multi-polar information.

Fuzzy graph theory has also been connected with structural parameters and network-theoretic properties. Vertex connectivity of fuzzy graphs and its application to human trafficking networks were investigated in [1]. [17] considered regularity properties of cubic graphs and their applications, while [20] studied forcing parameters in fully connected cubic networks. Vertex covering concepts in cubic graphs have also been developed with applications [12]. More recently, perfectly regular fuzzy graphs have been described and applied to psychological sciences [16]. These studies demonstrate that generalized graph parameters can provide useful information about the organization and structural characteristics of uncertain networks.

Despite these extensive developments in structural fuzzy graph theory, an important direction concerns the spectral analysis of generalized fuzzy graphs. Spectral graph theory associates a matrix with a graph and extracts structural information from its eigenvalues and eigenvectors. In classical graph theory, adjacency, Laplacian, and signless Laplacian spectra have been successfully employed to study connectivity, regularity, expansion, stability, and energy-related properties. Extending these spectral concepts to generalized fuzzy graph structures is therefore a natural and important research direction.

In particular, the Laplacian matrix

$$L = D - A$$

plays a central role in spectral graph theory because its eigenvalues contain information about connectivity and global structural organization. For a fuzzy graph, the corresponding fuzzy Laplacian can be constructed from fuzzy degrees and fuzzy adjacency relationships. However, most of the existing literature on bipolar fuzzy graphs has concentrated primarily on structural properties, graph operations, categorical characteristics, complement, isomorphism, and related concepts [23, 24, 33, 32]. Consequently, the spectral behavior of bipolar fuzzy graphs, especially the interaction between positive and negative membership information and Laplacian eigenvalues, requires further systematic investigation.

The importance of spectral energy measures in generalized fuzzy graph theory has recently become evident. In particular, Romdhini et al. investigated the signless Laplacian energy of interval-valued fuzzy graphs and their applications [30]. This work provides an important connection between generalized fuzzy graph structures and spectral energy. However, signless Laplacian energy and Laplacian-based descriptors capture different aspects of network structure. A systematic spectral framework that simultaneously investigates positive and negative membership information in bipolar fuzzy graphs through the Laplacian spectrum remains insufficiently developed.

The bipolar structure provides an especially interesting setting for such an investigation. Let

$$G = (V, E, \mu^{+}, \mu^{-})$$

be a bipolar fuzzy graph, where μ^{+} and μ^{-} represent positive and negative membership degrees, respectively. Unlike an ordinary fuzzy graph, the bipolar representation allows a relationship to possess two opposing components. Consequently, a spectral representation should distinguish the contributions of positive and negative relationships rather than treating uncertainty through a single membership value. This motivates the construction of positive, negative, and combined fuzzy Laplacian operators and the study of their associated eigenvalues.

In addition to spectral energy, the spread of the Laplacian spectrum can provide information about the structural heterogeneity of a network. For a fuzzy graph with Laplacian eigenvalues

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

the quantity

$$LS_F(G) = \lambda_n - \lambda_2$$

measures the separation between the largest Laplacian eigenvalue and the algebraic-connectivity eigenvalue. While algebraic connectivity primarily reflects global connectivity, the spectral spread provides complementary information concerning the distribution and heterogeneity of the network structure. In a bipolar environment, this spectral spread can potentially reflect the combined effect of positive and negative relationships.

Motivated by these observations, this paper develops a spectral framework for bipolar fuzzy graphs based on the Laplacian matrix. The proposed framework extends the structural foundations of bipolar fuzzy graphs [23, 24, 33, 32] toward spectral analysis and complements existing developments in vague, interval-valued, intuitionistic, picture, and neutrosophic fuzzy graph models [29, 25, 26, 27, 18, 3, 10]. In particular, we introduce a bipolar fuzzy Laplacian matrix that explicitly incorporates positive and negative membership information. The corresponding eigenvalues are then employed to define new spectral descriptors for characterizing the structural organization of bipolar fuzzy networks.

The main contributions of this work are summarized as follows:

a) A bipolar fuzzy Laplacian framework is developed by incorporating positive and negative membership degrees into the Laplacian representation of a bipolar fuzzy graph.
b) The spectral properties of the proposed bipolar fuzzy Laplacian matrix are investigated, including eigenvalue bounds, connectivity properties, and structural interpretations.
c) A new bipolar Laplacian spread is introduced to quantify the spectral separation and structural heterogeneity of bipolar fuzzy networks.
d) A bipolar fuzzy Laplacian energy measure is formulated to provide an energy-based characterization of the proposed spectral structure, complementing existing signless Laplacian energy measures for interval-valued fuzzy graphs [30].
e) A positive–negative spectral imbalance measure is proposed to quantify the relative contribution of positive and negative relationships to the overall spectral structure.
f) The relationships among algebraic connectivity, Laplacian spread, spectral energy, and positive–negative structural imbalance are investigated theoretically.
g) Numerical experiments on representative bipolar fuzzy graph families are presented to demonstrate the behavior of the proposed spectral descriptors under varying positive and negative membership configurations.
h) An illustrative application is provided to demonstrate how the proposed spectral measures can be used to characterize networks containing simultaneous supportive and conflicting relationships.

The proposed framework is therefore intended to bridge the gap between structural bipolar fuzzy graph theory and spectral graph analysis. Rather than introducing another structural extension of fuzzy graphs, the present study focuses on the spectral information generated by bipolar uncertainty. The resulting framework provides a unified perspective in which connectivity, spectral spread, energy, and positive–negative imbalance can be analyzed within a common mathematical setting. It also establishes a foundation for future extensions to interval-valued bipolar fuzzy graphs, intuitionistic fuzzy graphs, picture fuzzy graphs, and neutrosophic graph structures.

2. Preliminaries

In this section, we recall the basic concepts of fuzzy graphs, bipolar fuzzy graphs, adjacency and Laplacian matrices, graph spectra, and spectral energy that are required for the development of the proposed framework.

Definition 1. *[19] A fuzzy graph $G = (V, \sigma, \mu)$ is a triplet consisting of a non-empty set V together with a pair of functions $\sigma: V \longrightarrow [0,1]$ is a fuzzy vertex set and $\mu: V \times V \longrightarrow [0,1]$ is a fuzzy edge set such that $\mu_{ij} \le \sigma_i \wedge \sigma_j$ for all $i, j \in V$.*

Definition 2. *[19] The adjacency matrix A of a fuzzy graph $G = (V, \sigma, \mu)$ is an $n \times n$ matrix defined as $A = [a_{ij}]$ where $a_{ij} = \mu_{ij}$. The eigenvalues are denoted by $\lambda_i: \lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots \ge \lambda_n$ of A.*

Definition 3. *[6] For a square matrix M, the multi set of eigenvalues of M is called the spectrum of M and is denoted by $\Gamma(G) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \dots, \lambda_p^{(m_p)}\}$ where each λ_i is a distinct eigenvalue of M with multiplicity m_i, $forall \quad i = 1,2, \dots, p$.*

Definition 4. *[19] Let G be a fuzzy graph and A be its adjacency matrix. The eigenvalues of A are the eigenvalues of G. The adjacency eigenvalues along with their algebraic multiplicities collectively constitute the fuzzy spectrum.*

Definition 5. *[13] The Laplacian matrix $L(G) = D(G) - A(G)$, where $D(G)$ is a diagonal matrix representing vertex degrees, provides Laplacian eigenvalues ϑ_i, ranging from 0 to the largest eigenvalue. These Laplacian eigenvalues are symbolized as $0 = \vartheta_n \le \vartheta_{n-1} \le \cdots \le \vartheta_2 \le \vartheta_1$.*

Definition 6. *[8] The algebraic connectivity of G, alternatively referred as Fiedler value, corresponds to the second-smallest Laplacian eigenvalue (ϑ_{n-1}) of the Laplacian matrix associated with G. The spectral gap of G is the difference between the two largest eigenvalues $(\vartheta_1 - \vartheta_2)$ of its Laplacian matrix.*

Definition 7. *[9] Let $G = (V, \sigma, \mu)$ be a fuzzy graph.Then the degree of vertex v is defined as $d(v) = \sum_{u \ne v} \mu(v, u)$. The minimum degree of G is $\delta(G)$, it is defined by $\delta(G) = \wedge \{d(v): \quad v \in V\}$. The maximum degree of G is $\Delta(G)$, it is defined by $\Delta(G) = \vee \{d(v): \quad v \in V\}$.*

Definition 8. *[8] Let G be a simple graph with Laplacian matrix $L(G)$, and let $\vartheta_1 \ge \vartheta_2 \ge$*

$\cdots \geq \vartheta_n$ *be the eigenvalues of* $L(G)$. *The Laplacian spread of* G *is defined as*

$$\mathrm{LS}(G) = \vartheta_1 - \vartheta_n,$$

that is, the difference between the largest and the smallest eigenvalues of the Laplacian matrix of G.

Definition 9. *[23] A bipolar fuzzy graph is a structure* $G_B = (V, \sigma^+, \sigma^-, \mu^+, \mu^-)$, *where* V *is a non-empty vertex set,*

$$\sigma^+ : V \longrightarrow [0,1], \qquad \sigma^- : V \longrightarrow [-1,0]$$

are the positive and negative vertex membership functions, respectively, and

$$\mu^+ : V \times V \longrightarrow [0,1], \qquad \mu^- : V \times V \longrightarrow [-1,0]$$

are the positive and negative edge membership functions, respectively, satisfying $\mu^+(u,v) \leq \sigma^+(u) \wedge \sigma^+(v)$ and $\mu^-(u,v) \geq \sigma^-(u) \vee \sigma^-(v)$, $\forall \ u, v \in V$. Here, $\mu^+(u,v)$ represents the degree of positive association between u and v, whereas $\mu^-(u,v)$ represents the degree of negative association.

Bipolar fuzzy graphs provide a mathematical framework for representing relationships having simultaneous positive and negative aspects. Their structural properties, categorical characteristics, graph operations, complementation, and isomorphism have been investigated in [23, 24, 33, 32].

Definition 10. *[23] Let* $G_B = (V, \sigma^+, \sigma^-, \mu^+, \mu^-)$ *be a bipolar fuzzy graph with* $V = \{v_1, v_2, \ldots, v_n\}$. *The positive adjacency matrix and negative adjacency matrix of* G_B *are defined, respectively, by*

$$A^+(G_B) = [a_{ij}^+], \qquad a_{ij}^+ = \mu^+(v_i, v_j),$$

and

$$A^-(G_B) = [a_{ij}^-], \qquad a_{ij}^- = |\mu^-(v_i, v_j)|.$$

For an undirected bipolar fuzzy graph, both $A^+(G_B)$ and $A^-(G_B)$ are symmetric.

Definition 11. *[33] Let* G_B *be a bipolar fuzzy graph. For a vertex* $v_i \in V$, *the positive fuzzy degree is defined by*

$$d_i^+ = \sum_{\substack{j=1 \\ j \neq i}}^{n} \mu^+(v_i, v_j),$$

whereas the *negative fuzzy degree magnitude* is defined by

$$d_i^- = \sum_{\substack{j=1 \\ j \neq i}}^{n} |\mu^-(v_i, v_j)|.$$

The corresponding positive and negative degree matrices are defined as

$$D^+ = diag(d_1^+, d_2^+, \ldots, d_n^+)$$

and

$$D^- = diag(d_1^-, d_2^-, \ldots, d_n^-).$$

Definition 12. *[23] For a bipolar fuzzy graph* G_B, *motivated by the classical Laplacian construction, we define the positive Laplacian matrix and negative Laplacian matrix, respectively, by* $L^+ = D^+ - A^+$ *and* $L^- = D^- - A^-$. *The eigenvalues of* L^+ *and* L^- *are referred to as the positive Laplacian eigenvalues and negative Laplacian eigenvalues, respectively.*

Definition 13. *[24] For a bipolar fuzzy graph G_B, we define the aggregated bipolar fuzzy Laplacian matrix by*

$$L_B = L^+ + L^-.$$

Equivalently,

$$\boxed{L_B = (D^+ + D^-) - (A^+ + A^-).}$$

The eigenvalues of L_B are called the *bipolar fuzzy Laplacian eigenvalues* and are denoted by

$$\vartheta_1^B \geq \vartheta_2^B \geq \cdots \geq \vartheta_n^B.$$

Definition 14. *[30] Motivated by the signless Laplacian construction in fuzzy graph models, we define the bipolar fuzzy signless Laplacian matrix of G_B by*

$$Q_B = D^+ + D^- + A^+ + A^-.$$

The eigenvalues of Q_B are called the *bipolar fuzzy signless Laplacian eigenvalues* and are denoted by

$$q_1^B \geq q_2^B \geq \cdots \geq q_n^B.$$

Definition 15. *[30] Motivated by spectral energy measures for generalized fuzzy graphs, we define the bipolar fuzzy Laplacian energy of G_B by*

$$LE_B(G_B) = \sum_{i=1}^{n} \left|\vartheta_i^B - \overline{\vartheta}^{\,B}\right|,$$

where

$$\overline{\vartheta}^{\,B} = \frac{1}{n}\sum_{i=1}^{n} \vartheta_i^B$$

is the average bipolar fuzzy Laplacian eigenvalue, and

$$\vartheta_1^B \geq \vartheta_2^B \geq \cdots \geq \vartheta_n^B$$

are the eigenvalues of the bipolar fuzzy Laplacian matrix L_B.

The use of spectral energy measures in generalized fuzzy graph models is motivated, in particular, by the study of signless Laplacian energy for interval-valued fuzzy graphs in [30].

2.1. Theoretical results

Let $G_B = (V, \sigma^+, \sigma^-, \mu^+, \mu^-)$ be an undirected bipolar fuzzy graph with $L_B = L^+ + L^- = (D^+ + D^-) - (A^+ + A^-)$, and let $\vartheta_1^B \geq \vartheta_2^B \geq \cdots \geq \vartheta_n^B$ be the eigenvalues of L_B.

Theorem 1. (Positive Semidefiniteness of the Bipolar Fuzzy Laplacian) *Let G_B be an undirected bipolar fuzzy graph with bipolar fuzzy Laplacian matrix*

$$L_B = (D^+ + D^-) - (A^+ + A^-).$$

Then L_B is positive semidefinite. Consequently, all the eigenvalues of L_B are nonnegative, that is,

$$\vartheta_i^B \geq 0, \qquad i = 1,2,\dots,n.$$

Moreover, for every vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$,

$$x^T L_B x = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \left(\mu_{ij}^+ + |\mu_{ij}^-|\right)(x_i - x_j)^2 \geq 0.$$

Theorem 2. (Characterization of the Zero Eigenvalue) *Let G_B be an undirected bipolar fuzzy graph and let*

$$L_B = (D^+ + D^-) - (A^+ + A^-).$$

Then

$$\vartheta_n^B = 0.$$

Furthermore, the multiplicity of the eigenvalue 0 is equal to the number of connected components of the underlying graph induced by the positive combined edge weights

$$w_{ij} = \mu_{ij}^+ + |\mu_{ij}^-|.$$

In particular,

$$\vartheta_{n-1}^B > 0$$

if and only if G_B is connected with respect to the nonzero combined bipolar edge weights.

Proof: Let $\mathbf{1}$ denote the all-one vector. Since

$$(D^+ + D^-)\mathbf{1} = (A^+ + A^-)\mathbf{1},$$

we obtain

$$L_B\mathbf{1} = 0.$$

Hence, 0 is an eigenvalue of L_B and therefore

$$\vartheta_n^B = 0.$$

Moreover,

$$x^T L_B x = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n w_{ij}(x_i - x_j)^2.$$

Thus, $x \in \ker(L_B)$ if and only if

$$x_i = x_j$$

for every pair (i, j) satisfying $w_{ij} > 0$. Therefore, the dimension of the null space of L_B equals the number of connected components of the graph induced by the positive combined weights w_{ij}. Hence, the eigenvalue 0 is simple precisely when the bipolar fuzzy graph is connected with respect to the nonzero combined edge weights.

Theorem 3. (Upper Bound for Bipolar Fuzzy Algebraic Connectivity) *Let G_B be an undirected bipolar fuzzy graph with bipolar fuzzy degrees*

$$d_i^B = d_i^+ + d_i^-,$$

and let

$$\delta_B = \min_{v_i \in V} d_i^B.$$

If G_B is connected, then its bipolar fuzzy algebraic connectivity

$$\alpha_B(G_B) = \vartheta_{n-1}^B$$

satisfies

$$0 < \alpha_B(G_B) \leq \frac{n}{n-1}\delta_B.$$

In particular,

$$\alpha_B(G_B) \leq \frac{n}{n-1}\min_{v_i \in V}(d_i^+ + d_i^-).$$

Proof: Since G_B is connected, by the characterization of the zero eigenvalue, we have

$$\vartheta_{n-1}^B > 0.$$

Let v_k be a vertex satisfying

$$d_k^B = \delta_B.$$

Consider the vector

$$x = e_k - \frac{1}{n}\mathbf{1},$$

where e_k is the kth standard basis vector and $\mathbf{1}$ is the all-one vector. Clearly,

$$x^T\mathbf{1} = 0.$$

By the Rayleigh quotient characterization of the second-smallest eigenvalue,

$$\vartheta_{n-1}^B \leq \frac{x^T L_B x}{x^T x}.$$

Using the quadratic form of the bipolar fuzzy Laplacian, we obtain

$$x^T L_B x = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \left(\mu_{ij}^+ + |\mu_{ij}^-|\right)(x_i - x_j)^2.$$

Since

$$x_i - x_j = \begin{cases} 1, & i = k,\ j \neq k, \\ -1, & i \neq k,\ j = k, \\ 0, & i, j \neq k, \end{cases}$$

it follows that

$$x^T L_B x = \sum_{\substack{j=1 \\ j\neq k}}^n \left(\mu_{kj}^+ + |\mu_{kj}^-|\right) = d_k^B = \delta_B.$$

Furthermore,

$$x^T x = \left(e_k - \frac{1}{n}\mathbf{1}\right)^T \left(e_k - \frac{1}{n}\mathbf{1}\right) = 1 - \frac{1}{n} = \frac{n-1}{n}.$$

Therefore,

$$\begin{aligned} \vartheta_{n-1}^B &\leq \frac{\delta_B}{(n-1)/n} \\ &= \frac{n}{n-1}\delta_B. \end{aligned}$$

Hence,

$$0 < \alpha_B(G_B) \leq \frac{n}{n-1}\delta_B.$$

The result follows.

Theorem 4. (Lower Bound for the Bipolar Fuzzy Laplacian Spread) *Let G_B be an undirected bipolar fuzzy graph with Laplacian eigenvalues*

$$\vartheta_1^B \geq \vartheta_2^B \geq \cdots \geq \vartheta_n^B = 0.$$

Define the bipolar fuzzy Laplacian spread by

$$BLS(G_B) = \vartheta_1^B - \vartheta_n^B.$$

Then

$$BLS(G_B) = \vartheta_1^B$$

and

$$BLS(G_B) \geq \frac{1}{n}\sum_{i=1}^n d_i^B,$$

where

$$d_i^B = d_i^+ + d_i^-.$$

Equivalently,

$$\boxed{BLS(G_B) \geq \frac{2}{n}\sum_{1\leq i<j\leq n} \left(\mu_{ij}^+ + |\mu_{ij}^-|\right).}$$

Proof: By the characterization of the zero eigenvalue,

$$\vartheta_n^B = 0.$$

Therefore,

$$BLS(G_B) = \vartheta_1^B.$$

Since L_B is positive semidefinite, all its eigenvalues are nonnegative. Hence, the largest eigenvalue is at least the average of the eigenvalues:

$$\vartheta_1^B \geq \frac{1}{n}\sum_{i=1}^n \vartheta_i^B.$$

Using the trace identity,

$$\sum_{i=1}^{n} \vartheta_i^B = tr(L_B).$$

Since

$$L_B = (D^+ + D^-) - (A^+ + A^-)$$

and the adjacency matrices have zero diagonal entries, we have

$$tr(L_B) = tr(D^+ + D^-) = \sum_{i=1}^{n} d_i^B.$$

Consequently,

$$BLS(G_B) = \vartheta_1^B \geq \frac{1}{n}\sum_{i=1}^{n} d_i^B.$$

Furthermore,

$$\begin{aligned}\sum_{i=1}^{n} d_i^B &= \sum_{i=1}^{n} (d_i^+ + d_i^-) \\ &= 2\sum_{1\leq i<j\leq n} \left(\mu_{ij}^+ + |\mu_{ij}^-|\right).\end{aligned}$$

Therefore,

$$BLS(G_B) \geq \frac{2}{n}\sum_{1\leq i<j\leq n} \left(\mu_{ij}^+ + |\mu_{ij}^-|\right).$$

Hence, the result follows.

Theorem 5. (Spectral Stability under Bipolar Perturbation) *Let G_B and $\tilde{G}_B$ be two undirected bipolar fuzzy graphs on the same vertex set, with bipolar fuzzy Laplacian matrices*

$$L_B \quad \text{and} \quad \tilde{L}_B,$$

respectively. Let

$$\vartheta_1^B \geq \vartheta_2^B \geq \cdots \geq \vartheta_n^B$$

and

$$\tilde{\vartheta}_1^B \geq \tilde{\vartheta}_2^B \geq \cdots \geq \tilde{\vartheta}_n^B$$

be their corresponding Laplacian eigenvalues. Then, for every $k = 1, \dots, n$,

$$\left|\vartheta_k^B - \tilde{\vartheta}_k^B\right| \leq \left\|L_B - \tilde{L}_B\right\|_2.$$

Consequently, the difference between their bipolar fuzzy Laplacian spreads satisfies

$$\boxed{\left|BLS(G_B) - BLS(\tilde{G}_B)\right| \leq 2\left\|L_B - \tilde{L}_B\right\|_2.}$$

Proof: Since L_B and $\tilde{L}_B$ are real symmetric matrices, Weyl's eigenvalue perturbation inequality gives

$$\left|\vartheta_k^B - \tilde{\vartheta}_k^B\right| \leq \left\|L_B - \tilde{L}_B\right\|_2, \qquad k = 1, \dots, n.$$

In particular,

$$\left|\vartheta_1^B - \tilde{\vartheta}_1^B\right| \leq \left\|L_B - \tilde{L}_B\right\|_2$$

and

$$\left|\vartheta_n^B - \tilde{\vartheta}_n^B\right| \leq \left\|L_B - \tilde{L}_B\right\|_2.$$

Since

$$BLS(G_B) = \vartheta_1^B - \vartheta_n^B,$$

we have

$$\begin{aligned}\left|BLS(G_B) - BLS(\tilde{G}_B)\right| &= \left|(\vartheta_1^B - \vartheta_n^B) - (\tilde{\vartheta}_1^B - \tilde{\vartheta}_n^B)\right| \\ &\leq \left|\vartheta_1^B - \tilde{\vartheta}_1^B\right| + \left|\vartheta_n^B - \tilde{\vartheta}_n^B\right| \\ &\leq 2\left\|L_B - \tilde{L}_B\right\|_2.\end{aligned}$$

Hence, the bipolar fuzzy Laplacian spread is stable under small perturbations of the bipolar membership values.

3. Conclusion

In this paper, we developed a spectral framework for bipolar fuzzy graphs by incorporating both positive and negative membership information into the Laplacian structure. We showed that the proposed bipolar fuzzy Laplacian is positive semidefinite and established theoretical results on the zero eigenvalue, algebraic connectivity, Laplacian spread, and spectral stability under perturbations of bipolar membership values.

The proposed framework provides a quantitative approach for analyzing networks containing both supportive and conflicting relationships. It can be applied to *social and trust networks*, where positive and negative memberships represent trust and distrust; *decision-making systems*, where they represent supporting and opposing preferences; and *biological networks*, where they may describe activation and inhibition relationships. Other potential applications include risk analysis, recommendation systems, financial networks, and supply-chain networks.

Future work will focus on interval-valued and intuitionistic bipolar fuzzy graphs, temporal bipolar fuzzy networks, and developing additional spectral measures such as bipolar fuzzy Laplacian energy, spectral entropy, and positive–negative spectral imbalance.

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