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On Graceful Labeling of Some Bicyclic Graphs

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Abstract . A labeled graph G which can be gracefully numbered is said to be graceful. Labeling the nodes of G with distinct nonnegative integers and then labeling the e edges of G with the absolute differences between node values, if the graph edge numbers run from 1 to e, the graph G is gracefully numbered. In this paper, we have discussed the gracefulness of a few graphs formed from various combinations of some bicyclic graphs.

***Keywords*:** Labeling; Graceful graph; Bicyclic graph

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1. Introduction

We consider only finite, simple and undirected labeled graphs here. Labeled graphs form useful models for a wide range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing and database management

An *graceful labeling* f of a graph G with q edges is an injective function from the vertices of G to the set $\{0,1,2,\ldots,q\}$ such that when each edge xy is assigned the label $|f(x)-f(y)|$, the resulting edge labels are distinct and nonzero. The concept above was put forward by Rosa in 1967.

All trees are graceful by the Ringel-Kotzig conjecture. Among the trees known to be graceful are caterpillars, trees with at most 4-end vertices, trees with diameter at most 5, trees with at most 35 vertices, symmetrical trees, regular olive trees, lobsters, firecrackers, banana trees and bamboo trees.

Graphs consisting of any number of pair wise disjoint paths with common end vertices are called generalized theta graphs. Various labelings have been found for these graphs.

In this paper, some new classes of graphs have been constructed by combining some *theta* graphs, path graphs with the star graphs St(n), ($n \geq 1$). Our notations and terminology are as in [1]. We refer to [2] for some basic concepts.

2. Some results on the gracefulness of various bicyclic graphs

Definition 2.1. The theta graph θ (α, β, γ) consists of three edge disjoint paths of lengths α, β, and γ having the same end points. Let the theta graph θ (2, 2, 3) have the paths P_1:v_2,v_1,v_5;P_2:v_2,v_6,v_5 and P_3:v_2,v_3,v_4,v_5.

Definition 2.2. Let A be any graph. Let B and C be any tree graphs. A_{ij} @ (B,C) denotes a new graph formed by attaching a center vertex of B to a vertex v_i of A by means of an edge and a center vertex of C to a vertex v_j of A by means of an edge, where i and j are distinct.

Definition 2.3. A graph with two cycles having p vertices and (p + 1) edges is called a bicyclic graph.

Theorem 2.4. $\theta_{\alpha\beta}$ (2,2,3) @ (St(m),St(n)) is graceful, for m = n and m, n $\geq$ 1.

Proof. Consider the theta graph θ (2,2,3) with 6 vertices v_1,v_2,…,v_6 . Let St(m) be a star graph with (m +1) vertices w,w_1,w_2,…,w_m for m $\geq$ 1,where w is the center vertex and w_1,w_2,…,w_m are pendant vertices.

Label the star graph St(n) on (n+1) vertices, by naming the vertices u,u_1,u_2,…,$u_{n.}$ (n $\geq$ 1). Here u is the center vertex and the other vertices are pendant vertices. To form the bicyclic graph G = $\theta_{\alpha\beta}$ (2,2,3) @ (St(m),St(n)) attach the vertex w of St(m) with a vertex v_α (1 $\leq \alpha \leq$ 6) of θ (2,2,3) by an edge wv_α and attach the vertex u of St(n) with a vertex v_β (1 $\leq \beta \leq$ 6) of θ (2,2,3) by an edge wv_β, for distinct α and β. G has m + n + 8 vertices and m + n + 9 edges. Let m and n be equal.

The vertex set V(G) = {u, w, v_i, w_j, u_k / i = 1,2,…,6 , j = 1,2,…,m , k = 1,2,…,n}. The edge set E(G) ={v_i v_{i+1} / i=1,2,…,5} $\cup$ {v_i v_{i+4} / i = 1,2} $\cup$ {ww_j / j = 1,2,…,m} $\cup$ {u u_k / k=1,2,…,n}$\cup$ {v_i w, v_j u / i, j =1or 2 or 3 or 4 or 5 or 6 and i $\neq$ j }.

Let f be the labeling on the set of vertices of G and g be the induced labeling on the set of edges of G. The vertex label set of G can be written as A$\cup$B$\cup$C$\cup$I where A ={f(v_i) / i=1,2,…,6}, B = { f(w_j) / j=1,2,…, m}, C = { f(u_k) / k=1,2,…,n }, I = { f(u) , f(w) }.

The edge label set of G can be written as D$\cup$E$\cup$F$\cup$G$\cup$H where D={g(v_i v_{i+1}) / i=1,2,…,5}, E={g(v_i v_{i+4}) / i=1,2},F={g(ww_j) / j=1,2,…,m}, G = {g(u u_k) / k=1,2,…,n}, H={g(v_i w), g(v_j u) / i, j = 1 or 2 or 3 or 4 or 5 or 6 and i $\neq$ j }. Consider all the cases up to isomorphism.

Case 1. α = 1, β = 4.

Let the labeling f on the vertices of G be defined by f(v_i) = i + 1 for i = 1; f(v_i) = m + i + 5 for i = 2; f(v_i) = i – 3 for i = 3; f(v_i) = m + n + i +5 for i = 4; f(v_i) = m + i + 3 for i = 5; f(v_i) = i – 1 for i = 6; f(w_j) = j + 5 for 1 $\leq$ j $\leq$ m; f(u_k) = m + k + 8 for 1 $\leq$ k $\leq$ n; f(w) = m + 6; f(u) = 1.

The induced labeling g on the edges of G is defined by g(v_i v_{i+1}) = m + 5 for i = 1; g(v_i v_{i+1}) = m + 7 for i = 2; g(v_i v_{i+1}) = m + n + 9 for i = 3; g(v_i v_{i+1}) = m + 1 for i = 4; g(v_i v_{i+1}) = m + 3 for i = 5; g(v_i v_{i+4}) = m + 6 for i = 1; g(v_i v_{i+4}) = m + 2 for i = 2; g(ww_j) = m – j + 1 for 1 $\leq$ j $\leq$ m; g(uu_k) = m + k + 7 for 1 $\leq$ k $\leq$ n; g(wv_i) = m + 4 for i = 1; g(uv_j) = m + n + 8 for j = 4.

The vertex labels of G can be arranged in the following order.

A = { 0, 2, 5, m +7, m + 8, m + n + 9 }, B = {6,7, …,m + 5}, C = {m + 9,… , m + n + 8}, I = {1, m + 6 }.The set of vertex labels of G is A∪B∪C∪I = {0,1,2,5,…,m + n + 9}.

The edge labels of G can be arranged in the following order.

D = {m + 1, m + 3, m + 5, m + 7, m + n + 9 }, E = {m + 2, m + 6}, F = {1,2,…, m}, G = {m + 8,…, m + n + 7 }, H = {m + 4, m + n + 8}. The set of edge labels of G is D ∪ E ∪ F∪ G∪ H = {1, 2, 3,…,m + n +9 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta_{14}(2,2,3)$ @ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

Case 2. $\alpha = 1$, $\beta = 5$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i + 3$ for $i = 1$; $f(v_i) = m + n + i + 7$ for $i = 2$; $f(v_i) = i - 3$ for $i = 3$; $f(v_i) = i - 2$ for $i = 4$; $f(v_i) = m + n + i + 3$ for $i = 5$; $f(v_i) = i - 5$ for $i = 6$; $f(w_j) = j + 5$ for $1 \leq j \leq m$; $f(u_k) = m + k + 7$ for $1 \leq k \leq n$; $f(w) = 3$; $f(u) = 5$.

The induced labeling g on the edges of G is defined by $g(v_i\, v_{i+1}) = m + n + 5$ for $i = 1$; $g(v_i\, v_{i+1}) = m + n + 9$ for $i = 2$; $g(v_i\, v_{i+1}) = 2$ for $i = 3$; $g(v_i\, v_{i+1}) = m + n + 6$ for $i = 4$; $g(v_i\, v_{i+1}) = m + n + 7$ for $i = 5$; $g(v_i\, v_{i+4}) = m + n + 4$ for $i = 1$; $g(v_i\, v_{i+4}) = m + n + 8$ for $i = 2$; $g(ww_j) = j +2$ for $1 \leq j \leq m$; $g(uu_k) = m + k +2$ for $1 \leq k \leq n$; $g(wv_i) = 1$ for $i = 1$; $g(uv_j) = m + n +3$ for $j = 5$.

The vertex labels of G can be arranged in the following order. A = {0,1,2, 4,m + n +8,m + n + 9 }, B = {6,7,…,m + 5}, C = { m + 8,…,m + n + 7}, I = {3,5}. The set of vertex labels of G is A∪B∪C∪I = {0,1,2,…,m + 5, m + 8,…, m + n +9}.

The edge labels of G can be arranged in the following order.

D = {2, m+ n+ 5, m + n + 6, m + n + 7, m + n + 9}, E = {m+ n+ 4, m + n + 8}, F = {3,4,…,m+ 2}, G = {m + 3,…,m + n + 2}, H = {1,m + n + 3}. The set of edge labels of G is D ∪ E ∪ F∪ G∪ H = {1, 2, 3, … , m + n +9 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta_{15}(2,2,3)$ @ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

Case 3. $\alpha = 1$, $\beta = 6$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i + 3$ for $i = 1$; $f(v_i) = i$ for $i = 2$; $f(v_i) = m + n + i + 3$ for $i = 3$; $f(v_i) = i - 4$ for $i = 4$; $f(v_i) = m + n + i + 4$ for $i = 5$; $f(v_i) = i - 5$ for $i = 6$; $f(w_j) = j + 4$ for $1 \leq j \leq m$; $f(u_k) = m + k + 5$ for $1 \leq k \leq n$; $f(w) = m + n + 7$; $f(u) = m + n + 8$.

The induced labeling g on the edges of G is defined by $g(v_i\, v_{i+1}) = 2$ for $i = 1$; $g(v_i\, v_{i+1}) = m +n +4$ for $i = 2$; $g(v_i\, v_{i+1}) = m + n + 6$ for $i = 3$; $g(v_i\, v_{i+1}) = m + n + 9$ for $i = 4$; $g(v_i\, v_{i+1}) = m + n + 8$ for $i = 5$; $g(v_i\, v_{i+4}) = m +n + 5$ for $i = 1$; $g(v_i\, v_{i+4}) = 1$ for $i = 2$; $g(ww_j) = m +n - j + 3$for $1\leq j \leq m$; $g(uu_k) = n - k + 3$ for $1 \leq k \leq n$; $g(wv_i) = m + n + 3$ for $i = 1$; $g(uv_j) = m + n + 7$ for $j = 6$.

The vertex labels of G can be arranged in the following order.

A = {0,1,2,4,m +n +6,m +n +9}, B = {5,6,7,…,m + 4}, C = {m +6,…,m +n +5}, I = {m +n +7, m +n + 8}. The set of vertex labels of G is A∪B∪C∪I = {0,1,2,4,…, m +4,m +6,…,m +n +9}.

The edge labels of G can be arranged in the following order.

D = {2, m + n + 4, m + n + 6, m + n + 8, m + n + 9}, E = {1,m + n + 5}, F ={n + 3,…,m + n +2}, G = {3,…,m + n + 2 }, H = { m + n + 3, m + n + 7 }. The set of edge labels of G is D ∪ E ∪ F∪ G∪ H = {1, 2, 3, … , m + n +9 }.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence, G = $\theta_{16}(2,2,3)$ @ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

Case 4. $\alpha = 2$, $\beta = 4$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i - 1$ for i = 1; $f(v_i) = m + n + 6 + i$ for i = 2; $f(v_i) = i + 1$for i = 3; $f(v_i) = n + 2 + i$ for i = 4; $f(v_i) = m + n + i + 4$ for i = 5; $f(v_i) = i - 4$ for i = 6; $f(w_j) = n + j + 6$ for $1 \leq j \leq m$; $f(u_k) = k + 5$ for $1 \leq k \leq n$; f(w) = 3; f(u) = 5.

The induced labeling g on the edges of G is defined by $g(v_i\, v_{i+1}) = m + n + 8$ for i = 1; $g(v_i\, v_{i+1}) = m + n + 4$ for i = 2; $g(v_i\, v_{i+1}) = n + 2$ for i = 3; $g(v_i\, v_{i+1}) = n + 3$ for i = 4; $g(v_i\, v_{i+1}) = m + n + 7$ for i = 5; $g(v_i\, v_{i+4}) = m + n + 9$ for i = 1; $g(v_i\, v_{i+4}) = m + n + 6$ for i = 2; $g(ww_j) = n + j + 3$ for $1 \leq j \leq m$; $g(uu_k) = k$ for $1 \leq k \leq n$; $g(wv_i) = m + n + 5$ for i = 2; $g(uv_j) = n + 1$ for j = 4 .

The vertex labels of G can be arranged in the following order.

A = {0,2,4,n + 6,m + n + 8, m + n + 9 }, B = { n +7,m +n +6 }, C = {6,7,…, n + 5}, I ={3,5}. The set of vertex labels of G is A∪B∪C∪I = {0, 2,…,m + n +6, m + n + 8, m + n + 9}.

The edge labels of G can be arranged in the following order.

D = { n + 2, n + 3, m + n + 4, m + n + 7, m + n + 8 }, E = { m + n + 6, m + n + 9 }, F = {n + 4,…, m + n + 3 }, G = {1, 2,…, n }, H = { n +1, m + n +5}.The set of edge labels of G is D ∪ E ∪ F∪ G∪ H = {1, 2, 3, … , m + n +9 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta_{24}(2,2,3)$ @ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

Case 5. $\alpha = 2$, $\beta = 5$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i - 1$ for i = 1; $f(v_i) = m + n + i + 7$ for i = 2; $f(v_i) = n + i + 3$ for i = 3; $f(v_i) = m + n + i + 4$ for i = 4; $f(v_i) = m + n + i + 2$ for i = 5; $f(v_i) = i - 5$ for i = 6; $f(w_j) = j + 5$ for $1 \leq j \leq m$; $f(u_k) = m + k + 6$ for $1 \leq k \leq n$;f(w) = 4; f(u) = 3.

The induced labeling g on the edges of G is defined by $g(v_i\, v_{i+1}) = m + n + 9$ for i = 1; $g(v_i\, v_{i+1}) = m + 3$ for i = 2; $g(v_i\, v_{i+1}) = m + 2$ for i = 3; $g(v_i\, v_{i+1}) = 1$ for i = 4; $g(v_i\, v_{i+1}) = m + n + 6$ for i = 5; $g(v_i\, v_{i+4}) = m + n +7$ for i = 1; $g(v_i\, v_{i+4}) = m +n +8$ for i = 2; $g(ww_j) = j +1$ for $1 \leq j \leq m$; $g(uu_k) = m +k +3$ for $1 \leq k \leq n$; $g(wv_i) = m + n + 5$ for i = 2; $g(uv_j) = m + n + 4$ for j = 5.

The vertex labels of G can be arranged in the following order.

A = { 0,1, n + 6, m + n + 7, m +n +8, m + n + 9 }, B = {6,7, …, m + 5}, C = {m +7,…,m +n + 6 }, I = {3, 4 }.The set of vertex labels of G is A∪B∪C∪I = {0, 1,3,4, 6, … , m + n +9}.

The edge labels of G can be arranged in the following order.

D = {1, m + 2, m + 3, m + n + 6, m + n + 9 }, E = {m + n + 7, m + n + 8}, F = {2,3,…,m +1}, G = {m + 4,…, m + n + 3 }, H = { m + n + 4, m + n + 5 }.The set of edge labels of G is D $\cup$ E $\cup$ F$\cup$ G$\cup$ H = {1, 2, 3, … , m + n +9}.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta_{25}(2,2,3)$ @ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

Case 6. $\alpha = 3$, $\beta = 4$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i - 1$ for i = 1; $f(v_i) = m + n + i + 6$ for i = 2; $f(v_i) = n + i + 4$ for i = 3; $f(v_i) = i$ for i = 4; $f(v_i) = m + n + 4 + i$ for i = 5; $f(v_i) = i - 4$ for i = 6; $f(w_j) = m + j + 7$ for $1 \leq j \leq m$; $f(u_k) = k + 5$ for $1 \leq k \leq n$; f(w) = 3; f(u) = n + 6.

The induced labeling g on the edges of G is defined by $g(v_i\, v_{i+1}) = m + n + 8$ for i = 1; $g(v_i\, v_{i+1}) = n + 1$ for i = 2; $g(v_i\, v_{i+1}) = n + 3$ for i = 3; $g(v_i\, v_{i+1}) = m + n + 5$ for i = 4; $g(v_i\, v_{i+1}) = m + n + 7$ for i = 5; $g(v_i\, v_{i+4}) = m + n + 9$ for i = 1; $g(v_i\, v_{i+4}) = m + n + 6$ for i = 2; $g(ww_j) = m + j + 4$ for $1 \leq j \leq m$; $g(uu_k) = n - k + 1$ for $1 \leq k \leq n$; $g(wv_i) = n + 4$ for i = 3; $g(uv_j) = n + 2$ for j = 4.

The vertex labels of G can be arranged in the following order.

A = {0,2,4, n +7, m + n + 8, m + n + 9 }, B = { n + 8,…, m + n + 7}, C = {6,7,…, n + 5 }, I = {3, n + 6 }.The set of vertex labels of G is A$\cup$B$\cup$C$\cup$I = {0, 2, 3, 4, 6, … , m + n +9}.

The edge labels of G can be arranged in the following order.

D = { n + 1, n + 3, m + n + 5, m + n + 7, m + n + 8 }, E = { m + n + 6, m + n + 9}, F = {n + 5,…. , m + n + 4}, G = {1, 2,…, n }, H = {n + 2, n + 4 }. The set of edge labels of G is D $\cup$ E $\cup$ F$\cup$ G$\cup$ H = {1,2, 3,…, m + n +9 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta_{34}(2,2,3)$ @ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

Case 7. $\alpha = 4$, $\beta = 5$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i$ for i = 1; $f(v_i) = m + n + i + 7$ for i = 2; $f(v_i) = i - 3$ for i = 3; $f(v_i) = i - 2$ for i = 4, 6; $f(v_i) = m + n + i + 3$ for i = 5; $f(w_j) = j + 5$ for $1 \leq j \leq m$; $f(u_k) = m + k + 7$ for $1 \leq k \leq n$; f(w) = 3; f(u) = 5.

The induced labeling g on the edges of G is defined by $g(v_i\, v_{i+1}) = m + n + 8$ for i = 1; $g(v_i\, v_{i+1}) = m + n + 9$ for i = 2; $g(v_i\, v_{i+1}) = 2$ for i = 3; $g(v_i\, v_{i+1}) = m + n + 6$ for i = 4; $g(v_i\, v_{i+1}) = m + n + 4$ for i = 5; $g(v_i\, v_{i+4}) = m + n + 7$ for i = 1; $g(v_i\, v_{i+4}) = m + n + 5$ for i = 2; $g(ww_j) = j + 2$ for $1 \leq j \leq m$; $g(uu_k) = m + k + 2$ for $1 \leq k \leq n$; $g(wv_i) = 1$ for i = 4; $g(uv_j) = m + n + 3$ for j = 5.

The vertex labels of G can be arranged in the following order.

A = { 0,1,2,4,m + n + 8, m + n + 9}, B = { 6,…, m + 5}, C = { m + 8,…,m + n +7}, I= { 3, 5 }. The set of vertex labels of G is A$\cup$B$\cup$C$\cup$I = {0,1,… , m + 5, m + 8,…,m + n +9}.

The edge labels of G can be arranged in the following order.

D = {2, m +n +4,m + n +6, m +n +8, m +n +9}, E = { m + n + 5, m +n +7}, F = { 3,4,…, m + 2}, G = { m + 3,…, m + n + 2}, H ={1, m + n + 3}. The set of edge labels of G is D $\cup$ E $\cup$ F$\cup$ G$\cup$ H = {1, 2, 3, … , m + n +9 }.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence G = $\theta_{45}(2,2,3)$ @ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

Case 8. $\alpha = 4$, $\beta = 6$.
Let the labeling f on the vertices of G be defined by $f(v_i) = i - 1$ for i = 1, 6; $f(v_i) = m + n + i + 7$ for i = 2; $f(v_i) = i - 2$ for i = 3,5; $f(v_i) = m + n + i + 4$ for i = 4; $f(w_j) = j + 5$ for $1 \leq j \leq m$; $f(u_k) = m + k + 7$ for $1 \leq k \leq n$; f(w) = 2; f(u) = 4.

The induced labeling g on the edges of G is defined by $g(v_i\, v_{i+1}) = m + n + 9$ for i = 1; $g(v_i\, v_{i+1}) = m + n + 8$ for i = 2; $g(v_i\, v_{i+1}) = m + n + 7$ for i = 3; $g(v_i\, v_{i+1}) = m + n + 5$ for i = 4; $g(v_i\, v_{i+1}) = 2$ for i = 5; $g(v_i\, v_{i+4}) = 3$ for i = 1; $g(v_i\, v_{i+4}) = m + n + 4$ for i = 2; $g(ww_j) = j + 3$ for $1 \leq j \leq m$; $g(uu_k) = m + k + 3$ for $1 \leq k \leq n$; $g(wv_i) = m + n + 6$ for i = 4; $g(uv_j) = 1$ for j = 6.

The vertex labels of G can be arranged in the following order.

A = {0,1,3,5, m + n + 8, m + n + 9}, B ={6,7,…, m + 5}, C = {m +8,…,m + n + 7}, I ={2, 4}. The set of vertex labels of G is A ∪ B ∪ C ∪ I = {0,1,…, m + 5, m + 8,…,m + n +9}.

The edge labels of G can be arranged in the following order.

D = {2, m + n + 5, m + n + 7, m + n + 8, m + n + 9 }, E = { 3,m + n + 4}, F ={4,5,…, m+ 3}, G = {m + 4,…, m + n + 3}, H = {1, m + n + 6 }. The set of edge labels of G is D ∪ E ∪ F∪ G∪ H = {1,2,3,…,m + n +9}.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence G = $\theta_{46}(2,2,3)$ @ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

Case 9. $\alpha = 5$, $\beta = 6$.
Let the labeling f on the vertices of G be defined by $f(v_i) = i - 1$ for i = 1; $f(v_i) = m + n + i + 7$ for i = 2; $f(v_i) = i + 1$ for i = 3; $f(v_i) = i - 3$ for i = 4; $f(v_i) = m + n + i + 3$ for i = 5; $f(v_i) = m + n + i +1$ for i = 6; $f(w_j) = j + 5$ for $1 \leq j \leq m$; $f(u_k) = m + k + 6$ for $1 \leq k \leq n$; f(w) = 2; f(u) = 3.

The induced labeling g on the edges of G is defined by $g(v_i\, v_{i+1}) = m + n + 9$ for i = 1; $g(v_i\, v_{i+1}) = m + n + 5$ for i = 2; $g(v_i\, v_{i+1}) = 3$ for i = 3; $g(v_i\, v_{i+1}) = m + n + 7$ for i = 4; $g(v_i\, v_{i+1}) = 1$ for i = 5; $g(v_i\, v_{i+4}) = m + n + 8$ for i = 1; $g(v_i\, v_{i+4}) = 2$ for i = 2; $g(ww_j) = j + 3$ for $1 \leq j \leq m$; $g(uu_k) = m +k +3$ for $1 \leq k \leq n$; $g(wv_i) = m + n + 6$ for i = 5; $g(uv_j) = m + n + 4$ for j = 6.

The vertex labels of G can be arranged in the following order.

A = { 0,1,4, m +n +7, m + n + 8,m + n + 9 }, B = {6,7,…, m + 5}, C = { m + 7,…, m + n + 6}, I = {2, 3 }. The set of vertex labels of G is A∪B∪C∪I = {0,1,2,3,4, 6,… , m + 5, m + 7,… , m + n +9}.

The edge labels of G can be arranged in the following order.

D = {1, 3, m + n + 5, m + n + 7, m + n + 9 }, E = {2, m + n + 8}, F = {4,5,…, m + 3}, G = {m + 4,…,m + n + 3}, H = { m + n + 4, m + n + 6}. The set of edge labels of G is D ∪ E ∪ F∪ G∪ H = {1, 2, 3, … , m + n +9 }.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence G = $\theta_{56}(2,2,3)$ @ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

Definition 2.5. Let $l \geq 1$, $m \geq 1$, $n \geq 0$, be integers. Consider a path on l(odd) vertices $u_1, u_2, \ldots, u_l$. Let $d=(l+1)/2$ .Join the vertices u_1, u_2, u_d to a vertex t by means of the edges $u_1 t$, $u_d t$ and $u_l t$, to form a bicyclic graph. Attach a pendant vertex v_m of a path $v_1, v_2, \ldots, v_m$ to t and rename it as t. Let St(n) be a star graph with center w and pendant vertices $w_1, w_2, \ldots, w_n$. Fix the center w of St(n) with t and rename it as t. The new graph formed is called F(l,m,n) and it has (l+m+n) vertices and (l+m+n+1) edges.

Theorem 2.6. F(l,m,n) is graceful, for $l = 3$, $m \geq 1$ and $n \geq 0$.

Proof. Consider the graph G = F(l,m ,n) with $l = 3$, $m \geq 1$, $n \geq 0$. G has $(m + n + 3)$ vertices and $(m + n + 4)$ edges.

The vertex set $V(G) = \{ u_i, v_j, w_k, t \,/\, i = 1,2,\ldots,l,\ j = 1,2,\ldots,m-1,\ k = 1,2,\ldots,n \}$. The edge set $E(G) = \{tu_i / i=1,d,l\} \cup \{tv_{m-1}\} \cup \{tw_k / k=1,2,\ldots,n\} \cup \{ u_i u_{i+1} / i=1,2,\ldots, l-1\} \cup \{ v_j v_{j+1} / j=1,2,\ldots,m-2\}$

Let f be the labeling on the set of vertices of G and g be the induced labeling on the set of edges of G. The vertex label set of G can be written as $A \cup B \cup C \cup D$ where $A = \{f(u_i) / i=1,2,\ldots,l\}$, $B = \{ f(v_j) / j=1,2,\ldots, m-1\}$, $C = \{ f(w_k) / k=1,2,\ldots,n \}$, $D = \{ f(t)\}$.

The edge label set of G can be written as $F \cup G \cup H \cup I \cup J$ where $F=\{g(tu_i) / i=1,d,l\}$, $G=\{g(tv_{m-1})\}$, $H=\{g(tw_k) / k = 1,2,\ldots,n\}$ }, $I = \{g(u_i u_{i+1}) / i=1,2,\ldots,l-1\}$, $J=\{g(v_j v_{j+1}) / j = 1,2,\ldots,m-2\}$.

Let the labeling f on the vertices of G be defined by $f(u_i) = m + n + i + 2$ for $i = 1$; $f(u_i)= i - 2$ for $i = 2$; $f(u_i)= m + n + i + 1$ for $i = 3$; $f(w_k) = k$ for $1 \leq k \leq n$; $f(t) = m + n + 2$.

If m is odd, $f(v_j) = f(v_{2h+1}) =(2h +m +2n +5)/2$ for $h = 0,1,\ldots,(m-3)/2$; $f(v_j) = f(v_{2h}) =(-2h +m +2n +1)/2$ for $h = 1,2,\ldots,(m-1)/2$;

If m is even, $f(v_j) = f(v_{2h+1}) =(-2h +m +2n)/2$ for $h = 0,1,\ldots,(m-2)/2$; $f(v_j) = f(v_{2h}) =(2h +m +2n +4)/2$ for $h = 1,2,\ldots,(m-2)/2$;

The induced labeling g on the edges of G is defined by $g(tu_i) = i$ for $i = 1$; $g(tu_i) = m + n + i$ for $i = 2$; $g(tu_i) = i - 1$ for $i = 3$; $g(tv_j) = j + 2$ for $j = m - 1$; $g(tw_k) = m + n - k + 2$ for $1 \leq k \leq n$; $g(u_i u_{i+1}) = m + n + 3$ for $i = 1$; $g(u_i u_{i+1}) = m + n + 4$ for $i = 2$; $g(v_j v_{j+1}) = j + 2$ for $1 \leq j \leq m - 2$;

The vertex labels of G can be arranged in the following order.

$A = \{0, m + n + 3, m + n + 4\}$, If m is odd, $B = \{ n + 1,\ldots,(2n + m - 1)/2, (2n + m + 5)/2,\ldots,m + n + 1\}$, $C = \{1,2,\ldots, n\}$, If m is even, $B = \{ n + 1,\ldots,(2n + m)/2, (2n + m + 6)/2,\ldots,m + n + 1\}$, $C = \{1,2,\ldots, n\}$, $D = \{ m+ n + 2 \}$.The set of vertex labels of G is $A \cup B \cup C \cup D = \{0, 1,2, 3, \ldots, (2n + m - 1)/2, (2n + m + 5)/2,\ldots, m + n +3\}$ if m is odd. The set of vertex labels of G is $A \cup B \cup C \cup D = \{0, 1,2, 3, \ldots, (2n + m)/2, (2n + m + 6)/2,\ldots,m + n +3\}$ if m is even.

The edge labels of G can be arranged in the following order.

$F = \{ 1,2, m + n + 2\}$, $G = \{ m + 1\}$, $H = \{m + 2,\ldots,m + n + 1\}$, $I = \{m + n + 3, m + n + 4 \}$, $J = \{3,4,\ldots,m \}$. The set of edge labels of G is $F \cup G \cup H \cup I \cup J = \{1,2, 3,\ldots, m+ n +4 \}$.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence, G = F(l,m ,n) with $l=3$, $m \geq 1$, $n \geq 0$, is a graceful graph.

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