Generalized Penalty Method for a Class of Variational-hemivariational Inequality

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Abstract. In this paper, we consider a class of variational-hemivariational inequality problems with constraints in a reflexive Banach space. This inequality problem involves two nonlinear operators and two nondifferentiable functionals. We introduce the penalty parameter and the penalty operator and change the initial problem into the penalty one, and then use the generalized penalty method to prove the existence result of the solution to the objective inequality.

Keywords: Variational-hemivariational inequality, Nonlinear operators, Generalized penalty method, Existence result.

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1. Introduction

It is very important for us to study contact problems which are always described by variational or hemivariational inequalities that are usually arisen in Mechanics, Physics and Engineering for contact processes are very common phenomena both in life and industry. The theories of variational inequalities, which involve arguments of monotonicity and convexity that including properties of the subdifferential of a convex function, were firstly studied in the sixties and have been widely developed since then, cf. ([1]-[6]). The conception of hemivariational inequalities have been introduced in the early 1980s by Panagiotopoulos’ pioneering works. The studies of this class of inequalities are usually based on properties of the subdifferential in the sense of Clarke and defined for locally Lipschitz functions which may be nonconvex, cf. ([7]-[12]).

Recently, variational-hemivariational inequalities have been studied by more and more researchers. This class of inequalities involves both convex and nonconvex functions, see ([13]-[19]). Our main in this paper is to present a generalized penalty method in the study of a class of variational-hemivariational inequality. The penalty method is effective in the numerical solution of constrained problems. It is a useful tool in proving the existence of the solution to constrained problems, see ([20]-[22]). And the generalized penalty method is the generalization of the penalty method, cf. ([23,24]). The penalty method has also been used to research the history-dependent variational or hemivariational
inequalities, in which convergence of the penalty method is shown as the penalty parameter goes to zero.

The rest of the paper is organized as follows. In Section 2, we recall some basic notions and definitions. In Section 3, we introduce a class of variational-hemivariational inequality, and prove the existence result through the generalized penalty method.

2. Preliminaries
In this section, we recall some definitions and results we need in this paper. More details can be found in the references [25].

For a normed space $V$, we denote by $V^*$ its topological dual. We use the notations $\| \cdot \|_V$ and $\| \cdot \|_{V^*}$ for the norms of the spaces $V$ and $V^*$, respectively. $\langle \cdot , \cdot \rangle_{V^* \times V}$ represents the dual pairing between $V^*$ and $V$. Also, the symbols $\to$ and $\rightharpoonup$ stand for the strong and weak convergence in various spaces, respectively.

We shall consider single-valued operator $A : V \to V^*$. The following definitions hold.

Definition 1. An operator $A : V \to V^*$ is called pseudomonotone, if it is bounded and $u_n \to u$ in $V$ together with $\limsup\limits_{n \to \infty} \langle Au_n , u_n - u \rangle \leq 0$ imply $\langle Au , u - v \rangle \leq \liminf\limits_{n \to \infty} \langle Au_n , u_n - v \rangle$ for all $u \in V$.

Proposition 1. For a reflexive Banach space $V$, the following statements hold.
(a) If the operator $A : V \to V^*$ is bounded, demicontinuous and monotone, then $A$ is pseudomonotone.
(b) If $A, B : V \to V^*$ are pseudomonotone operators, then the sum $A + B : V \to V^*$ is pseudomonotone.

Definition 2. For a locally Lipschitz function $f : V \to \mathbb{R}$, we denote by $j^0(u ; v)$ the generalized (Clarke) directional derivative of $f$ at the point $u \in V$ in the direction $v \in V$ defined by

$$f^0(u ; v) = \limsup\limits_{y \to u, \lambda \to 0} \frac{f(y + \lambda v) - f(y)}{\lambda}$$

The generalized gradient or subdifferential of $f$ at $u$, denoted by $\partial f(u)$, is a subset of the dual space $V^*$ given by $\partial f(u) = \{ \xi \in V^* | f^0(u ; v) \geq \langle \xi , v \rangle_{V^* \times V}, \forall v \in V \}$.

Proposition 2. Let $V$ be a real Banach Space, $f : V \to \mathbb{R}$ be a locally Lipschitz function. For all $(u, v) \in V \times V$, $(u_n , v_n) \in V \times V$ such that $(u_n , v_n) \to (u, v)$ in $V \times V$, we have limsup $f^0(u_n ; v_n) \leq f^0(u ; v)$.

3. The main result
Let $\Omega$ be an open bounded subset of $\mathbb{R}^d$ with boundary $\partial \Omega$ which is Lipschitz continuous. $\Gamma$ be a measurable subset of $\partial \Omega$. We denote by $x$ a generic point in $\Gamma$ and $m(\Gamma)$ the $(d - 1)$ dimensional measure of $\Gamma$. Given an integer $s > 0$. We use notation $V$ for a closed subset of $H^s(\Omega; \mathbb{R}^d)$, where $H = L^2(\Omega; \mathbb{R}^d)$. We denote $\gamma : V \to L^2(\Gamma; \mathbb{R}^d)$ the trace operator, $\| \gamma \|$ the norm in the space $(V, L^2(\Gamma; \mathbb{R}^d))$. $(V, H, V^*)$ forms evolution triple of spaces and the embedding $V \subset H$ is compact, and $K$ is a subset of $V$. Given operators
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\( A: V \to V^* \), functions \( \varphi, j: \Gamma \times \mathbb{R}^s \) and a functional \( f: V \to \mathbb{R} \), we consider the following problem. 

Problem \((P)\). Find an element \( u \in K \) such that
\[
(\langle Au, v - u \rangle + \int_{\Gamma} \varphi(yv) - \varphi(yu) \, d\Gamma + \int_{\Gamma} j^0(yu; yv - yu) \, d\Gamma) \geq (f, v - u) \quad \forall v \in K.
\]

We introduce the following hypotheses.

\( H(K) \): \( K \) is a nonempty, closed and convex subset of \( V \).

\( H(A) : V \to V^* \) is

- (a) pseudomonotone and there exists \( \alpha > 0 \) such that \( \langle Av, v \rangle_{V^* \times V} \geq \alpha \| v \|_{V^*} \quad \forall v \in V ; \)
- (b) strongly monotone, i.e., there exists \( m_A > 0 \) such that
  \[
  \langle Av_1 - Av_2, v_1 - v_2 \rangle_{V^* \times V} \geq m_A \| v_1 - v_2 \|_V \quad \forall v_1, v_2 \in V .
  \]

\( H(\varphi) : \varphi: \Gamma \times \mathbb{R}^s \to \mathbb{R} \) is such that

- (a) \( \varphi(\cdot, \xi) \) is measurable on \( \Gamma \) for all \( \xi \in \mathbb{R}^s \) and there exists \( \bar{\varphi} \in L^2(\Gamma; \mathbb{R}^s) \) such that \( \varphi(\cdot, \xi) \in L^2(\Gamma) ; \)
- (b) \( \varphi(x, \cdot) \) is convex for a.e. \( x \in \Gamma ; \)
- (c) there exists \( L_\varphi > 0 \) such that for all \( \xi_1, \xi_2 \in \mathbb{R}^s , \)
  \[
  |\varphi(x, \xi_1) - \varphi(x, \xi_2)| \leq L_\varphi \| \xi_1 - \xi_2 \|_{\mathbb{R}^s} \quad \text{a.e. } x \in \Gamma .
  \]

\( H(j) : j: \Gamma \times \mathbb{R}^s \to \mathbb{R} \) is such that

- (a) \( j(\cdot, \xi) \) is measurable on \( \Gamma \) for all \( \xi \in \mathbb{R}^s \) and there exists \( \bar{\varphi} \in L^2(\Gamma; \mathbb{R}^s) \) such that \( j(\cdot, e(\cdot)) \in L^2(\Gamma) ; \)
- (b) \( j(x, \cdot) \) is locally Lipschitz on \( \mathbb{R}^s \) for \( x \in \Gamma ; \)
- (c) \( \| \partial j(x, \xi) \|_{\mathbb{R}^s} \leq c_0 + c_1 \| \xi \|_{\mathbb{R}^s} \) for a.e. \( x \in \Gamma , \) for all \( \xi \in \Gamma \) with \( c_0, c_1 \geq 0 ; \)
- (d) \( j^0(x, \xi_1; \xi_2 - \xi_1) + j^0(x, \xi_2; \xi_1 - \xi_2) \leq \beta \| \xi_1 - \xi_2 \|_{\mathbb{R}^s}^2 \) for a.e. \( x \in \Gamma , \) all \( \xi_1, \xi_2 \in \mathbb{R}^s \) with \( \beta \geq 0 ; \)
- (e) \( j^0(x, \xi_1 - \xi) \leq d(1 + \| \xi \|_{\mathbb{R}^s}) \) for all \( \xi \in \mathbb{R}^s \) a.e. \( x \in \Gamma \) with \( d \geq 0 \).

\( H(f) : f \in V^* . \)

\( H(s) : \beta \| y \|_{\mathbb{R}^s}^2 < m_A . \)

Note that Problem \((P)\) is governed by a set of constraints \( K \). Therefore, it is useful to approximate it by a penalty method. Here we introduce the following generalized penalty problem.

Problem \((P_n)\). Find an element \( u_n \in V \) such that
\[
(\langle Au_n, v - u_n \rangle + \frac{1}{\lambda_n}(P_n u_n, v - u_n) + \int_{\Gamma} \varphi(yv) - \varphi(yu_n) \, d\Gamma + \int_{\Gamma} j^0(yu_n; yv - yu_n) \, d\Gamma) \geq (f, v - u_n) \quad \forall v \in V.
\]

For the study, we introduce the following assumptions.

\( H(\lambda_n) : \)

- (a) \( \lambda_n > 0 \) for all \( n \in \mathbb{N} ; \)
- (b) \( \lambda_n \to 0 \) as \( n \to \infty . \)

\( H(P_n) : P_n: V \to V^* \) is bounded, demicontinuous, monotone and coercive for all \( n \in \mathbb{N} . \)

\( (H_1) : \) For each \( v \in K \), there exists a sequence \( \{ \nu_n \} \subset V \) such that \( P_n \nu_n = 0_{V^*} \) for each \( n \in \mathbb{N} \) and \( \nu_n \to v \in V \) as \( n \to \infty . \)
Theorem 1. Assume $H(A), H(\varphi), H(j), H(f), H(s), H(\lambda_n)(a)$ and $H(P_n)$. Then Problem 
($P_n$) has a unique solution $\mathbf{u} \in \mathbf{V}$.

Proof. Let $n \in \mathbb{N}$. Assumptions $H(\lambda_n), H(P_n)$ and Proposition 1 (a) imply that the 
operator $\frac{1}{\lambda_n} P_n : \mathbf{V} \to \mathbf{V}^*$ is pseudomonotone. Assumption $H(A)$ on the operator $A$ and 
Proposition 1 (b) shows that the operator $A_n : \mathbf{V} \to \mathbf{V}^*$ defined by $A_n = A + \frac{1}{\lambda_n} P_n$ is 
pseudomonotone, too. From that $P_n$ is monotone and $\lambda_n > 0$, using assumption $H(A)$ we 
deduce that $A_n$ is strongly monotone with constant $m_A$. We can conclude from above that the 
operator $A_n$ satisfies condition $H(A)$, too. Similar to the, we can prove that Problem 
($P_n$) has a unique solution $u_n$. $\square$

Theorem 2. Assume $H(A), H(\varphi), H(j), H(f), H(s), H(\lambda_n), H(P_n), (H_1), (H_2)$ and ($H_3$). 
Then Problem ($P$) has a solution.

Proof. We prove this theorem in the following several steps. 

Step 1. The sequence $\{u_n\}$ is bounded. Let $v \in K$. Assume that $n \in \mathbb{N}$ is fixed, then 
assumptions $H(j)(c), H(j)(d), (H_2)$ and Proposition 2 guarantee that

$$
\int f(x) \varphi(v_n - u_n) dx \leq \int f(x) \varphi(v_n - u_n) dx + \int f(x) \varphi(v_n - y u_n) dx
$$

where $\xi \in \partial \varphi(v_n)$. Therefore,

$$
\int f(x) \varphi(v_n - y u_n) dx \leq \beta \| u_n - v_n \|_{\mathbf{V}}^2 + \int \max(\xi, u_n - v_n) \| u_n - v_n \|_{\mathbf{V}}
$$

On the other hand, we use $H(f)(a)$ and $H(\varphi)(a)$ to see that

$$
\int f(x) \varphi(v_n) - \varphi(y u_n) dx \leq \beta \| u_n - v_n \|_{\mathbf{V}}^2 + (c_0 m(\Gamma) + c_1 \| \gamma \|) \| u_n - v_n \|_{\mathbf{V}}.
$$

Next, we test with $v = u_n \in V$ in (3.1.) and take into account the fact that $P_n v_n = 0_{\mathbf{V}^*}$ and 
hypotises $H(A), (H_1)$ and $H(P_n)$ to see that

$$
m_A \| u_n - v_n \|_{\mathbf{V}}^2 \leq \langle A v_n, v_n - u_n \rangle + \frac{1}{\lambda_n} \langle P_n u_n - P_n v_n, v_n - u_n \rangle
$$

By virtue of (3.2) and (3.3), we find that
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\[ m_A \| u_n - v_n \| \leq \| Av_n - f \| \| v \| \| u_n - v_n \| + L_\varphi \| \gamma \| \| u_n - v_n \| \]

\[ + \beta \| \gamma \|^2 \| u_n - v_n \| + (c_0 m(\Gamma) + c_1 \| \gamma \|) \| u_n - v_n \|. \]

Therefore,

\[ (m_A - \beta \| \gamma \|^2) \| u_n - v_n \| \leq \| Av_n - f \| \| v \| + L_\varphi \| \gamma \| + (c_0 m(\Gamma) + c_1 \| \gamma \|) \| v \|. \]

We use \((H_2)\) to see that \(\{v_n\}\) is bounded. Combine \(H(A)(a), H(f)\) and \(H(s)\), we know that there exists a positive constant \(C\) such that \(\| u_n - v_n \| < C\), i.e., \(\{u_n\}\) is bounded in \(V\).

Since \(V\) is reflexive Banach space, then there exists an element \(\tilde{u} \in V\) such that, the subsequence of \(\{u_n\}\), we also presented by \(\{u_n\}\), converges weakly to \(\tilde{u}\), i.e.,

\[ u_n \rightharpoonup \tilde{u} \in V. \]  

\textbf{Step 2.} \(\tilde{u} \in K\). We use (3.1) to see that

\[ \frac{1}{\lambda_n} \langle P_n u_n, u_n - v \rangle \leq \langle A u_n - A v, v - u_n \rangle + \langle A v - f, v - u_n \rangle + \int_{\Gamma} \varphi (\gamma v) - \varphi(\gamma u_n) \, d\Gamma + \int_{\Gamma} (\gamma u_n; \gamma v - \varphi u_n) \, d\Gamma. \]

From \(H(A)\), we know that \(A\) is monotone, then

\[ \frac{1}{\lambda_n} \langle P_n u_n, u_n - v \rangle \leq \| Av - f \| \| u_n - v \| + L_\varphi \| \gamma \| \| u_n - v \| \]

\[ + (c_0 m(\Gamma) + c_1 \| \gamma \|) \| u_n - v \|. \]

We now use the boundedness of \(\{u_n\}\) to see that, there exists a positive constant \(C_1(v)\) depend on \(v\) such that

\[ \frac{1}{\lambda_n} \langle P_n u_n, u_n - v \rangle \leq C_1(v). \]

We use \(H(\lambda_n)\) to see that

\[ \limsup_{n \to \infty} \langle P_n u_n, u_n - v \rangle \leq 0. \]

Let \(v = \tilde{u} \in V\) in (3.5), we get

\[ \limsup_{n \to \infty} \langle P_n u_n, u_n - \tilde{u} \rangle \leq 0. \]

(3.6)

Then, using (3.4), (3.6) and \((H_2)(a)\), we have

\[ \liminf_{n \to \infty} \langle P_n u_n, u_n - v \rangle \geq \langle P \tilde{u}, \tilde{u} - v \rangle \text{ for all } v \in V. \]

(3.7)

The inequalities (3.5) and (3.7) together imply that \(\langle P \tilde{u}, \tilde{u} - v \rangle \leq 0 \quad \forall v \in V\). Let \(v = \tilde{u} - w, w \in V\), we have \(\langle P \tilde{u}, w \rangle = 0, w \in V\). Thus, \(P \tilde{u} = 0\), and by \((H_2)(b), \tilde{u} \in K\).

\textbf{Step 3.} \(\tilde{u} \in K\) is a solution to Problem \((P)\). Let \(v\) be a given element in \(V\). From \((H_2)\), we have that for each \(v \in K\), there exists a sequence \(\{v_n\} \subset V\) such that \(P_n v_n = 0\), for each \(n \in N\). Then, let \(v = v_n\) in (3.1), we have

\[ \langle A u_n, u_n - v_n \rangle \leq \frac{1}{\lambda_n} \langle P_n u_n - P_n v_n, v_n - u_n \rangle + \int_{\Gamma} \varphi(\gamma v_n) - \varphi(\gamma u_n) \, d\Gamma \]

\[ + \int_{\Gamma} (\gamma u_n; \gamma v_n - \varphi u_n) \, d\Gamma + (f, u_n - v_n). \]

Using \(H(P_n)\), we infer that

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\[
\langle Au_n, u_n - v_n \rangle \leq \langle f, u_n - v_n \rangle + \int_{\Gamma} \varphi(yv_n) - \varphi(yu_n)\,d\Gamma \\
+ \int_{\Gamma} j^0(yu_n; yv_n - yu_n)\,d\Gamma.
\]

(3.8)

Then, using (34) and assumptions \((H_2)(a),(H_3)\) we find that

\[
\limsup_{n \to \infty} \int_{\Gamma} \varphi(yv_n) - \varphi(yu_n)\,d\Gamma \leq \int_{\Gamma} \limsup_{n \to \infty} (\varphi(yv_n) - \varphi(yu_n))\,d\Gamma \\
\leq \int_{\Gamma} \varphi(yv - y\bar{u})\,d\Gamma.
\]

(3.9)

Moreover,

\[
\limsup_{n \to \infty} \int_{\Gamma} j^0(yu_n; yv_n - yu_n)\,d\Gamma \leq \int_{\Gamma} \limsup_{n \to \infty} j^0(yu_n; yv_n - yu_n)\,d\Gamma \\
\leq \int_{\Gamma} j^0(y\bar{u}; yv - y\bar{u})\,d\Gamma.
\]

(3.10)

Next,

\[
\limsup_{n \to \infty} \langle f, u_n - v_n \rangle = \langle f, \bar{u} - v \rangle.
\]

(3.11)

From (3.8)-(3.11), we have that

\[
\limsup_{n \to \infty} \langle Au_n, u_n - v_n \rangle \leq \int_{\Gamma} \varphi(yv) - \varphi(y\bar{u})\,d\Gamma + \int_{\Gamma} j^0(y\bar{u}; yv - y\bar{u})\,d\Gamma + \langle f, \bar{u} - v \rangle.
\]

(3.12)

Next, \(H(A), (H_1),(3.4)\) and imply that \(\langle Au_n, v - u_n \rangle \to 0\), as \(n \to \infty\). Hence, writing

\[
\langle Au_n, u_n - v_n \rangle = \langle Au_n, u_n - v \rangle + \langle Au_n, v - u_n \rangle.
\]

We deduce that

\[
\limsup_{n \to \infty} \langle Au_n, u_n - v_n \rangle = \limsup_{n \to \infty} \langle Au_n, u_n - v \rangle.
\]

This inequality combine with inequality (3.12) yields

\[
\limsup_{n \to \infty} \langle Au_n, u_n - v \rangle \leq \int_{\Gamma} \varphi(yv) - \varphi(y\bar{u})\,d\Gamma + \int_{\Gamma} j^0(y\bar{u}; yv - y\bar{u})\,d\Gamma + \langle f, \bar{u} - v \rangle.
\]

(3.13)

for all \(v \in K\). Now, choosing \(v = \bar{u} \in K\) in (3.13) and using Proposition 2 we obtain that

\[
\liminf_{n \to \infty} \langle Au_n, u_n - v \rangle \geq \langle \bar{A}u, \bar{u} - v \rangle.
\]

(3.14)

for all \(v \in V\). We use (3.13) and (3.14) to see that

\[
\langle \bar{A}u, \bar{u} - v \rangle \leq \int_{\Gamma} \varphi(yv) - \varphi(y\bar{u})\,d\Gamma + \int_{\Gamma} j^0(y\bar{u}; yv - y\bar{u})\,d\Gamma + \langle f, \bar{u} - v \rangle.
\]

for all \(v \in K\). This means \(\bar{u}\) is a solution to Problem\((P)\), i.e., \(\bar{u} = u\). \(\square\)

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REFERENCES


