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A Weak Contraction in a *G*-Complete Fuzzy Metric Space

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Abstract. In this work our aim is to establish a weakened version of a contraction mapping principle in a fuzzy metric space with a partial ordering. The weak contraction is primarily considered on specific chains. When it is considered on the whole space it generalizes a contraction mapping theorem. The result is supported with an example.

Keywords: Fixed point, G-completeness, fuzzy metric space

AMS Mathematics Subject Classification (2010): 54H25

1. Introduction

In this paper, we consider the following definition of a fuzzy metric space given by George and Veeramani [4].

Definition 1.1. (George and Veeramani [4]) The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non- empty set, $*$ is a continuous $t-$norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t+s)$.

where $*:[0,1]^2 \to [0,1]$ is a continuous $t-$ norm if the following properties are satisfied:

(i) $*$ is associative and commutative. (ii) $a*1 = a$ for all $a \in [0,1]$,
(iii) $a*b \le c*d$ whenever $a \le c$ and $b \le d$, for all $a, b, c, d \in [0,1]$.

Definition 1.2. (George and Veeramani [4]) Let $(X, M, *)$ be a fuzzy metric space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ iff $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(ii)A sequence $\{x_n\}$ in X is a $G-$Cauchy sequence if $M(x_n, x_{n+p}, t) \to 1$ as $n \to \infty$, for all $t > 0$ and positive integer p.

(iii) X is G-complete if every G-Cauchy sequence is convergent.

Lemma 1.3. [8] If for a sequence $\{x_n\}$, $M(x_n, x_{n+1}, t) \to 1$ as $n \to \infty$, then $\{x_n\}$ in X is a convergent sequence.

Lemma 1.4. [7] M is continuous in all the variables and monotone in the third variable.

A mapping $f : X \to X$. Let $(X, \leq)$ be a partially ordered set is said to be monotone increasing if for any $x_1, x_2 \in X$, $x_1 \leq x_2$ implies $fx_1 \leq fx_2$.

Orbit of a function $f : X \to X$ at the point $x \in X$ is the set $\{x, fx, f^2x, f^3x, \ldots\ldots, f^nx, \ldots\ldots\}$ where f^nx is the $n-$th iterate of f.

Definition 1.5. Let $(X, M, *)$ be a complete fuzzy metric space. Let C be a subset of X. Let $f : C \to C$ be a self mapping which satisfies the following inequality :

$$\psi(M(fx, fy, t)) \leq \psi(M(x, y, t)) - \emptyset(M(x, y, t)) \tag{1.1}$$

where $x, y \in X, t > 0$ and $\psi, \phi : (0,1] \to [0, \infty)$ are two functions such that,

$(i)\psi$ is continuous and monotone decreasing with $\psi(t) = 0$ if and only if $t = 1$

$(ii)\phi$ is continuous with $\phi(s) = 0$ if and only if $s = 1$

Then f is called a weak contraction on C.

Weak contractions are intermediate to contractions and nonexpansive mappings. The purpose here is to extend the fuzzy contraction principle by proving the same for weak contractions. Several works on weak contractions are in [1,2,3,6] .

2. Main Results

Theorem 2.1. Let $(X, \leq)$ be a partially ordered set and $(X, M, *)$ be a G-complete fuzzy metric space which has the property that whenever $\{x_n\}$ is a monotone increasing sequence in X converging to a point z, it follows that $x_n \leq z$ for all n. Let $f : X \to X$ be a mapping with the monotone property on X. Let there exist $x_0 \in X$ such that $x_0 \leq fx_0$. Then the orbit $O(x_0, f)$ is a chain with x_0 as the least element. If f is weakly contractive on every chain containing $O(x_0, f)$ with x_0 as the least element, then f has a fixed point.

Proof: Let $x_0 \in X$ and $\{x_n\}$ be the sequence defined by $x_{n+1} = fx_n = f^nx_0$, $n \geq 1$.

Since f has monotone property on X, and the fact that $x_0 \leq fx_0 = x_1$, we have that

$$x_0 \leq x_1 \leq x_2 \ldots \ldots . \leq x_n \leq x_{n+1} \leq \cdots \tag{2.1}$$

Then $O(f, x_0)$ is a chain.

If $x_{n+1} = x_n$, then f has a fixed point. So we assume

$$x_{n+1} \neq x_n \qquad \text{for all } n \geq 0 \tag{2.2}$$

Putting $x = x_{n-1}$ and $y = x_n$ in (1.1) we obtain, for all $n \geq 1, t > 0$,

$$\psi(M(fx_{n-1}, fx_n, t)) \leq \psi(M(x_{n-1}, x_n, t)) - \phi(M(x_{n-1}, x_n, t))$$

that is,

$$\psi(M(x_n, x_{n+1}, t) \leq \psi(M(x_{n-1}, x_n, t) - \phi(M(x_{n-1}, x_n, t) \tag{2.3}$$

From the above inequality, for all $n \geq 1, t > 0, \psi(M(x_n, x_{n+1}, t) \leq \psi(M(x_{n-1}, x_n, t)$ which, by the monotone decreasing property of ψ, implies that for all $n \geq 1, t > 0$,

$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t)$, that is, $\{M(x_n, x_{n+1}, t)\}$ is a monotone increasing sequence in X. This sequence being bounded by 1, for every choice $t > 0$, there exists $a(t)$ such that

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = a(t) \leq 1 \tag{2.4}$$

Now taking $n \to \infty$ in (2.3) we obtain, for all $t > 0, \psi(a(t)) \leq \psi(a(t)) - \phi(a(t))$, which is a contradiction unless $a(t) = 1$, for all $t > 0$.

Thus we conclude that for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1 \tag{2.5}$$

Then by lemma 1.3 it is established that $\{x_n\}$ is a Cauchy sequence and hence it is convergent in the complete fuzzy metric space X to a point z, that is,

$$\lim_{n\to\infty} x_n = z \in X \tag{2.6}$$

Next we show that z is fixed point of f. By our construction, $\{x_n\}$ is a monotone increasing sequence and hence, by our assumption, $x_n \leq z$ for all $n \geq 0$.

Then it follows that $\{x_n : n \geq 0\} \cup \{z\}$ is a chain containing $O(f, x_0)$ with x_0 as the least element, by a condition of the theorem, f is a weak contraction on this chain.

Then for all $n \geq 1, t > 0$,

$$\psi(M(x_n, fz, t)) = \psi(M(fx_{n-1}, fz, t)) \leq \psi(M(x_{n-1}, z, t)) - \phi(M(x_{n-1}, z, t))$$

Letting $n \to \infty$ in the above inequality, using the properties of ψ, ϕ and (2.6) we obtain

$$\psi(\lim_{n\to\infty} M(x_n, fz, t)) \leq \psi(1) - \phi(1) = 0, \tag{2.7}$$

that is, by a property of ψ,

$$\lim_{n\to\infty} M(x_n, fz, t) = 1 \quad \text{for all } t > 0, \tag{2.8}$$

which implies that $\qquad x_n \to fz \quad as \quad n \to \infty. \qquad$ (2.9)

Since the topology in the fuzzy metric space is a Hausdorff topology, we conclude from (2.6) and (2.9) that $z = fz$.

Corollary 2.1. (Fuzzy Banach Contraction mapping Theorem) [5]
Let $f : X \to X$, where $(X, M, *)$ is a G-complete fuzzy metric space, be such that

$$\left(\frac{1}{M(fx, fy, t)} - 1\right) \leq k\left(\frac{1}{M(x, y, t)} - 1\right) \tag{2.10}$$

where $x, y \in X, t > 0$ and $0 < k < 1$.

Then T has a fixed point.

Proof: Let $\psi(s) = \frac{1-s}{s}$ and $\phi(s) = \frac{(1-k)(1-s)}{s}$ where $0 < s \leq 1$. Then we see that (2.10) implies (1.1) for all $x, y \in X$ and $t > 0$ and with the above choices of ψ and ϕ.

The corollary then follows by an application of the theorem.

3. Conclusion:

In this paper, the concept of weak contraction in fuzzy metric spaces is introduced. It is proved that these mappings have fixed points if the space is G- complete. There are several scopes of its possible extentions which can be explored. One such problem is to consider the problem in complete fuzzy metric spaces rather than in G- complete spaces. This problem is purported to be taken up in one of our future works.

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