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All the Solutions of the Diophantine Equation $p^3 + q^y = z^3$ with Distinct Odd Primes p, q when $y > 3$

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Abstract. In this paper, we consider the equation $p^3 + q^y = z^3$ in which p, q assume distinct odd primes and z is a positive integer. Then, for all possible integers $y > 3$, the equation $p^3 + q^y = z^3$ has no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds.

In this paper, we consider the equation $p^3 + q^y = z^3$ in which p, q are distinct odd primes and z is a positive integer. The value y is a positive integer. We now provide a short survey of the equation $p^3 + q^y = z^3$ when $y = 1, 2$ and 3.

When $y = 1$, we have shown [4] that the equation $p^3 + q = z^3$ has infinitely many solutions. The first four solutions of the equation when p, q are primes and $y = 1$ are:

$$3^3 + 37 = 4^3, \qquad 11^3 + 397 = 12^3,$$

$$13^3 + 547 = 14^3, \qquad 17^3 + 919 = 18^3.$$

When $y = 2$, we have established [5] that the equation $p^3 + q^2 = z^3$ has exactly four solutions. These are:

$$7^3 + 13^2 = (2^3)^3, \qquad 7^3 + (7^2)^2 = (2\cdot7)^3,$$

$$7^3 + (3\cdot7^2)^2 = (2^2\cdot7)^3, \qquad 7^3 + (3\cdot7^2\cdot13)^2 = (2\cdot7\cdot11)^3.$$

Quite surprisingly in all the above solutions of $p^3 + q^2 = z^3$, we have $p = 7$, where only in the first solution q is prime.

Let $y = 3$. In 1637, Fermat (1601 – 1665) stated that the Diophantine equation $x^n + y^n = z^n$, with integral $n > 2$, has no solutions in positive integers x, y, z. This is known as Fermat's "Last Theorem". In 1995, 358 years later, the validity of the Theorem was established and published by A. Wiles. Thus, the equation $p^3 + q^3 = z^3$ has no solutions in positive integers p, q, z.

Therefore, in this paper we consider possible values y which satisfy $y > 3$. This is done in the following Section 2.

2. All the solutions of $p^3 + q^y = z^3$ when p, q are distinct odd primes, and $y > 3$

In this section, we show that the equation $p^3 + q^y = z^3$ with distinct odd primes, p, q and $y > 3$ has no solutions.

Theorem 2.1. Suppose that p, q are distinct odd primes. Let z be a positive integer. Then for all possible values $y > 3$ the equation $p^3 + q^y = z^3$ has no solutions.

Proof: We shall assume that for some value $y > 3$, the equation $p^3 + q^y = z^3$ has a solution and reach a contradiction.

By our assumption, $q^y = z^3 - p^3$ or

$$q^y = (z - p)(z^2 + zp + p^2). \tag{1}$$

From (1) it follows that

$$z - p = q^A, \qquad z^2 + zp + p^2 = q^B, \qquad A < B, \qquad A + B = y \tag{2}$$

where A, B are non-negative integers, and all conditions in (2) must be satisfied simultaneously.

Let $A \ge 1$. Then $B \ge 3$ and $y \ge 4$. We have from (2) that $z = p + q^A$, and hence

$$z^2 + zp + p^2 = (p + q^A)^2 + (p + q^A)p + p^2 = 3p^2 + 3p\cdot q^A + (q^A)^2 = q^B. \tag{3}$$

It then follows from (3) that $q \mid 3p^2$. Hence, either $q = 3$ or $q \mid p^2$ which is impossible. When $q = 3$, we have from (3)

$$3p^2 + 3p\cdot 3^A + 3^{2A} = 3^B \qquad \text{or after simplification}$$

$$p^2 + p\cdot 3^A + 3^{2A-1} = 3^{B-1}. \tag{4}$$

Since $p \ne 3$ (p, q are distinct), it follows that (4) is impossible. Thus $A \ngeq 1$.

Let $A = 0$. Then $y = B \ge 4$. From (2) and (3) we obtain

$$z = p + 1, \qquad z^2 + zp + p^2 = 3p^2 + 3p + 1 = q^y, \qquad y \ge 4.$$

Denote $q^y - (3p^2 + 3p) = t$. We will now show that $t \ne 1$.

In Table 1, the primes p, q are distinct odd primes and $y \ge 4$. For each prime p, the prime q and the value y are chosen in such a way that they ensure the smallest possible value t where $q^y - (3p^2 + 3p) = t$.

All the Solutions of the Diophantine Equation $p^3 + q^y = z^3$ with Distinct Odd Primes p, q when $y > 3$

Table 1.

p	$3p^2 + 3p$	q	y	q^y	$q^y – (3p^2 + 3p) = t$
3	36	5	4	625	589
5	90	3	5	243	153
7	168	3	5	243	75
11	396	5	4	625	229
13	546	5	4	625	79
17	918	3	7	2187	1269
19	1140	3	7	2187	1047
23	1656	3	7	2187	531
29	2610	5	5	3125	515
31	2976	5	5	3125	149
37	4218	3	8	6561	2343
41	5166	3	8	6561	1395
43	5676	3	8	6561	885

As a consequence of the data presented in Table 1, unequivocally, it then follows that the value t is not equal to 1. In Table 1 we have considered the first thirteen consecutive primes p, and accordingly for each p, the respective prime q and value y as mentioned earlier. For all these primes p, the number t satisfies $t \geq 75$. If D denotes the number of digits of the number t, then for all numbers t we have that $D \geq 2$. Observing that no value t even has one digit ($D = 1$), not to say the least of all values $D = 1$, namely $t = 1$, it follows that $t = 1$ is never attained.
We can therefore state that the equation $p^3 + q^y = z^3$ has no solutions.

This concludes the proof of Theorem 2.1. □

Final remark. In this paper, we have provided a short concise summary on the equation $p^3 + q^y = z^3$ when p, q are distinct odd primes and $y = 1, 2, 3$. Some solutions were exhibited when $y = 1$ and $y = 2$. We have also established closure to the above equation for all possible values $y > 3$ when p, q are distinct odd primes.
We note that to the best of our knoledge, other authors have not considered equations of the kind $p^3 + q^y = z^3$. It is therefore obvious, that no references concerning such equations can be provided.

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