On $\beta^*$-Connectedness and $\beta^*$-Disconnectedness and their Applications

R. Ramesh¹, A. Vadivel² and D. Sivakumar³

¹Department of Mathematics, Pope John Paul II College of Education Reddiarpalayam, Puducherry – 605010.
Corresponding Author e-mail: rameshroshitha@gmail.com
²Mathematics Section (FEAT), Annamalai University, Annamalainagar – 608002
e-mail: avmaths@gmail.com
³Department of Mathematics (DDE), Annamalai University, Annamalainagar – 608 002
e-mail: sivakumardmaths@yahoo.com

Received 10 October 2014; accepted 20 November 2014

Abstract. In this paper, by using $\beta^*$-closed sets we study the concept of $\beta^*$-separated sets. With this concept we study the notion of $\beta^*$-connected sets and strongly $\beta^*$-connected sets. We give some properties of such concepts with some $\beta^*$-separation axioms and compact spaces. Finally, we construct a new topological space on a connected graph.

Keywords: $\beta^*$-separated sets, $\beta^*$-connected sets

AMS Mathematics Subject Classification (2010): 54B05, 54B10, 54C10, 54D18, 90D42

1. Introduction

Connectedness [1] is a well-known notion in topology. Numerous authors studied connectedness. In [2], $P$-spaces and external disconnectedness are studied. Connectedness in [4–6] are used to expand some topological spaces. In [13], authors proved that neither first countable nor Čech-complete spaces are maximal Tychonoff connected. Many other topologists defined and studied connectedness in bitopological spaces [3, 12]. It is important to study some types of connectedness in digital spaces. A point with integer coordinates is called a digital point. The problem of finding a topology for the digital plane and the digital 3-space is of importance in image processing and more generally in all situations where spatial relations are modeled on a computer. In all these applications it is essential to have a data structure on the computer which shares as many as possible features with the real topological situation. Connectedness and compactness are powerful tools in topology but they have many dissimilar properties. The concept of Hausdorff spaces is almost an integral part of compactness. Investigations into the properties of cut points of topological spaces which are connected, compact and Hausdorff date back to the 1920s. Connectedness together with compactness with the assumption of Hausdorff has been studied in [15] from the view point of cut points. In [7], authors studied some types of connected topological spaces. Recently Palanimani [9] introduced and studied a new class of sets called $\beta^*$-closed sets in topological spaces. Since then these concepts have used to define and investigate many topological properties. The aim of this paper is
On $\beta^r$-Connectedness and $\beta^r$-Disconnectedness and their Applications

to study $\beta^r$-connectedness. Also digital spaces are examined in the context of these new concepts. However, our main interest shall be digital spaces that are also topological spaces.

2. Preliminaries

Throughout the present paper, the space $(X, \tau)$ and $(Y, \sigma)$ always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Here we present some of the definitions, which are used in our study.

**Definition 2.1.** A subset $A$ of a topological spaces $(X, \tau)$ is called a

(i) generalized closed (briefly, $g$-closed) [8] if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

(ii) $\beta^r$-closed [9] is $\text{Cl}(\text{Int}(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open in $(X, \tau)$.

The Complements of the above mentioned closed sets are their respective open sets. We denote the collection of all $g$-closed (resp. $\beta^r$-closed) sets by $\text{GC}(X)$ (resp. $\beta^C(X)$). We set $\text{GC}(X,x) = \{U : x \in U \in \text{GC}(X)\}$ (resp. $\beta^C(X,x) = \{U : x \in U \in \beta^C(X)\}$). The $\beta^r$-closure of a set $A$, denoted by $\beta^C(A)$, is the intersection of all $\beta$-closed sets containing $A$. $\beta^C(A)$ is the smallest $\beta$-closed set containing $A$. The $\beta^r$-interior of a set $A$ denoted by $\beta^r\text{Int}(A)$, is the union of all $\beta^r$-open sets contained in $A$. $\beta^C(A)$ is the largest $\beta^r$-open set contained in $A$. The family of all $\beta^r$-open (resp. $\beta^r$-closed) sets in a space $X$ will be denoted by $\beta^r\text{O}(X)$ (resp. $\beta^C(X)$).

**Proposition 2.1.** [9] (i) The union of any family of $\beta^r$-open sets is a $\beta^r$-open set.

(ii) The intersection of an open and a $\beta^r$-open set is a $\beta^r$-open set.

**Lemma 2.1.** [9] The $\beta^r$-closure of a subset $A$ of $X$, denoted by $\beta^C(A)$, is the set of all $x \in X$ such that $O \cap A \neq \emptyset$ for every $O \in \beta^r \text{O}(X,x)$, where $\beta^r \text{O}(X,x) = \{U : x \in U \in \beta^r \text{O}(X,\tau)\}$.

**Definition 2.2.** The $\beta^r$-boundary of a set $A$ of a space $X$ is defined by $\beta^r\text{bd}(A) = \beta^C(A) - \beta^C(X - A)$.

**Definition 2.3.** A space $X$ is said to be $\beta^r$-connected if $X$ cannot be expressed as the union of two disjoint nonempty $\beta^r$-open sets of $X$.

**Lemma 2.2.** Let $A$ be a subset of a topological space $X$. Then $A \in \beta^r \text{O}(X)$ if and only if $\beta^C(A)$ is $\beta^r$-clopen in $X$ (i.e., $\beta^r$-open and $\beta^r$-closed).

**Definition 2.4.** [11] A subset $N \subseteq X$ is called a $\beta^r$-neighborhood (briefly $\beta^r$-nbd) of a point $x \in X$ if there exists a $\beta^r$-open set $U \subseteq N$ such that $x \in U \subseteq N$.

3. $\beta^r$-Separateness and $\beta^r$-Connectedness

**Definition 3.1.** Two subsets $A$ and $B$ in a space $X$ are said to be $\beta^r$-separated if and only if $A \cap \beta^C(B) = \emptyset$ and $\beta^C(A) \cap B = \emptyset$. From the fact that $\beta^C(A) \subseteq \text{Cl}(A)$, for every subset $A$ of $X$, every separated set is $\beta^r$-separated. But the converse may not be true as shown in the following example.
**Remark 3.1.** Each two $\beta^*$-separated sets are always disjoint, since $A \cap B \subseteq A \cap B^* \cap Cl(B) = \phi$. The converse may not be true in general.

**Example 3.2.** In Example 3.1, $\beta^* O(X) = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{d\}, \{a, d\}, \{a, b\}, \{b, c, d\}\}$. The subsets $\{c\}, \{a, d\}$ are $\beta^*$-separated but not separated.

**Theorem 3.2.** Let $A$ and $B$ be nonempty sets in a space $X$. The following statements hold:

(i) If $A$ and $B$ are $\beta^*$-separated and $A_1 \subseteq A$ and $B_1 \subseteq B$, then $A_1$ and $B_1$ are so.

(ii) If $A \cap B = \phi$ such that each of $A$ and $B$ are both $\beta^*$-closed ($\beta^*$-open), then $A$ and $B$ are $\beta^*$-separated.

(iii) If each of $A$ and $B$ are both $\beta^*$-closed ($\beta^*$-open) and if $H = A \cap (X - B)$ and $G = B \cap (X - A)$, then $H$ and $G$ are $\beta^*$-separated.

**Proof:**

(i) Since $A \subseteq A$, then $\beta^* Cl(A_1) \subseteq \beta^* Cl(A)$. Then $B \cap \beta^* Cl(A) = \phi$ implies $A_1 \cap \beta^* Cl(A_1) = \phi$ and $B_1 \cap \beta^* Cl(A_1) = \phi$. Similarly, $A_1 \cap \beta^* Cl(B_1) = \phi$. Hence $A_1$ and $B_1$ are $\beta^*$-separated.

(ii) Since $A = \beta^* Cl(A)$ and $B = \beta^* Cl(B)$, and $A \cap B = \phi$, then $\beta^* Cl(A) \cap B = \phi$ and $\beta^* Cl(B) \cap A = \phi$. Hence $A$ and $B$ are $\beta^*$-separated.

(iii) If $A$ and $B$ are $\beta^*$-open, then $X - A$ and $X - B$ are $\beta^*$-closed. Since $H \subseteq X - B$, $\beta^* Cl(H) \subseteq \beta^* Cl(X - B) = X - B$ and so $\beta^* Cl(H) \cap B = \phi$. Thus $G \cap \beta^* Cl(G) = \phi$. Similarly, $H \cap \beta^* Cl(H) = \phi$. Hence $H$ and $G$ are $\beta^*$-separated.

**Theorem 3.3.** The sets $A$ and $B$ of a space $X$ are $\beta^*$-separated if and only if there exist $U$ and $V$ in $\beta^* O(X)$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$, $B \cap U = \phi$.

**Proof:** Let $A$ and $B$ be $\beta^*$-separated sets. Set $U = X - \beta^* Cl(A)$ and $U = X - \beta^* Cl(B)$. Then $U, V \in \beta^* O(X)$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$, $B \cap U = \phi$. On the other hand, let $U, V \in \beta^* O(X)$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$, $B \cap U = \phi$. Since $X - V$ and $X - U$ are $\beta^*$-closed, then $\beta^* Cl(A) \subseteq X - V \subseteq X - B$ and $\beta^* Cl(B) \subseteq X - U \subseteq X - A$. Thus $\beta^* Cl(A) \cap B = \phi$ and $\beta^* Cl(B) \cap A = \phi$.

**Definition 3.2.** A point $x \in X$ is called a $\beta^*$-limit point of a set $A \subseteq X$ if every $\beta^*$-open set $U \subseteq X$ containing $x$ contains a point of $A$ other than $x$.

**Theorem 3.3.** Let $A$ and $B$ be nonempty disjoint subsets of a space $X$ and $E = A \cup B$. Then $A$ and $B$ are $\beta^*$-separated if and only if each of $A$ and $B$ is $\beta^*$-closed ($\beta^*$-open) in $E$.

**Proof:** Let $A$ and $B$ be $\beta^*$-separated sets. By Definition 3.1, $A$ contains no $\beta^*$-limit points of $B$. Then $B$ contains all $\beta^*$-limit points of $B$ which are in $A \cup B$ and $B$ is $\beta^*$-closed in $A \cup B$. Therefore $B$ is $\beta^*$-closed in $E$. Similarly $A$ is $\beta^*$-closed in $E$.

**Definition 3.3.** A subset $S$ of a space $X$ is said to be $\beta^*$-connected relative to $X$ if there is not exist two $\beta^*$-separated subsets $A$ and $B$ relative to $X \cup S = A \cup B$. Otherwise, $S$ is said to be $\beta^*$-disconnected.
On $\beta^*$-Connectedness and $\beta^*$-Disconnectedness and their Applications

By Definition 3.3., one can show that each $\beta^*$-connected set is connected. The converse may not be true in general as shown in the below examples. In other words, each disconnected is $\beta^*$-disconnected.

**Example 3.3.** Any space with indiscrete topology is connected but not $\beta^*$-connected since $\beta^*$-open sets establish a discrete topology.

**Example 3.4.** Let $X = \{a, b, c, d\}$ with a topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. The subset $\{a, b, c\}$ is connected but not $\beta^*$-connected.

**Theorem 3.4.** Let $A$ and $B$ be subsets in a space $X$ such that $A \subseteq B \subseteq \beta^*C(A)$. If $A$ is $\beta^*$-connected, then $B$ is $\beta^*$-connected.

**Proof:** If $B$ is $\beta^*$-disconnected, then there exist two $\beta^*$-separated subsets $U$ and $V$ relative to $X$ such that $B = U \cup V$. Then either $A \subseteq \text{Uor}A \subseteq V$. Without loss of generality, let $A \subseteq U$. As $A \subseteq U \subseteq B \subseteq \beta^*C(A) \subseteq \beta^*C(B) \subseteq \beta^*C(U)$. Also $\beta^*C(B) = B \subseteq \beta^*C(U)$. This implies $\beta^*C(U)$. So $U$ and $V$ are not $\beta^*$-separated and $B$ is $\beta^*$-connected.

**Theorem 3.5.** If $E$ is $\beta^*$-connected, then $\beta^*C(E)$ is $\beta^*$-connected.

**Proof:** By contradiction, suppose that $\beta^*C(E)$ is $\beta^*$-disconnected. Then there are two $\beta^*$-separated sets $G$ and $H$ in $X$ such that $\beta^*C(E) = G \cup H$. Since $E = (G \cap E) \cup (H \cap E)$ and $\beta^*C(G \cap E) \subseteq \beta^*C(G)$ and $\beta^*C(H \cap E) \subseteq \beta^*C(H)$ and $G \cap H = \phi$, then $\beta^*C(G \cap E) \cap H = \phi$. Hence $(\beta^*C(G \cap E)) \cap (H \cap E) = \phi$. Similarly $(\beta^*C(H \cap E)) \cap (G \cap E) = \phi$. Therefore $E$ is $\beta^*$-disconnected.

**Lemma 3.1.** Let $A \subseteq B \subseteq C$ such that $A$ be a nonempty $\beta^*$-connected set in a space $X$ and $B$, $C$ are $\beta^*$-separated. Then only one of the following conditions holds:

(i) $A \subseteq B \cap \text{Cand} A \cap C = \phi$.

**Proof:** Since $A \cap C = \phi$, then $A \subseteq B$. Also, if $A \cap B = \phi$, then $A \subseteq C$. Since $A \subseteq B \cap C$, then both $A \cap B = \phi$ and $A \cap C = \phi$ cannot hold simultaneously. Similarly, suppose that $A \cap B \neq \phi$ and $A \cap C \neq \phi$, then, by Theorem 3.5, (i), $A \cup B$ and $A \cap C$ are $\beta^*$-separated such that $A = (A \cap B) \cup (A \cap C)$ which contradicts with the $\beta^*$-connectedness of $A$. Hence one of the conditions (i) and (ii) must be hold.

**Definition 3.4.** [10], [11] A function $f : X \to Y$ is said to be:

(i) $\beta^*$-continuous if the inverse image of each open set in $Y$ is $\beta^*$-open in $X$.

(ii) $\beta^*$-open if the image of each open set in $X$ is $\beta^*$-open in $Y$.

(iii) $\beta^*$-closed if the image of each closed set in $X$ is $\beta^*$-closed in $Y$.

**Lemma 3.2.** Let $f : X \to Y$ be a $\beta^*$-continuous function. Then $\beta^*C(f^{-1}(B)) \subseteq f^{-1}(\beta^*C(B))$, for each $B \subseteq Y$.

**Proof:** Let $A$ be subset of $(X, \tau)$. Let $B = f(A)$ be subset of $Y$. Then $\beta^*C(B)$ is closed in $Y$. Since $f$ is $\beta^*$-continuous, $f^{-1}(\beta^*C(B))$ is $\beta^*$-closed in $X$ and $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\beta^*C(B))$ that is $f^{-1}(\beta^*C(B))$ is $\beta^*$-closed subset of $X$ containing $A$. By Definition of $\beta^*$-closed sets implies $\beta^*C(A) \subseteq f^{-1}(\beta^*C(B))$. Hence $\beta^*C(f^{-1}(B)) \subseteq f^{-1}(\beta^*C(B))$. 


R. Ramesh, A. Vadivelan, and D. Sivakumar

**Theorem 3.6.** For a $\beta^*$-continuous function $f : X \to Y$, if $K$ is $\beta^*$-connected in $X$, then $f(K)$ is connected in $Y$.

**Proof:** Suppose that $f(K)$ is disconnected in $Y$. There exist two separated sets $P$ and $Q$ of $Y$ such that $f(K) = P \cup Q$. Set $A = K \cap f^{-1}(P)$ and $B = K \cap f^{-1}(Q)$. Since $f(K) \cap P \neq \phi$, then $K \cap f^{-1}(P) \neq \phi$ and so $A \neq \phi$. Similarly, $B \neq \phi$. Since $P \cap Q = \phi$, then $A \cap B = K \cap f^{-1}(P \cap Q) = \phi$ and so $A \cap B = \phi$. Since $f$ is $\beta^*$-continuous, then by Lemma 3.2., $\beta^* Cl(f^{-1}(Q)) \subset f^{-1}(Cl(Q)) \text{ and } B \subset f^{-1}(Q)$, then $\beta^* Cl(B) \subset f^{-1}(Cl(Q))$.

Since $P \cap Cl(Q) = \phi$, then $A \cap f^{-1}(Cl(Q)) \subset f^{-1}(P \cap f^{-1}(Cl(Q)) = \phi$ and then $A \cap \beta^* Cl(B) = \phi$. Thus $A$ and $B$ are $\beta^*$-separated.

**Corollary 3.1.** For a $\beta^*$-continuous function $f : X \to Y$, if $K$ is disconnected in $X$, then $f(K)$ is $\beta^*$-disconnected in $Y$.

**Proof:** Obvious.

**Theorem 3.7.** For a bijective $\beta'$-closed $f : X \to Y$, if $K$ is $\beta'$-connected in $Y$, then $f^{-1}(K)$ is connected in $X$.

**Proof:** The proof is similar to that of Theorem 3.6. Thus we omit it.

**Definition 3.5.** A function $f : X \to Y$ is said to be:

(i) $\beta^*$-Irresolute if for each point $x \in X$ and each $\beta^*$-open set $V$ of $Y$ containing $f(x)$, there exists a $\beta^*$-open set $U$ of $X$ containing $x$ such that $f(U) \subset V$.

(ii) $\beta^*$-Irresolute [10] if $f^{-1}(V) \in \beta^* O(Y)$ for every $V \in \beta^* O(Y)$.

(iii) $M-\beta^*$-open if $f(V) \in \beta^* O(Y)$ for every $V \in \beta^* O(X)$.

(iv) $M-\beta^*$-closed if $f(V) \subset \beta^* C(Y)$ for every $V \in \beta^* C(X)$.

(v) Strongly $\beta^*$-Irresolute if $f^{-1}(V) \in \beta^* O(X)$ for every open set $V \subset Y$.

(vi) Strongly $M-\beta^*$-open if $f(V) \in \beta^* O(Y)$ for every open set $V \subset X$.

(vii) Strongly $M-\beta^*$-closed if $f(V) \in \beta^* C(Y)$ for every closed set $V \subset X$.

**Lemma 3.3.** A function $f : X \to Y$ is a $\beta \beta^*$-Irresolute if and only if $\beta Cl(f^{-1}(B)) \subset f^{-1}(\beta(Cl(B)) \subset f^{-1}(Cl(B))$, for each $B \subset Y$.

**Proof:** Follows from the Definition 3.5.

**Theorem 3.8.** Let $f : (X, \tau) \to (Y, \sigma)$ be a $\beta'$-Irresolute function. If $K$ is $\beta'$-connected in $X$, then $f(K)$ is $\beta'$-connected in $Y$.

**Proof:** By using Definition 3.20 and Lemma 3.3, it is a direct consequence of Theorem 3.6.

**3.1. Strongly $\beta'$-Connectedness in compact spaces**

**Definition 3.1.1.** A space $X$ is strongly $\beta'$-connected if and only if it is not a disjoint union of countably many but more than one $\beta'$-closed set i.e. if $E_i$ are nonempty disjoint closed sets of $X$, then $X \neq E_1 \cup E_2 \cup ...$. Otherwise, $X$ is said to be strongly $\beta'$-disconnected. Note the similarity between Definition 4.1 and that of $\beta'$-connectedness. If $X$ is $\beta'$-connected, and $E_i$ and $E_2$ are any two nonempty disjoint closed sets of $X$, then $X \neq E_1 \cup E_2$.

**Lemma 3.1.1.** For any surjective $\beta'$-Irresolute function $f : X \to Y$. The image $f(X)$ is strongly $\beta'$-connected if $X$ is strongly $\beta'$-connected.
On $\beta'$-Connectedness and $\beta'$-Disconnectedness and their Applications

**Proof:** Suppose $f(X)$ is strongly $\beta'$-disconnected, by Definition 4.1, it is a disjoint union of countably many but more than one $\beta'$-closed sets. Since $f$ is $\beta'$-irresolute, then the inverse image of $\beta'$-closed sets are still $\beta'$-closed, $X$ is also a disjoint union of $\beta'$-closed sets. Therefore, $f(X)$ is strongly $\beta'$-connected.

**Theorem 3.1.1.** A space $X$ is strongly $\beta'$-connected if there exists a constant surjective $\beta'$-irresolute function $f : X \to D$, where $D$ denote to a discrete space of $X$.

**Proof:** Let $X$ be strongly $\beta'$-connected and $f : X \to D$ be a surjective $\beta'$-irresolute function, then by Lemma 3.1.1., $f(X)$ is strongly $\beta'$-connected. The only strongly $\beta'$-connected subset of $D$ are the one-point spaces. Hence $f$ is constant. Conversely, suppose $X$ is a disjoint union of countably many but more than one $\beta'$-closed sets, $X = \bigcup E_i$. Then define $f : X \to D$ by taking $f(x) = i$ whenever $x \in E_i$. This $f$ is a surjective $\beta'$-irresolute and not constant. So $X$ is strongly $\beta'$-connected. Strongly $\beta'$-connectedness is a stronger notion of $\beta'$-connectedness. In other words, given a $\beta'$-connected space, we can make it strongly $\beta'$-connected by adding some conditions. But what conditions should be added is the difficulty. Our starting point is $\beta'$-connected spaces, thus a $\beta'$-continuum may be useful. The concept of a $\beta'$-continuum is defined on a $\beta'$-connected set.

**Definition 3.1.2.** A compact $\beta'$-connected set is called a $\beta'$-continuum.

**Definition 3.1.3.** A space $X$ is called:
1. $\beta T_1$ if for each $x, y \in X$, $x \neq y$, there exist two disjoint $\beta'$-open sets $U$ and $V$ such that $x \in U$, $y \notin U$, and $x \notin V$, $y \in V$.
2. $\beta T_0$ if for each $x, y \in X$, $x \neq y$, there exist two disjoint $\beta'$-open sets $U$ and $V$ such that $x \in U$, $y \notin V$ and $U \cap V = \emptyset$.
3. $\beta$-normal for any pair of disjoint $\beta'$-closed sets $F_1$ and $F_2$, there exist disjoint $\beta'$-open sets $U$ and $V$ such that $F_1 \subseteq U$ and $F_2 \subseteq V$ such that $U \cap V = \emptyset$.

**Lemma 3.1.2.** If $A$ is any $\beta'$-continuum in a $\beta T_2$ space $X$ and $B$ is any $\beta'$-open set such that $A \cap B \neq \emptyset$, then every component of $(A \cap Cl(B)) \cap \beta' - bd(B) \neq \emptyset$.

**Proof:** It is obvious by Definitions 2.2., 3.1.2 and 3.1.3.

**Theorem 3.1.2.** Let $X$ be a compact $\beta T_2$-space. Then $X$ is $\beta'$-connected if and only if $X$ is strongly $\beta'$-connected.

**Proof:** It is clear that if $X$ is strongly $\beta'$-connected, then $X$ is $\beta'$-connected. Now, suppose that $X$ is a compact $\beta T_2$ $\beta'$-connected space and it is strongly $\beta'$-disconnected, then $X$ is a union of a countably many but more than one $\beta'$-closed sets. Then $X = \bigcup K_i$, where $K_i$ are $\beta'$-closed disjoint sets. Since a compact $\beta T_2$-space is $\beta'$-normal, then $X$, by Definition 3.1.3., is a $\beta'$-normal space. So there exist a $\beta'$-open sets $U$ such that $K_2 \subseteq U$ and $\beta Cl(U) \cap K_2 = \emptyset$. Let $X_1$ be a component of $\beta Cl(U)$ which intersects $K_2$. Then $X_1$ is compact and $\beta'$-connected. Now by Lemma 3.1.2., $X_1 \cap \beta'-bd(U) \neq \emptyset$, i.e. $X_1$ contains a point $p \in \beta'-bd(U)$ such that $p \notin U$ and $p \notin K_1$. Hence $X_1 \cap K_i \neq \emptyset$ for some $i > 2$. Let $K_{n_2}$ be the first $K_i$ for $i > 2$ which intersects $X_1$, and let $V$ be a $\beta'$-open set satisfying $K_{n_2} \subseteq V$ and $\beta Cl(V) \cap K_2 = \emptyset$. Then let $X_2$ be a component of $X_1 \cap \beta Cl(V)$ which contains a point of $K_{n_2}$. Again we have $X_2 \cap \beta' - bd(V) \neq \emptyset$, and $X_2$ contains some
Proof: Let $O$ be a $\beta$-open $\beta$-nbhd of a point $x \in X$. Then there exists a compact $\beta$-nbhd $V$ of $x$ lying inside $O$. Let $C$ be a $\beta$-connected component of $V$ containing $x$. Since $V$ is a $\beta$-nbhd of $x$ and $X$ is locally $\beta$-connected, $C$ is $\beta$-nbhd of $x$. Since $C$ is $\beta$-closed in $V$ and $V$ is compact, then $C$ is compact. So $C$ is a compact $\beta$-connected $\beta$-nbhd of $x$ lying inside $O$. By Theorem 3.1.2., $C$ is strongly $\beta$-connected.

**Theorem 3.1.4.** Let $X$ be a locally compact $\beta T_2$-space. If $X$ is locally $\beta$-connected and $\beta$-connected, then $X$ is strongly $\beta$-connected.

**Proof:** This follows from Theorems 3.1.2 and 3.1.3.

**Lemma 3.1.3.** For a space $X$ the following statements are equivalent:

(i) $X$ is a $\beta T_1$-space.

(ii) For any point $x \in X$, the singleton set $\{x\}$ is $\beta$-closed.

**Corollary 3.1.1.** strongly $\beta$-connected $\beta T_1$-space having more than one point is uncountable.

**Proof:** By Lemma 3.1.3., a one-point set in a $\beta T_1$-space is $\beta$-closed. Thus by Definition 3.1.1., a $\beta T_1$-space cannot have countably many but more than one point.
On $\beta^*$-Connectedness and $\beta^*$-Disconnectedness and their Applications