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Analysis of Numerical Methods for Differential-Algebraic Equations: The one Step Methods

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Abstract. Differential algebraic equations have wide applications in the field of engineering and science where the mathematical models form the descriptor systems. Analysis and modeling of the solutions of such systems need to handle with the different equations related to the systems. A time-domain discretization, for example, finite difference, finite volume, etc., may lead to DAEs of descriptor forms. Numerical approaches for solving those differential-algebraic equations (DAEs) can be divided into two classes: direct discretizations and reformulation method (e.g., index reduction), combined with discretization. There are some limitations of direct discritization method. In this paper, the numerical methods have been discussed for strangeness-free problems with example.

***Keywords*:** Differential algebraic equations (DAEs), discretization, strangeness-free DAEs, convergent, index reduction, approximate solution, restrictive structures.

AMS Mathematics Subject Classification (2010): 65L80

1. Introduction

Numerical approaches for the solution of differential-algebraic equations (DAEs) can be divided roughly into two classes: (i) direct discretizations of the given system and (ii) methods which involve a reformulation (e.g., index reduction), combined with discretization. The advantage for a direct discretizations method is that it is relatively cheaper than the reformulation method, it usually requires less input from the user, and it involves less user intervention compared to the reformulation method. But the problem with the direct discretizations methods is that, they are essentially limited in their utility to index-1(strangeness-free) and semi-explicit index-2 DAE systems [4] and [6]. Fortunately, most DAEs encountered in practical applications either are index-1(strangeness-free) or, if higher, can be expressed as a combination of more restrictive structures of DAEs coupled with constraints and can be solved easily [4,6]. In this paper we will discuss the numerical methods for strangeness-free problems.

2. Methodology

Let us consider the numerical solution of initial value problems of differential algebraic system [4] of the form

$$F(t,x,\dot{x})=0,\ x(t_0)=x_0 \tag{1}$$

in the interval $I=[t_0,T]\subset R$. We denote by $t_0<t_1<t_2\ldots<t_N=T$ grid points in the interval I and by x_i approximations to the solution $x(t_i)$.We concentrate on a fixed step size, i.e., we use $t_i=t_0+ih,\ i=0,1,\ldots,N$ and $T-t_0=Nh$.We here note that this N is not the Nilpotent part of Weierstra ß canonical form [5]. We can distinguish the two meanings of N from the context.

A *discretization method* for the solution of (1) is given by the iteration

$$\aleph_{i+1}=\xi(t_i,\aleph_i;h) \tag{2}$$

where they $\aleph_i$ are the elements in some R^n, together with $\aleph(t_i)\in R^n$ representing the actual solution at t_i.We are interested in the conditions that will make the methods convergence in the sense that $\aleph_N$ tends to $\aleph(t_N)$ when h tends to zero.

Definition 2.1. The discretization method (2) is said to be *consistent of order p* if

$$\|\aleph(t_{i+1})-\xi(t_i,\aleph(t_i);h)\|\le Ch^{p+1} \tag{3}$$

with a constant C independent of h.

Definition 2.2. The discretization method (2) is said to be *stable* if there exist a vector norm $\|.\|$ such that

$$\|\xi(t_i,\aleph(t_i);h)-\xi(t_i,\aleph_i;h)\|\le(1+hk)\|\aleph(t_i)-\aleph_i\| \tag{4}$$

In this vector norm, with a constant K independent of h.

Definition 2.3. The discretization method (2) is said to be *convergent of order p* if

$$\|\aleph(t_N)-\aleph_N\|\le Ch^p \tag{5}$$

with a constant C independent of h, provided that

$$\|\aleph(t_0)-\aleph_0\|\le\tilde{C}h^p \tag{6}$$

with a constant $\widetilde{C}$ independent of h.

Theorem 2.4. If the discretization method (2) is stable and consistent of order p, it is convergent of order p.

Proof. We have

$$\begin{aligned}\|\aleph(t_{i+1})-\aleph_{i+1}\|&=\|\aleph(t_{i+1})-\xi(t_i,\aleph(t_i);h)+\xi(t_i,\aleph(t_i);h)-\aleph_{i+1}\|\\&\le\|\aleph(t_{i+1})-\xi(t_i,\aleph(t_i);h)\|+\|\xi(t_i,\aleph(t_i);h)-\aleph_{i+1}\|\\&\le Ch^{p+1}+(1+hk)\|\aleph(t_i)-\aleph_i\|.\end{aligned}$$

It follows that

$$\|\aleph(t_N)-\aleph_N\|\le Ch^{p+1}+(1+hk)\|\aleph(t_{N-1})-\aleph_{N-1}\|$$

$$\leq C h^{p+1} + (1+hk)(C h^{p+1} + (1+hk)\| \aleph(t_{N-2}) - \aleph_{N-2} \|$$

$$\leq C h^{p+1} + (1 + (1+hk) + (1+hk)^2 + \ldots + (1+hk)^{N-1}) +$$
$$(1+hk)^N \| \aleph(t_0) - \aleph_0 \|$$

$$\leq C h^{p+1} \frac{(1+hk)^N - 1}{(1+hk) - 1} + (1+hk)^N \tilde{C} h^p$$

$$\leq h^p \left(\frac{C}{k} \right) (1+hk)^N - h^p \left(\frac{C}{k} \right) + (1+hk)^N \tilde{C} h^p$$

$$= h^p \left(\frac{C}{k} \right) \left(1 + N.(hk) + \frac{N(N-1)}{2!} (hk)^2 + \ldots + (hk)^N \right)$$
$$- h^p \left(\frac{C}{k} \right) + (1+hk)^N \tilde{C} h^p$$

$$\leq \left(\frac{C}{k} + \tilde{C} \right) \exp(Nhk) h^p \text{ (since, } (T - t_0) = Nh)$$

Therefore

$$\| \aleph(t_N) - \aleph_N \| = \left(\frac{C}{k} + \tilde{C} \right) \exp(k(T - t_0)) h^p . \qquad \square$$

3.The Kronecker product

When we study Runge-Kutta methods, it is convenient to describe the structure of matrices via the Kronecker product of two matrices. The Kronecker product of $R = [r_{ij}] \in \mathbf{C}^{k,l}$ with $S \in \mathbf{C}^{m,n}$ is defined as the block matrix $R \otimes S = [r_{ij} S] \in \mathbf{C}^{km,\ln}$. Its main properties are given by the following lemma.

Lemma 3.1. The Kronecker product has the following properties:
i. Let matrices *U*, *V* and *R*, *S* such that the product *UR*, *VS* exist. Then,

$$(U \otimes V)(R \otimes S) = UR \otimes VS \tag{7}$$

ii. Let $R \in \mathbf{C}^{k,l}$ and $S \in \mathbf{C}^{m,n}$. Considering $R \otimes S$ as a block matrix consisting of blocks of size $m \times n$, we define so-called perfect shuffle matrices $\prod_1$ and $\prod_2$ with respect to the rows and columns, respectively by the following process:
Take the first row/column of the first block, then the first row/column of the second block, and so on until the first row/column of the last block; continue in the same way with the second row/column of every block, until the last row/column of every block. With the so obtained permutation matrices $\prod_1$ and $\prod_2$, we get

$$\prod_1^T (R \otimes S) \prod_2 = S \otimes R \tag{8}$$

If $k = l$ and $m = n$, then $\prod_1 = \prod_2$ and the $S \otimes R$ is similar to $R \otimes S$.

4. Solution methods

4.1. One-step methods

A one-step method for the computation of numerical approximation x_i to the values $x(t_i)$ of a solution of x of an ordinary differential equation [1] $\dot{x} = f(t,x)$ has the form

$$x_{i+1} = x_i + h\phi(t_i, x_i; h) \tag{9}$$

where ϕ is called the *increment function.* In the context of ordinary differential equations, a one-step method is called *consistent* of order p, if under the assumption that $x_i = x(t_i)$, the local *descritization error* $x_{i+1} - x(t_{i+1})$ satisfies

$$\| x(t_{i+1}) - x_{i+1} \| \le Ch^{p+1} \tag{10}$$

with a constant C that is independent of h. Using (9), (10) is equivalent to

$$\| x(t_{i+1}) - x(t_i) - h\phi(t_i, x(t_i); h) \| \le Ch^{p+1} \tag{11}$$

Now we set $\aleph_i = x_i$, $\aleph(t_i) = x(t_i)$, and $\xi(t_i, \aleph(t_i); h) = x_i + h\varphi(t_i, x_i; h)$, the one-step method (9) can be arranged as a general discretization method.

Since

$$\| \xi(t_i, \aleph(t_i); h) - \xi(t_i, \aleph_i; h) \| = \| (\aleph(t_i) + h\phi(t_i, \aleph(t_i); h)) - (\aleph_i + h\phi(t_i, \aleph_i; h)) \|$$
$$\le (1 + hk) \| \aleph(t_i) - \aleph_i \|$$

where k is the Lipschitz condition of ϕ with respect to its second argument, these methods are stable. Hence consistency implies convergence for one-step method.

4.2. Runge-Kutta methods for DAEs.

As we mention in our introduction, we will discuss the one-step methods using Runge-Kutta methods [3]**.** The general form of an s-stage Runge-Kutta method for the solution of $\dot{x} = f(t,x)$, $x(t_0) = x_0$ is given by

$$x_{i+1} = x_i + h\sum_{j=1}^{s} \gamma_j \dot{X}_{i,j} \tag{12}$$

where
$$\dot{X}_{i,j} = f(t_i + \alpha_j h, X_{i,j}), \qquad j = 1, 2, \ldots, s, \tag{13}$$

and so-called internal stages $X_{i,j}$ are given by

$$X_{i,j} = x_i + h\sum_{l=1}^{s} \beta_{jl} \dot{X}_{i,l}, \qquad j = 1, 2, \ldots, s. \tag{14}$$

The coefficients $\alpha_{j,}$ β_{jl} and γ_j determine the particular method and are conveniently displayed in a *Butcher tableau*

$$\begin{array}{c|c} A & B \\ \hline & C^{\mathrm{T}} \end{array}$$

with $B = \lfloor \beta_{jl} \rfloor$, $A = \lfloor \alpha_j \rfloor$, and $C = \lfloor \gamma_j \rfloor$. The coefficients are assumed to be satisfy the condition

$$\gamma_j = \sum_{l=1}^{s} \beta_{jl}, \quad j = 1, 2, \ldots, s, \tag{16}$$

which implies that the RK method is consistence and invariant regarding autonomization.

Theorem 4.3. If the coefficients $\alpha_{j,}$ β_{jl} and γ_j of the RK method given by (12), (13), and (14) satisfy the conditions

$$\begin{aligned} B(p)&: \ \sum_{j=1}^{s} \gamma_j \alpha_j^{k-1} = \frac{1}{k}, \ k = 1, \ldots, p, \\ C(q)&: \ \sum_{l=1}^{s} \beta_{jl} \alpha_l^{k-1} = \frac{1}{k} \alpha_j^{k}, \qquad j = 1, \ldots, s, \quad k = 1, \ldots, q \\ D(r)&: \ \sum_{j=1}^{s} \gamma_j \alpha_j^{k-1} \beta_{jl} = \frac{1}{k} \gamma_l \left(1 - \alpha_l^{k}\right), \quad l = 1, \ldots, s, \quad k = 1, \ldots, r, \end{aligned} \tag{17}$$

with $p \le q + r + 1$ and $p \le 2q + 2$, then the method is *consistent* and hence *convergent of order p.*

Now using the RK methods we can generalize differential-algebraic equations of the form (1) by defining x_{i+1} as the solution of (12) and (14) together with

$$F\left(t_i + \alpha_j h, X_{i,j}, \dot{X}_{i,j}\right) = 0, \ j = 1, \ldots, s. \tag{18}$$

Of course, the above relations only define a method if we can show that they define a unique x_{i+1} with respect to x_i at least for sufficiently small step size h. If this happens, then we can analyze the convergence properties of the resulting methods. We will do this by considering these methods as general discretization methods.

5. The general discretization methods on DAEs.

Let us consider the linear differential-algebraic equations with constant coefficients

$$E\dot{x} = Ax + f(t), \quad x(t_0) = x_0.$$

In this case, the RK method has the form (12), with $\dot{X}_{i,l}$ obtained using the solution of the linear system

$$E\dot{X}_{i,j} = A X_{i,j} + f\left(t_i + \alpha_j h\right).$$

Together with equation (14) it becomes

$$E\,\dot{X}_{i,j} = A\left(x_i + h\sum_{l=1}^{s}\beta_{jl}\dot{X}_{i,l}\right) + f\left(t_i + \alpha_j h\right),\ j = 1,\ldots,s.$$

$$\text{Or, } \left(E - h\sum_{l=1}^{s}\beta_{jl}\,\dot{X}_{i,l}\,A\right)\dot{X}_{i,j} = Ax_i + f\left(t_i + \alpha_j h\right)$$

$$\text{or, } \begin{bmatrix} E-h\beta_{1,1}A & -h\beta_{1,2}A & \ldots & \ldots & -h\beta_{1,s}A \\ -h\beta_{2,1}A & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & -h\beta_{s-1,s}A \\ -h\beta_{s,1}A & \ldots & \ldots & -h\beta_{s,s-1}A & E-h\beta_{s,s}A \end{bmatrix} \dot{X}_i = Z_i \qquad (19)$$

where

$$\dot{X}_i = \begin{bmatrix} \dot{X}_{i,1} \\ \dot{X}_{i,2} \\ \ldots \\ \ldots \\ \dot{X}_{i,s} \end{bmatrix} \quad \text{and } Z_i = \begin{bmatrix} Ax_i + f\left(t_i + \alpha_1 h\right) \\ Ax_i + f\left(t_i + \alpha_2 h\right) \\ \ldots \\ \ldots \\ Ax_i + f\left(t_i + \alpha_s h\right) \end{bmatrix}.$$

Using the Kronecker product as introduced at the beginning of the discussion, we can rewrite (19) as

$$\left(I_s \otimes E - hB \otimes A\right)\dot{X}_i = Z_i\,. \qquad (20)$$

For non square coefficient matrix E, A this system is not uniquely solvable for arbitrary right hand sides. But even in the square cases, if the pair (*E, A*) is not regular, the coefficient matrix in (20) is singular. Therefore, the Pair (*E, A*) should be regular for a well-defined method. Since for (non singular) matrices P, Q of appropriate dimensions

$$\left(I_s \otimes P\right)\left(I_s \otimes E - hB \otimes A\right)\left(I_s \otimes Q\right) = \left(I_s \otimes PEQ - hB \otimes PAQ\right),$$

we may assume for the analysis that the pair (E , A) is Weierstra β canonical form

$$\left(E, A\right) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right),$$

with *J, N* in Jordan canonical form and *N* is nilpotent. We can now separate the system into two subsystems, one a differential part and another is an algebraic part. The RK methods are well studied for ordinary differential equations. For this reason we will consider the algebraic part, i.e., the nilpotent part of the form

$$N\,\dot{x} = x + f(t)\,. \qquad (21)$$

Furthermore, since *J* is in Jordan canonical form, the system again can be separated. Thus, for the analysis, we may assume that N in (21) consist of a single nilpotent Jordan

block of size ν. We know that the solution of $x = -\sum_{j=0}^{v-1} N^j f^{(j)}$, independent of any initial values.

In this case the linear system (20) has the form

$$\left(I_s \otimes N - hB \otimes I_v\right)\dot{X}_i = Z_i . \tag{22}$$

Using the perfect shuffle matrix Π of Lemma 1, we find that

$$\Pi^T \left(I_s \otimes N - hB \otimes I_v\right)\Pi = N \otimes I_s - I_v \otimes hB = \begin{bmatrix} -hB & I_s & \dots & \\ \dots & & & \\ \dots & \dots & & I_s \\ \dots & \dots & & -hB \end{bmatrix}. \tag{23}$$

Hence, to obtain a reasonable method, it is necessary that *B* is nonsingular, which implies that we are restricted to *implicit Runge-Kutta methods*. We now have the following theorem for the local order condition.

Theorem 5.1. Consider the differential-algebraic equation (21) with ν = ind(*N*, *I*). Apply a RK method with coefficients *A*, *B*, and *C*, and assume that *B* is invertible. If $k_j \in N$, $j = 1, 2, \dots, v,$ exist such that

$$\begin{aligned} &\gamma^T B^{-k} e = \gamma^T B^{-j} \alpha^{j-k} / (j-k)!, && k = 1, 2, \dots, j-1, \\ &\gamma^T B^{-j} \alpha^k = k! / (k-j+1)!, && k = j, j+1, \dots, k_j, \end{aligned} \tag{24}$$

where $e = [1, \dots, 1]^T$ of appropriate *size* and $\alpha^j = \left[\alpha_1^j, \dots, \alpha_s^j\right]$, then the *local error satisfies*

$$x\left(t_{i+1}\right) - x_{i+1} = O\left(h^{k_v - v + 2}\right) + O\left(h^{k_{v-1} - v + 3}\right) + \dots + O\left(h^{k_1 + 1}\right) \tag{25}$$

An important class of Runge-Kutta methods for differential-algebraic equations is so-called *stiffly accurate* [2] *Runge-Kutta* methods. These are defined to satisfy $\gamma_j = \beta_{sj}$ for all $j = 1, \dots, s$. Writing this as $\gamma^T = e_s^T B$ with $e_s^T = \left[0 \dots 0 \quad 1\right]^T$ of appropriate size, we then obtain from (16) that $\alpha_s = e_s^T B e = \lambda^T e.$ Since $\lambda^T e = 1$ for consistent Runge-Kutta methods, it follows that $\alpha_s = 1$. Moreover, we get that $\gamma^T B^{-1} e = e_s^T \; e = 1$, implying that k_1 in (24) is infinite. Hence, the methods show that the regular linear differential-algebraic equations with constants coefficients and the ordinary differential equation have the same order of consistency.

Theorem 5.2. Consider a Runge-Kutta method consisting of (12), (14), and (19) with invertible B applied to a linear differential-algebraic equation with constant coefficient

[4] of the form $E\dot{x} = Ax + f(t), \quad x(t_0) = x_0$ with a regular pair (E, A) and $v = ind(E, A)$. Furthermore, let, $k_j \geq j, \quad j = 1, \ldots, v$, according to Theorem 3, and let

$$\left|1 - \gamma^T B^{-1} e\right| < 1. \tag{26}$$

Then the Runge-Kutta method is convergent of order

$$\min_{1 \leq j \leq v} \{p, k_j - v + 2\}, \tag{27}$$

where p is the order of the method when applied to ordinary differential equations.

Theorem 5.2. Shows that, even if the RK method satisfies (26) it is not stable in the sense of definition of *direct discretization method* for higher index problems, i.e. for problems with $v > 1$. A related effect is that the RK method may lead to an unstable recursion for the numerical solutions x_j. We will make this fact easier with the following example.

6. Example

Consider the linear differential-algebraic equation

$$\begin{bmatrix} 0 & 0 \\ 1 & \eta t \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -\eta t \\ 0 & -(1+\eta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{28}$$

with a parameter $\eta \in \Re$.

We solve the problem performing a change of basis via

$$x = T\tilde{x} \tag{29}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; T = \begin{bmatrix} -\eta t & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}, \text{ for } \eta \in \Re.$$

Now from (29), we get

$$\dot{x} = \dot{T}\,\tilde{x} + T\,\dot{\tilde{x}}$$

Therefore, the differential-algebraic equation $E\dot{x} = Ax + f(t)$ has the form

$$E\left(\dot{T}\tilde{x} + T\dot{\tilde{x}}\right) = A\left(T\tilde{x}\right) + f(t)$$

$$\text{Or, } \left(ET\right)\dot{\tilde{x}} = \left(AT - E\dot{T}\right)\tilde{x} + f(t) \tag{30}$$

Here, we have

$$E = \begin{bmatrix} 0 & 0 \\ 1 & \eta t \end{bmatrix}, \quad \dot{T} = \begin{bmatrix} -\eta & 0 \\ 0 & 0 \end{bmatrix}$$

A simple computation shows that

$$ET = \begin{bmatrix} 0 & 0 \\ 1 & \eta t \end{bmatrix} \cdot \begin{bmatrix} -\eta t & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E\dot{T} = \begin{bmatrix} 0 & 0 \\ 1 & \eta t \end{bmatrix} \cdot \begin{bmatrix} -\eta & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\eta & 0 \end{bmatrix}, \text{ and}$$

$$AT = \begin{bmatrix} -1 & -\eta t \\ 0 & -(1+\eta) \end{bmatrix} \cdot \begin{bmatrix} -\eta t & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -(1+\eta) & 0 \end{bmatrix}$$

Hence,

$$AT - E\dot{T} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Substituting these values in (30), we get an equivalent constant coefficient system

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}. \tag{31}$$

Obviously, (31) and thus (28) has the strangeness index $\mu = 1$ independent of η. Now, from (31) we get

$$\begin{cases} 0 = -\tilde{x}_2 + f_1(t) \\ \dot{\tilde{x}}_2 = -\tilde{x}_1 + f_2(t) \end{cases} \text{ Or, } \begin{cases} \tilde{x}_2 = f_1(t) \\ \tilde{x}_1 = f_2(t) - \dot{\tilde{x}}_2 = f_2(t) - \dot{f}_1(t) \end{cases} \tag{32}$$

Therefore, the system (28) together with (29) and (32) gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\eta t & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_2(t) - \dot{f}_1(t) \\ f_1(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$

Solving the above system, we have

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f_1(t) - \eta t \left(f_2(t) - \dot{f}_1(t) \right) \\ f_2(t) - \dot{f}_1(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{33}$$

without specifying any initial values

Now we will apply the Implicit Euler method on system (28).The system (28) can be written as

$$0 = x_1(t) - \eta t\, x_2(t) + f_1(t),$$
$$\dot{x}_1(t) + \eta t\, \dot{x}_2(t) = -(1+\eta) x_2(t) + f_2(t). \tag{34}$$

Applying the implicit Euler method gives

$$0 = x_{1,i+1} - \eta t_{i+1}\, x_{2,i+1} + f_1(t_{i+1}),$$
$$\frac{1}{h}(x_{1,i+1} - x_{1,i}) + \eta t_{i+1} \frac{1}{h}(x_{2,i+1} - x_{2,i}) = -(1+\eta) x_{2,i+1} + f_2(t_{i+1}).$$

We solve the first equation in two consecutive steps $x_{1,i}$ and $x_{1,i+1}$ as follows

$$\eta t_{i+1}\, x_{2,i+1} = -x_{1,i+1} + f_1(t_{i+1}) \text{ , and } x_{1,i} = -\eta t_i\, x_{2,i} + f_1(t_i)$$

Inserting these into the second equation and simplifying, we obtain

$$(1+\eta)x_{2,i+1} = \eta x_{2,i} + f_2(t_{i+1}) - \frac{1}{h} f_1(t_{i+1}) + \frac{1}{h} f_1(t_i); \text{ (Since, } t_{i+1} - t_i = h)$$

Therefore, the recursion for $x_{2,i}$ becomes

$$x_{2,i+1} = \frac{\eta}{(1+\eta)} x_{2,i} + \frac{1}{(1+\eta)}\left(f_2(t_{i+1}) - \frac{1}{h}(f_1(t_{i+1}) - f_1(t_i)) \right),$$

which is obviously divergent for $\eta < -\frac{1}{2}$. This finally shows that the RK method may determine a unique numerical solution.

7. Conclusion

In this paper, the numerical methods have been reviewed for solving strangeness-free DAE problems. These problems arise in many areas of science and engineering, particularly in the areas where the periodic control is deserved, Electrical circuits and networks, aerospace realm, control of industrial processes and communication systems, modeling of filters and electrical networks, and many more [2, 4] and [7] . Specially, we discuss the Runge-Kutta methods for DAEs. The method is based on discretization of DAEs in the time domain. There are other popular methods exist to solve such DAEs [2, 3, 4]. They remain as our future tasks to explore.

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