Secondary κ-Kernel Symmetric Fuzzy Matrices

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Abstract. In this paper, characterizations of secondary κ- kernel symmetric fuzzy matrices are obtained. Relation between s- κ- kernel symmetric, s- kernel symmetric, κ- kernel symmetric and kernel symmetric matrices are discussed. Necessary and sufficient conditions are determined for a matrix to be s- κ- kernel symmetric.

Keywords: Fuzzy matrices, kernel symmetric, s-κ- kernel symmetric

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1. Introduction
All matrices considered in this paper are fuzzy matrices, that is, matrices over a fuzzy algebra $\mathcal{F}$ with support $[0, 1]$ under max-min operations. A fuzzy matrix $A$ is range symmetric if $R(A) = R(A^T)$ and kernel symmetric if $N(A) = N(A^T)$. It is well known that for complex matrix, the concept of range and kernel symmetric are same. However this fails for fuzzy matrices. This motivated us to study on s- κ- kernel symmetric matrices. Lee [1] has initiated the study of secondary symmetric matrices, that is matrices whose entries are symmetric about the secondary diagonal. Cantoni and Paul [2] have studied persymmetric matrices, that is matrices which are symmetric about both the diagonals and their applications to communication theory. Hill and Waters [3] have developed a theory of κ-real and κ-hermitian matrices as a generalization of s-real and s-hermitian matrices. A development of κ- kernel symmetric fuzzy matrices is made by Meenakshi and Jayashree [5] analogous to that of k-real and k-hermitian of a complex matrix [3].

Throughout let κ-be a fixed product of disjoint transpositions in $S_n = \{1, 2, ..., n\}$ and $K$ be the associated permutation matrix. A matrix $A=\left( a_{ij} \right) \in \mathcal{F}_n$ is κ-symmetric if $a_{ij} = a_{k(i)k(j)}$ for $i, j = 1$ to $n$. Meenakshi and krishnamoorthy[6] have introduced the concept of s-k hermitian matrices as a generalization of secondary hermitian and hermitian matrices. In this paper, we extend the concept of s- κ- kernel symmetric fuzzy matrices as a particular case of the results on complex matrices found in [7].

2. Preliminaries
Throughout let $\mathcal{F}$ be the permutation matrix with units in its secondary diagonal and let ‘κ’ be a fixed product of disjoint transpositions in $S_n = \{1, 2, ..., n\}$ and $K$ be the
associated permutation matrix. For \( x = (x_1, x_2, \ldots, x_n)^T \in \mathcal{F}_n \) let us define the function
\[
\mathcal{R}(x) = (x_{\kappa(1)}, x_{\kappa(2)}, \ldots, x_{\kappa(n)})^T \in \mathcal{F}_n.
\]
Since \( K \) is involutory, it can be verified that the associated permutation matrix satisfy the following properties.

(P.2.1) \( KK^T = K^TK = I_n, K = K^T, K^2 = I \) and \( \mathcal{R}(x) = Kx \)

By the definition of \( V \),
(P.2.2) \( V = V^T, VV^T = V^TV = I_n \) and \( V^2 = I \)
(P.2.3) \( N(A) = N(AV), N(A) = N(AK) \)
(P.2.4) \( (AV)^T = VA^T, (VA)^T = A^T V \)

If \( A^* \) exists, then
(P.2.5) \( (AV)^+ = VA^+, (VA)^+ = A^+ V \)

**Definition 2.1.** [4] \( A \in \mathcal{F}_n \) is kernel symmetric if and only if \( N(A) = N(A^T) \).

**Lemma 2.1.** [[4] P. 119] For \( A \in \mathcal{F}_n \) and a permutation matrix \( P \), \( N(A) = N(B) \) if and only if \( N(PAP^T) = N(PBP^T) \).

**Lemma 2.2.** [5] A matrix \( A \in \mathcal{F}_n \) is \( \kappa \)-kernel symmetric \( \iff \) \( KA \) is kernel symmetric \( \iff \) \( AK \) is kernel symmetric.

3. Secondary \( \kappa \)-kernel symmetric fuzzy matrices

**Definition 3.1.** A matrix \( A \in \mathcal{F}_n \) is s-symmetric if and only if \( A = VA^TV \).

**Definition 3.2.** A matrix \( A \in \mathcal{F}_n \) is s-kernel symmetric if \( N(A) = N(VA^TV) \).

**Definition 3.3.** A matrix \( A \in \mathcal{F}_n \) is s- \( \kappa \)-kernel symmetric if \( N(A) = N(KVA^TK) \).

**Lemma 3.1.** A matrix \( A \in \mathcal{F}_n \) is s-kernel symmetric \( \iff \) \( VA \) is kernel symmetric \( \iff \) \( AV \) is kernel symmetric.

**Proof.**

\( A \) is s-kernel symmetric
\[
\iff N(A) = N(VA^TV) \quad \text{[By Definition 3.2]}
\]
\[
\iff N(AV) = N((AV)^T) \quad \text{[By P.2.2]}
\]
\[
\iff AV \text{ is kernel symmetric}
\]
\[
\iff N(VA^TV) = N(VA^TV) \quad \text{[By Lemma 2.1]}
\]
\[
\iff N(VA) = N((VA)^T) \quad \text{[By P.2.2]}
\]
\[
\iff VA \text{ is kernel symmetric.}
\]

**Remark 3.1.** In particular when \( \kappa(i) = i \) for \( i = 1, 2, \ldots, n \) then the associated permutation matrix \( K \) reduces to the identity matrix and Definition (3.3) reduces to \( N(A) = N(VA^TV) \) which implies that \( A \) is s-kernel symmetric matrices.

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Remark 3.2. For \( \kappa(i) = n - i + 1 \), the corresponding permutation matrix \( K \) reduces to \( V \) and Definition (3.3) reduces to \( N(A) = N(A^T) \) which implies that \( A \) is kernel symmetric.

Remark 3.3. We note that s- \( \kappa \)-symmetric matrix is s-\( \kappa \)-symmetric for if \( A \) is s-\( \kappa \)-symmetric then \( A = KVA^TVK \) Hence \( N(A) = N(KVA^TVK) \) which implies that \( A \) is s-\( \kappa \)-kernel symmetric. However the converse need not be true. This is illustrated in the following example.

Example 3.1. For \( \kappa = (1,2) \), \( A = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.5 \end{bmatrix} \) is symmetric

\[
KVA^TVK = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0.6 \\ 0.6 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.6 \\ 0.6 & 1 \end{bmatrix} \neq A
\]

Here \( A = KVA^T K \) therefore \( A \) is symmetric, \( \kappa \)-symmetric, s-\( \kappa \)-kernel symmetric but not s-\( \kappa \)-symmetric.

Example 3.2. For \( \kappa = (1,2) \), \( V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \)

\( A = \begin{bmatrix} 0.4 & 0.5 \\ 0.5 & 0.4 \end{bmatrix} \) is symmetric, s-\( \kappa \)-symmetric and hence therefore s-\( \kappa \)-kernel symmetric.

Example 3.3. For \( \kappa = (1,2)(3) \) \( K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) and \( V = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \) here \( K \neq V, K \neq I \) and \( KV \neq VK \).

Now \( A = \begin{bmatrix} 0.5 & 1 & 0 \\ 0.5 & 0.3 & 0 \end{bmatrix} \) is s-\( \kappa \)-kernel symmetric but not s-\( \kappa \)-symmetric.

\[
KVA^TVK = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.3 \\ 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq A
\]
Hence $A$ is not s-$\kappa$-symmetric. But $N(A) = N(KVA^TKVA) = \{0\}$.

**Theorem 3.1.** For $A \in F_n$ the following are equivalent

1. $A$ is s- $\kappa$-kernel Symmetric
2. $KVA$ is kernel symmetric
3. $AKV$ is kernel symmetric
4. $AVK$ is kernel symmetric
5. $VKA$ is kernel symmetric
6. $VKA$ is $\kappa$-kernel symmetric
7. $AV$ is $\kappa$-kernel symmetric
8. $AK$ is s-kernel symmetric
9. $KA$ is s-kernel symmetric
10. $N(AV) = N(KVA^T)$
11. $N(A) = N(KVA^T)$

**Proof:**

$(1) \iff (4) \iff (5) \iff (9)$

$A$ is s-$\kappa$-kernel symmetric

$\iff N(A) = N(KVA^TVK)$

$\iff N(A) = N(KVA^T)$

$\iff N(A) = N((AVK)^T)$

$\iff AVK$ is kernel symmetric

$\iff VKA$ is kernel symmetric

$\iff KA$ is s-kernel symmetric

Thus $(1) \iff (4) \iff (5) \iff (9)$ hold.

$(2) \iff (6)$

$KVA$ is kernel symmetric

$\iff VAK$ is $\kappa$-kernel symmetric

Thus $(2) \iff (6)$ hold.

$(2) \iff (10)$

$KVA$ is kernel symmetric

$\iff N(KVA) = N((KVA)^T)$

$\iff N(KVA) = N(A^T)$

$\iff N(A) = N((AVK)^T)$

Thus $(2) \iff (10)$ hold.

$(4) \iff (11)$

$AVK$ is kernel symmetric

$\iff N(AV) = N((AVK)^T)$

$\iff N(A) = N(KVA^T)$

Thus $(4) \iff (11)$ hold.

$(1) \iff (4) \iff (7)$

$A$ is s-$\kappa$-kernel symmetric

$\iff N(A) = N(KVA^TVK)$

$\iff N(A) = N((AVK)^T)$

$\iff N(AV) = N((AVK)^T)$
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$\iff \mathbf{AVK}$ is kernel symmetric
$\iff \mathbf{AV}$ is $\kappa$-kernel symmetric. Thus (1) $\iff$ (4) $\iff$ (7) hold.

(3) $\iff$ (8)

$\mathbf{AV}$ is kernel symmetric $\iff \mathbf{AK}$ is s- $\kappa$-kernel symmetric.

Hence the Theorem.

In Particular for $\mathbf{K} = \mathbf{I}$, the above Theorem reduces to the equivalent condition for a matrix to be secondary kernel symmetric.

**Corollary 3.1.** For $\mathbf{A} \in \mathcal{F}_n$, the following are equivalent

1. $\mathbf{A}$ is s-kernel symmetric
2. $\mathbf{VA}$ is kernel symmetric
3. $\mathbf{AV}$ is kernel symmetric
4. $\mathbf{N}(\mathbf{A}^T) = \mathbf{N}(\mathbf{VA})$
5. $\mathbf{N}(\mathbf{A}) = \mathbf{N}(\mathbf{VA}^T)$

**Lemma 3.2.** Let $\mathbf{A} \in \mathcal{F}_n$, if $\mathbf{A}^+$ exists $\iff (\mathbf{KA})^+$ exists $\iff (\mathbf{VKA})^+$ exists.

**Proof:**

$\mathbf{A}^+$ exists $\iff (\mathbf{KA})^+$ exists $\iff \mathbf{KA} = (\mathbf{KA})(\mathbf{KA})^T(\mathbf{KA})$

$\iff \mathbf{VKA} = (\mathbf{VKA})(\mathbf{VKA})^T(\mathbf{VKA})$

$\iff (\mathbf{VKA})^T = (\mathbf{VKA})$ [1]

$\iff (\mathbf{VKA})^+$ exists.

**Lemma 3.2.** Let $\mathbf{A} \in \mathcal{F}_n$, if $\mathbf{A}^+$ exists $\iff (\mathbf{KA})^+$ exists $\iff (\mathbf{VKA})^+$ exists.

**Proof:**

$\mathbf{A}^+$ exists $\iff (\mathbf{KA})^+$ exists $\iff \mathbf{KA} = (\mathbf{KA})(\mathbf{KA})^T(\mathbf{KA})$

$\iff \mathbf{VKA} = (\mathbf{VKA})(\mathbf{VKA})^T(\mathbf{VKA})$

$\iff (\mathbf{VKA})^T = (\mathbf{VKA})[1]$

$\iff (\mathbf{VKA})^+$ exists.

**Remark 3.4.** For $\mathbf{A} \in \mathcal{F}_n$, $\mathbf{A}^+$ exists $\iff (\mathbf{KVA})^+$ exists.

**Theorem 3.2.** Let $\mathbf{A} \in \mathcal{F}_n$. Then any two of the following conditions imply the other one.

1. $\mathbf{A}$ is $\kappa$-kernel symmetric
2. $\mathbf{A}$ is s- $\kappa$-kernel symmetric
3. $\mathbf{N}(\mathbf{A}^T) = \mathbf{N}(\mathbf{KAV})$

**Proof:**

(1) and (2) $\Rightarrow$ (3)

$\mathbf{A}$ is s- $\kappa$-kernel symmetric $\Rightarrow \mathbf{N}(\mathbf{A}) = \mathbf{N}(\mathbf{AVK}^T)$ [By Theorem 3.1]
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\[ A \text{ is } \kappa \text{-kernel symmetric} \Rightarrow N(KAK) = N(VAV^T) \quad \text{[By Lemma 2.1]} \]

Hence (1) and (2) hold.

\[ (1) \text{ and (3)} \Rightarrow (2) \]

\[ A \text{ is } \kappa \text{-kernel symmetric} \Rightarrow N(KAK) = N(A^T) \]

Thus (2) hold.

(2) and (3) \Rightarrow (1)

\[ A \text{ is } s \text{- } \kappa \text{-kernel symmetric} \Rightarrow N(KAK) = N(KAV^T) \quad \text{[By Lemma 2.1]} \]

Thus (1) hold. Hence the theorem.

REFERENCES