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Schur-Complement in Block Matrices Over an Incline

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Abstract. Inclines are additively idempotent semirings in which product is always less than or equal to each factor. For a pair of regular elements in an incline, we have constructed a regular 2×2 order matrix over that incline such that its Schur-complement is also an element in that incline. As an application it is shown that the sums, parallel sums and product of regular elements are regular in a commutative incline.

Keywords: Incline, Regular Incline, Incline Matrices

AMS Mathematics Subject Classifications (2010): 15A09, 16Y60

1. Introduction

The concept of incline was introduced by Cao. The notion of inclines and their applications are described comprehensively in Cao, Kim and Roush [1]. Inclines are the generalization of Boolean algebra, fuzzy algebra, and distributive lattice. Boolean matrices, fuzzy matrices and lattice matrices are the prototypical examples of the incline matrices. All applications of Boolean algebra, fuzzy algebra to automata theory, design of switching circuits, logic of binary relation, medical diagnosis, Markov chains, information system and clustering are instances in which inclines can be applied. Besides, inclines are applied to nervous system, probable reasoning, finite state machine, psychological measurement, dynamical programming and decision theory.

Kim and Roush have surveyed and outlined the algebraic properties of incline and of matrices over an incline [6]. A sub-set A of R is said to be a sub incline if A is closed under addition and multiplication. An incline R is said to be a regular incline, if each element of R is regular, that is ,for each a in R, the equation axa=a has a solution, and such a solution is called a generalized inverse (g-inverse) of a. In our previous paper [8], we have established that an element a in R is regular if and only if a is idempotent. In [4], Hartwig has studied on existence and construction of various g-inverses associated with an element in a *-regular ring and has also presented some methods to compute the g-inverses of 2×2 matrices over a ring. In [7], Meenakshi has discussed the regularity for block fuzzy matrices by extending the concept of Schur-complements for fuzzy matrices.

The concept of parallel sum for matrices originated by Rao and Mitra in [9] was later developed for elements in a ring by Hartwig and Shoaf in [5] and they have also discussed about the invariance triplets $a\,(a^n)^-a$ and $a\,(a+b)^-b$ for elements a, b in a prime regular ring.

The Schur-complement of a 2×2 matrix over incline is an element in that incline and this motivates the present study. In this paper, some results on Schur-complement in a 2×2 matrix over an incline and invariant conditions are discussed. In particular, certain conditions are determined for the invariance of $ca^{-}b$, for all g-inverses of a.

2. Preliminaries

In this section, we shall present some definitions, and required results.

Definition 2.1. An incline is a non-empty set R with binary operations addition and multiplication denoted as +, • (we usually suppress the dot in 'a•b' and write ab) defined on R•R → R such that for all x, y, z∈R,

$x+y = y+x$, $x+(y+z) = (x+y)+z$, $x(y+z) = xy+xz$, $(y+z)x = yx+zx$,
$x(y z) = (xy) z$, $x+x = x$, $x+xy = x$, $y+xy = y$.

An incline R is said to be commutative if xy = yx for all x, y in R.

(R, ≤) is an incline with order relation "≤" defined on R such that for x, y∈ R, x ≤ y if and only if x + y = y. If x ≤ y, then y is said to dominate x.

Throughout let 'R' denotes an incline with order relation '≤'. Let R^n denotes the n×n matrices over the incline R.

Property 2.2. For x, y in an incline R, $x + y \geq x$ and $x + y \geq y$.

Property 2.3. For x, $y \in$R, $xy \leq x$ and $xy \leq y$.

Lemma 2.4. [8] Let $a \in$R be regular. Then $a = ax = xa$ for all $x \in a\{1\}$,the set of g-inverses of a.

Lemma 2.5. [8] For $a \in$R, a is regular if and only if a is idempotent.

Theorem 2.6. [8] Let a, b, $c \in$R and a is regular, then the following hold:

(i) $b = ba \Rightarrow Rb \subseteq Ra$

(ii) $c = ac \Rightarrow cR \subseteq aR$.

Let $A = (a_{ik}) \in R^n$ and $B = (b_{ik}) \in R^n$, the product $AB = C = (c_{ik}) \in R^n$ is defined as $c_{ik} = \sum_j a_{ij}\, b_{jk}$ for any i, k $(1 \leq i, k \leq n)$.

For any $A = (a_{ik}) \in R^n$ and $B = (b_{ik}) \in R^n$, $A \leq B$ if $a_{ik} \leq b_{ik}$ for all i, j. It follows that if $A \leq B$ then $AC \leq BC$ and $CA \leq CB$ for all $C \in R^n$.

Definition 2.7. $A \in R^n$ is said to be regular if there exists a matrix $X \in R^n$ such that $AXA = A$. Then X is called a generalized inverse, in short g-inverse (or) 1-inverse of A and is denoted as A^{-}. Let A{1} denotes the set of all 1-inverses of A.

3. Main Results

Definition 3.1. For a block matrix over an incline of the form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ if A is regular then the Schur-complement of A in M denoted as M/A is defined as M/A = D – $CA^{-}B$ and this depends on A^{-}.

Therefore M/A is well defined if and only if $CA^{-}B$ is invariant for all choice of g-inverses of A. By M/A is an incline matrix we mean that $CA^{-}B$ is invariant and $D \geq CA^{-}B$ (or) equivalently $D + CA^{-}B = D$.

In [7], regularity of a block fuzzy matrix M in terms of A and M/A are obtained analogous to that for complex matrices. Here we discuss the invariance of the product $ca^{-}b$,for all choice of g-inverses of a, and the existence of Schur complements for a 2x2 incline matrix, whose diagonal elements are regular.

Definition 3.2. A pair of elements a, b $\in$R are said to be parallel summable, if (a+b) is regular and $a(a+b)^{-}b$ is invariant for all choice of $(a+b)^{-} \in (a+b)\{1\}$ and is denoted as a:b.

Lemma 3.3. Let R be a regular incline and a, b, c $\in$ R. If $Rc \subseteq Ra$ (or) $bR \subseteq aR$ then $ca^{-}b$ is invariant for all choice of g-inverses of a.

Proof. Since a is regular in R, by Lemma 2.5, a itself is a g-inverse of a and by Theorem 2.7 we have,

$$Rc \subseteq Ra \quad \text{(or)} \quad bR \subseteq aR \quad \Rightarrow \quad c = ca \quad \text{(or)} \quad b = ab$$
$$ca^{-}b = caa^{-}b = cab \quad \text{or} \quad ca^{-}b = ca^{-}ab = cab.$$

Thus $ca^{-}b$ is invariant for all choice of g-inverses of a.

Theorem 3.4. Let R be regular incline. If a, b, c $\in$R, then the following are equivalent:

(i) $Rc \subseteq Ra$ or $bR \subseteq aR$
(ii) $c = ca$ or $b = ab$
(iii) $ca^{-}b$ is invariant for all choice of $a^{-} \in a\{1\}$.

Proof. To prove this Theorem it is enough to prove the following implication:

(iii)$\Rightarrow$(ii) $\Rightarrow$(i) $\Rightarrow$(iii)

(i) $\Leftrightarrow$ (ii) :This equivalence is a part of Theorem 3.16 in [8].

(i) $\Rightarrow$ (iii) :This follows from Lemma (3.3).

(iii) $\Rightarrow$ (ii): If $ca^{-}b$ is invariant, then cxb = cyb for all g-inverse x, y of a. By the incline axiom since, $c = c + ca$, we have $c \geq ca$.

Suppose $c > ca$, then $ca^{-}b > caa^{-}b = cab$
$$\Rightarrow \quad ca^{-}b > cab$$
This contradicts the invariance of $ca^{-}b$. Hence $c = ca$. Thus (ii) holds.

Other alternatives can be proved in the same manner.

Definition 3.5. For the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over an incline with order relation '≤' if a is regular then the Schur-complement of a in M, denoted as M/a is defined as M/a =d – ca^{-}b and this depends on a^{-}. Thus M/a is well defined if and only if ca^{-}b is invariant for all $a^{-} \in a\{1\}$.

M/a is an element in an incline if ca^{-}b is invariant and d ≥ ca^{-}b (or) equivalently d + ca^{-}b = d. In this case we say that M/a exists.

Theorem 3.6. Let a, b, c, d∈R are regular elements. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^2$

and X = $\begin{pmatrix} a^{-} & a^{-}bd^{-} \\ d^{-}ca^{-} & d^{-} \end{pmatrix} \in R^2$ for some $a^{-} \in a\{1\}$ and $d^{-} \in d\{1\}$, A is regular

and X ∈A{1} if any one of the following hold:

(i) Rc ⊆ Ra and Rb ⊆ Rd
(ii) bR ⊆ aR and cR ⊆ dR
(iii) A/a exists, Rc ⊆ Ra and bR ⊆ aR
(iv) A/d exists, Rb ⊆ Rd and cR ⊆ dR

Proof.

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = \begin{pmatrix} a^{-} & a^{-}bd^{-} \\ d^{-}ca^{-} & d^{-} \end{pmatrix}$

by Lemma 2.4, Proposition 2.5, $ad^{-} = a = a^2$ and $dd^{-} = d = d^2$ by using the Definition 2.1 on computation

$$AXA = \begin{pmatrix} a + bd^{-}c & ab + bd^{-}ca^{-}b + bd \\ ca + ca^{-}bd^{-}c + dc & ca^{-}b + d \end{pmatrix} \tag{3.1}$$

(i) Since Rc ⊆ Ra and Rb ⊆ Rd, by Theorem 2.6 we have

$$c = ca, \quad b = bd \tag{3.2}$$

Using equation (3.2) in equation (3.1), we have the 11th entry of AXA becomes $a + bd^{-}ca = a$ by Definition 2.1. In the same manner we have 12th entry of AXA = b, 21th entry of AXA = c and the 22th entry of AXA = d.

Therefore, AXA=$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

Hence X ∈A{1}.

(ii) This can be proved in the same manner as that of (i) and hence omitted.

(iii) Let A/a exists by Definition (3.5) we have d+ $ca^{-}b$ = d

Since R$c \subseteq$ Ra and b$R \subseteq a$R by Theorem 2.6 it follows that c = ca and b = ab, using this in equation (3.1) we have the 11th entry of AXA = a+ $bd^{-}ca$ = a by Definition 2.1. Similarly 12th entry of AXA reduces to b and 21th entry of AXA reduces to c.

Therefore, AXA=$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

(iv) This can be proved in a similar manner as that of (iii). Hence omitted.

Definition 3.7. A pair of elements a, b $\in$R are said to be parallel summable, if (a+b) is regular and a(a+b)$^{-}$b is invariant for all choice of (a+b)$^{-}$ $\in$(a+b){1} and is denoted as a:b.

Theorem 3.8. If a, b $\in$R are regular elements then the following hold:

(i) (a+b) is regular

(ii) a and b are parallel summable

(iii) The parallel sum of a and b, a:b = ab is regular (when a and b commutes)

(iv) $M = \begin{pmatrix} (a+b) & b \\ a & (a+b) \end{pmatrix} \in R^2$ is regular

(v) M/(a+b) exists.

Proof. Since a and b are regular by Lemma 2.4 and Lemma 2.5 we have, $a = a^2 = a^{-}a$ and $b = b^2 = b^{-}b$.

(i) Let a, b are regular elements in R then by Lemma 2.5 both a and b are idempotent elements

Now, (a +b) (a +b) = a^2 + ba + ab +b^2

=(a + ba) + (ab +b)

= (a +b)

Again by Lemma 2.5, (a +b) is regular.

(ii) Since a is regular by Lemma 2.5 $a^2 = a$.

By the above statement (i) (a+b) is regular hence (a+b) itself is one choice of (a+b)$^{-}$ and by Definition 2.1

a (a+b) = a^2 + ab = a + ab = a

Similarly we have (b+a)b = b

Then by Theorem 2.6,

R$a \subseteq$ R(a + b) and bR $\subseteq$ (a + b)R

By Lemma 3.3, $a(a+b)^{-}b$ is invariant for all choice of $(a+b)^{-}$. Thus a and b are parallel summable and (ii) holds.

(iii) Since a, b are parallel summable then $a\,(a+b)^{-}b$ is invariant, by taking $(a+b)$ as a g-inverse of $(a+b)$

$$
\begin{aligned}
a{:}b = a\,(a+b)^{-}b &= a\,(a+b)\,b \\
&= (a^2 + ab)b \\
&= (a + ab)b \qquad \text{(By Lemma 2.5)} \\
&= ab \qquad \text{(By Definition 2.1)}
\end{aligned}
$$

Therefore $a{:}b = ab$.

Since a and b commutes

$$
\begin{aligned}
(ab)\,(ab) &= a\,(ab)\,b \\
&= a^2b^2 = ab.
\end{aligned}
$$

Therefore (ab) is idempotent, by Lemma (2.5) (ab) is regular.

Thus (iii) holds.

(iv) $M = \begin{pmatrix} (a+b) & b \\ a & (a+b) \end{pmatrix}$ since $a\,(a+b) = a$ and $b\,(a+b) = b$ by Theorem 2.6, $aR \subseteq (a+b)R$ and $bR \subseteq (a+b)R$ holds.

Therefore by Theorem 3.6 (ii) M is regular.

(v) Let us consider $M = \begin{pmatrix} (a+b) & b \\ a & (a+b) \end{pmatrix}$

By (ii) $a\,(a+b)^{-}b$ is invartiant and by Properties 2.2 and 2.3 we have,

$ab \le a \le (a+b)$ and $ab \le b \le (a+b)$

$M/(a+b) = (a+b) - a{:}b = (a+b) - ab \ge 0$

Hence, $M/(a+b)$ is an element in R and M/a exists.

Thus (v) holds.

The above Theorem 3.8 leads to the following corollary:

Corollary 3.1.9. If R is a commutative incline then the sums, parallel sums and products of regular elements are regular in R.

Let R be a commutative incline. Then the set of all regular elements in R forms a sub-incline.

For a, $b \in I$,the set of all regular elements in R, by Theorem 3.8(ii) and (iii) $(a+b) \in I$ and $ab = a{:}b \in I$ (is regular).

Hence I is closed under addition and multiplication in the incline R.

Therefore I is a sub-incline.

4. Conclusion

The main results in the present paper are the generalization of results on Schur-Complements in fuzzy matrices. We propose to establish these results for n×n incline matrices.

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