A Further Study on $L$-Fuzzy Covering Rough Sets

Yao-liang Xu$^1$, Dan-dan Zou$^2$ and Ling-qiang Li$^3$

Department of Mathematics, Liaocheng University
Liaocheng – 252059, Liaocheng, China
Email: $^2$zoudandan0706@126.com, $^3$lixingqiang0614@126.com
Corresponding author. Email: $^1$xuyaoliang06@126.com

Received 4 April 2022; accepted 28 May 2022

Abstract. For $L = (L,*,→)$ a complete residuated lattice, a type of $L$-fuzzy covering rough sets was defined by Li [8] in 2017. In this paper, a further study on rough sets was given. Precisely, a single axiomatic characterization of the $L$-fuzzy covering rough sets was presented, and the relationships between the $L$-fuzzy covering rough sets and $L$-fuzzy relation rough sets were established.

Keywords: $L$-fuzzy rough set; $L$-fuzzy covering; Residuated lattice; Axiomatic characterization

AMS Mathematics Subject Classification (2010): 03E72, 94D05

1. Introduction

Rough set theory [15,16], proposed by Pawlak in 1982, is a new tool for dealing with uncertain and incomplete knowledge. The classical Pawlak rough sets are based on equivalent relations, which greatly limits the scope of rough set theory and applications. Hence, many kinds of generalized rough sets were proposed [7, 11, 12, 24-28]. The core concept of (generalized) rough set theory is a pair of approximation operators. There are generally two different approaches to studying those operators: the constructive approach and the axiomatic approach. In the constructive approach, the binary relation, covering, and neighborhood (system) in the domain of discourse are regarded as the original concepts, and the lower (upper) approximation operator is constructed from them. In the axiomatic approach, a pair of abstract approximation operators are put as the initial concepts, then find an axiom set (or even a single axiom) to ensure the existence of binary relation, covering and neighborhood (system) to reproduce the initial approximation operators by the constructive approach.

Fuzzy covering (relation) rough sets are vital generalized rough sets [2-4, 10, 11, 13]. Especially, complete residuated lattice-valued fuzzy covering (relation) rough sets have attracted much attention for their many-valued logic background [1, 8, 9, 14, 17-22] (complete residuated lattice can be regarded as the truth table of many-valued logic [5]). In 2017, consider $L = (L,*,→)$ a complete residuated lattice, Li [8] introduced a type of $L$-fuzzy covering rough sets and used axiom sets to characterize them. However, a more interesting single axiomatic characterization has not been given. In addition, it is known
Yao-liang Xu, Dan-dan Zou and Ling-qiang Li

from [28] that the covering rough sets and relation rough sets are interrelated closely. Nowadays, the relationships between $L$-fuzzy covering rough sets and $L$-fuzzy relation rough sets have not been clarified. In this paper, we shall give a further study on $L$-fuzzy covering rough set around the above two problems.

The arrangement of this paper is as follows. In Section 2, We recall some the basic concepts and symbols. In Section 3, we give a single axiomatic characterization of the $L$-fuzzy covering approximation operators. In Section 4, we clarify the relationship between $L$-fuzzy covering approximation operators and $L$-fuzzy relation approximation operators. In Section 5, we conclude this paper.

2. Preliminaries

In this section, we shall recall some notions and notation for later use.

2.1. Complete lattice $L$ and $L$-fuzzy sets

Let $X$ be a nonempty set. And we use $P(X)$ to denote the power set of $X$, i.e., $P(X) = \{ A \mid A \subseteq X \}$.

A complete residuated lattice is an algebra $L = (L, \land, \lor, *, 0, 1)$ fulfills:

1. $L = (L, \land, \lor, 0, 1)$ is a complete lattice with the least (resp., largest) element $0$ (resp., 1),
2. $(L, *, 1)$ is a commutative monoid with $1$ as the unit element,
3. $*$ distributes over arbitrary joins, that is, $\forall a, a_j (j \in J) \in L, a * (\lor a_j) = \lor (a * a_j)$.

The binary operation $\to : L \times L \to L$ determined by $a \to b = \lor \{ c \in L \mid a * c \leq b \}$ is called the residuated implication w.r.t. $*$.

A mapping $A : X \to L$ is called an $L$-fuzzy set on $X$ [6]. All $L$-fuzzy sets on $X$ are denoted by $X^L$. For $A \in P(X)$, we also use $A$ to denote the $L$-fuzzy set values at $x \in A$ and $0$ otherwise.

An $L$-fuzzy set $R$ on $X \times X$ is called an $L$-fuzzy relation on $X$.

Proposition 2.1. Let $L$ be a complete residuated lattice.

1. $1 \to a = a, 1 * a = a, a \leq b \to a, a \leq b \Leftrightarrow a \to b = 1$.
2. $(a * b) \to c = a \to (b \to c) = b \to (a \to c)$.
3. $\quad a \to \bigwedge_{j \in J} b_j = \bigwedge_{j \in J} (a \to b_j)$.
4. $\lor a_j \to b = \bigwedge_{j \in J} (a_j \to b)$.
5. $a * \bigwedge_{j \in J} b_j \leq \bigwedge_{j \in J} (a * b_j)$.
6. $\bigvee_{j \in J} (a \to b_j) \leq a \to \bigvee_{j \in J} b$,.
7. $\bigvee_{j \in J} (a_j \to b) \leq \bigwedge_{j \in J} a_j \to b$.

If we have $(a \to 0) \to 0 = a$ for all $a \in L$, then we say the complete residuated lattice $L$ is regular.

Definition 2.1. [8] For $A, B \in L^X$, we define
A Further Study on L-Fuzzy Covering Rough Sets

\[ S_{\rightarrow}(A, B) = \bigwedge_{x \in X} \left( A(x) \rightarrow B(x) \right), \]

\[ I_{\leftarrow}(A, B) = \bigvee_{x \in X} \left( A(x) \ast B(x) \right), \]

and call them the \( \rightarrow \)-subsetshood degree and \( \ast \)-intersection degree of \( A, B \), respectively.

**Lemma 2.1.** [8] For \( A, B \in L^X, a \in L \).

(O1) \( S_{\rightarrow}(A, B) = 1 \iff A \leq B \). (O2) \( S_{\rightarrow}(A, a \rightarrow B) = a \rightarrow S_{\rightarrow}(A, B) \).

(O3) \( S_{\rightarrow}(A, a \ast B) \geq a \ast S_{\rightarrow}(A, B) \).

2.2 \( L \)-fuzzy relation rough sets via \( \ast \) and \( \rightarrow \)

**Definition 2.2.** [19,20] Let \( R \) be an \( L \)-fuzzy relation on \( X \). For any \( A \in L^X \), the pair of \( L \)-fuzzy sets \((R_{\rightarrow}, R_{\leftarrow})\) defined by \( \forall x \in X \),

\[ R_{\rightarrow}(A)(x) = \bigwedge_{y \in X} \left( R(x, y) \rightarrow A(y) \right) = S_{\rightarrow}(R(x), A), \]

\[ R_{\leftarrow}(A)(x) = \bigvee_{y \in X} \left( R(x, y) \ast A(y) \right) = I_{\leftarrow}(R(x), A), \]

is said to be the \( L \)-fuzzy relation rough set of \( A \). The associated mappings \( R_{\rightarrow} \) and \( \bar{R}_{\leftarrow} \) on \( L^X \) are called the lower and upper \( L \)-fuzzy relation approximation operators, respectively.

2.2. Covering rough sets and \( L \)-fuzzy covering rough sets

**Definition 2.3.**[10] (1) Let \( C \in P(X) \). If \( \forall C = X \) and \( K \neq \emptyset \) for each \( K \in C \), then \( C \) is called a covering on \( X \). (2) Let \( C \subseteq L^X \). If \( \forall C = 1 \) and \( K \neq 0 \) for each \( K \in C \), then \( C \) is called an \( L \)-fuzzy covering on \( X \).

**Definition 2.4.**[28] Let \( C \) be a covering on \( X \). For any \( x \in X \), the family

\[ Md(x) = \{ x \in K \in C \mid \forall S \in C, x \in S \cap S \subseteq K \Rightarrow K = S \} \]

is called the minimal description of \( x \). Furthermore, \( C \) is called unary if for any \( x \in X \), \( |Md(x)| = 1 \), i.e., there is only one element in \( Md(x) \). It is easily observed that \( C \) is unary iff \( \cap\{ K \in C : x \in K \} \subseteq C \) for all \( x \in X \).

**Definition 2.5** [12] Let \( C \) be a covering on \( X \). For \( A \in P(X) \), the pair of subsets \((C(A), \bar{C}(A))\) defined by

\[ C(A) = \cup\{ K \mid K \in C, K \subseteq A \}, \bar{C}(A) = \cap\{ K' \mid K \in C, A \subseteq K' \}, \]

where \( K' = X - K \) is said to be the covering rough set of \( A \). The associated mappings \( C \) and \( \bar{C} \) on \( P(X) \) are called the lower and upper covering approximation operators, respectively.

In [8], Li defined a type of \( L \)-fuzzy covering rough sets, which can be regarded as the generalization of that in Definition 2.5.
Definition 2.6. [8] Let \( C \) be an \( L \)-fuzzy covering on \( X \) and \( A \in L^X \). Then the pair of \( L \)-fuzzy sets \((C_+(A), \overline{C}_+(A))\) defined by:
\[
C_+(A) = \bigvee_{K \in C} (K \ast S_+(K, A)), \quad \overline{C}_+(A) = \bigwedge_{K \in C} (K \rightarrow I_+(K, A))
\]
is said to be the \( L \)-fuzzy covering rough sets of \( A \), and \( C_+ \) (resp., \( \overline{C}_+ \)) is called the \( L \)-fuzzy covering lower (resp., upper) approximation operator.

Li also used axiom sets to characterize the lower and upper \( L \)-fuzzy covering approximation operators.

Proposition 2.2. [8] For a mapping \( p : L^X \rightarrow L^X \), there is an \( L \)-fuzzy covering \( C \) on \( X \) s.t. \( p = C_+ \) iff \( p \) satisfies
\[
(L1) \quad p(1) = 1, \quad (L2) \quad A \leq B \Rightarrow p(A) \leq p(B) \quad \text{for all} \ A, B \in L^X, \quad (L3) \quad p(A) \leq A \quad \text{for all} \ A \in L^X, \quad (L4) \quad p(A) \leq pp(A), \quad (L5) \quad a \ast p(A) \leq p(a \ast A).
\]

Proposition 2.3. [8] Let \( L \) be a regular complete residuated lattice and \( h : L^X \rightarrow L^X \) be a mapping. Then there is an \( L \)-fuzzy covering \( C \) on \( X \) s.t. \( h = C_+ \) iff \( h \) satisfies
\[
(U1) \quad h(0) = (0), \quad (U2) \quad A \leq B \Rightarrow h(A) \leq h(B) \quad \text{for all} \ A, B \in L^X, \quad (U3) \quad h(A) \geq A \quad \text{for all} \ A \in L^X, \quad (U4) \quad h(A) \geq hh(A) \quad \text{for all} \ A \in L^X, \quad (U5) \quad a \rightarrow h(A) \geq h(a \rightarrow A).
\]

3. The single axiomatic characterizations on \( L \)-fuzzy covering approximation operators
In this section, we shall use a single axiom to characterize the \( L \)-fuzzy covering approximation operators.

3.1. On lower approximation operator \( C_+ \)
A mapping \( \Phi : L^X \rightarrow L^X \) is called order-preserving if for any \( A, B \in L^X, A \leq B \) implies \( \Phi(A) \leq \Phi(B) \).

Lemma 3.1. Let \( \Phi : L^X \rightarrow L^X \) be an order-preserving mapping. Then the following two conditions are equivalent.
\[
(1) \quad a \ast \Phi(A) \leq \Phi(a \ast A) \quad \text{for any} \quad a \in L, A \in L^X.
\]
\[
(2) \quad a \rightarrow \Phi(A) \geq \Phi(a \rightarrow A) \quad \text{for any} \quad a \in L, A \in L^X.
\]
Proof: \((1) \Rightarrow (2)\). It follows by \( a \ast \Phi(a \rightarrow A) \leq \Phi(a \ast (a \rightarrow A)) \leq \Phi(A) \), which means \( \Phi(a \rightarrow A) \leq a \rightarrow \Phi(A) \).

\((2) \Rightarrow (1)\). It follows by \( a \rightarrow \Phi(a \ast A) \geq \Phi(a \rightarrow (a \ast A)) \geq \Phi(A) \), which means \( \Phi(a \ast A) \leq a \ast \Phi(A) \). □
Remark 3.1. From Lemma 3.1, it is noted that the condition (L5) in Proposition 2.2 can be restated equivalently as: $p(a \to A) \leq a \to p(A)$ for all $a \in L, A \in L^X$.

Lemma 3.2. Let $p : L^X \to L^X$ be a mapping. Then (L2)+(L5) $\Rightarrow$ (POD): $\forall A, B \in L^X, S_\rightarrow(A, B) \leq S_\rightarrow(p(A), p(B))$.

Proof. Let $A, B \in L^X$ and $a = S_\rightarrow(A, B)$. Then

$$\forall x \in X, a \leq A(x) \to B(x) \Rightarrow \forall x \in X, a * A(x) \leq B(x) \Rightarrow a * A \leq B$$,

by (L2)

$$\Rightarrow h(a * A) \leq h(B)$$, by (L5) $\Rightarrow a * h(A) \leq h(a * A) \leq h(B) \Rightarrow a \leq S_\rightarrow(h(A), h(B))$.

$\Leftarrow$. Let $A, B \in L^X$ and $a \in L$.

If $A \leq B$, then it follows by Lemma 2.1(O1) and (POD) that

$$1 = S_\rightarrow(A, B) \leq S_\rightarrow(h(A), h(B)) \Rightarrow h(A) \leq h(B),$$

which means (L2) holds. In addition, note that

$$1 = S_\rightarrow(a \to A, a \to A) = a \to S_\rightarrow(a \to A, A) \leq a \to S_\rightarrow(h(a \to A), h(A)) = S_\rightarrow(h(a \to A), a \to h(A)),$$

which means $h(a \to A) \leq a \to h(A)$, i.e., (L5) holds. $\square$

Theorem 3.1. (The characterization by a single axiom) Let $p : L^X \to L^X$ be a mapping. Then there is an $L$-fuzzy covering $C$ on $X$ s.t. $p = C_\rightarrow$ iff it satisfies (POM): for any index set $T$ and any $A, B_i (i \in T) \in L^X$,

$$p(1) \wedge \bigwedge_{i \in T} S_\rightarrow(p(A_i), B_i) = \bigwedge_{i \in T} S_\rightarrow(p(A_i), p(B_i)).$$

Proof: From Proposition 2.2, we need only to verify that (POM)$\iff$ (L1)-(L5).

$\Rightarrow$. (L1): Take $T = \emptyset$ in (POM), we get $p(1) = 1$ since $\wedge \emptyset = 1$ for $\emptyset \subseteq L$.

For any $A, B \in L^X$, put $T = \{1\}$ and $A_1 = A, B_1 = B$, then applying (L1) in (POM), we have that $(POM^+): S_\rightarrow(p(A), B) = S_\rightarrow(p(A), p(B))$.

(L3): Take $A = B$ in $(POM^+)$, it follows by Lemma 2.1 (O1) that $S_\rightarrow(p(A), A) = 1$, i.e., $p(A) \leq A$.

(L2)+(L5): Applying (L3) in $(POM^+)$ we get

$$S_\rightarrow(A, B) \leq S_\rightarrow(p(A), B) = S_\rightarrow(p(A), p(B)),$$

i.e., (POD) holds. It follows by Lemma 3.2 that (L2) and (L5) holds.

(L4): Take $B = p(A)$ in $(POM^+)$, it follows by Lemma 2.1(O1) that

$$S_\rightarrow(p(A), pp(A)) = 1,$$

i.e., $p(A) \leq pp(A)$.

$\Leftarrow$. At first, we prove that (L2)-(L5) implies $(POM^+)$. Indeed, for any $A, B \in L^X$,
Yao-liang Xu, Dan-dan Zou and Ling-qiang Li

\[ S_\gamma(p(A), B)^{(L_2)+(L_5)=(POD)} \leq S_\gamma(pp(A), p(B))^{(L_4)} \leq S_\gamma(p(A), p(B)) \leq S_\gamma(p(A), B)^{(L_3)}. \]

Hence, \( S_\gamma(p(A), B) = S_\gamma(p(A), p(B)), \) i.e., \((POM^+)\) holds. Then together \((POM^+)\) and \((L1)\) we obtain \((POM)\). □

3.2. On upper approximation operator \(\overline{C}_\to\)

The following lemma is just a restatement of Lemma 3.2 by replacing \(p\) with \(h\).

**Lemma 3.3.** Let \(h: L^X \to L^X\) be a mapping. Then \((U2)+(U5)\)\(\iff\) \((HOD): \forall A, B \in L^X, S_\gamma(A, B) \leq S_\gamma(h(A), h(B)).\)

**Remark 3.2.** From Lemma 3.3, it is noted that the condition \((U5)\) in Proposition 2.3 can be restated equivalently as: \(h(a \ast A) \geq a \ast h(A)\) for all \(a \in L, A \in L^X\).

**Theorem 3.2.** (The characterization by a single axiom) Let \(L\) be a regular complete residuated lattice and \(h: L^X \to L^X\) be a mapping. Then there is an \(L\)-fuzzy covering \(C\) on \(X\) s.t. \(h=\overline{C}_\to\) iff it satisfies \((HOM)\): for any index set \(T\) and any \(A_t, B_t (t \in T) \in L^X\),

\[ h(0) \vee \bigvee_{t \in T} S_\gamma(A_t, h(B_t)) = \bigvee_{t \in T} S_\gamma(h(A_t), h(B_t)). \]

**Proof:** From Proposition 2.3, we need only to verify that \((HOM) \iff (U1)-(U5).\)

\(\Rightarrow.\) (U1): Take \(T = \emptyset\) in \((HOM)\), we get \(h(0) = 0\) since \(\bigvee_{T} = 0\) for \(\emptyset \subseteq L\).

For any \(A, B \in L^X\), put \(T = \{1\}\) and \(A_1 = A, B_1 = B\), then applying (U1) in \((HOM)\), we have that \((HOM^-) : S_\gamma(A, h(B)) = S_\gamma(h(A), h(B)).\)

(U3): Take \(A = B\) in \((HOM^-)\), it follows by Lemma 2.1 (O1) that \(S_\gamma(A, h(A)) = 1\), i.e., \(A \leq h(A)\).

(U2)+(U5): Applying (U3) in \((HOM^-)\), we get \(S_\gamma(A, B) \leq S_\gamma(A, h(B)) = S_\gamma(h(A), h(B)),\) i.e., \((HOD)\) holds. It follows by Lemma 3.3 that \((U2)\) and \((U5)\) holds.

(U4): Take \(A = h(B)\) in \((HOM^-)\), it follows by Lemma 2.1 (O1) that \(S_\gamma(hh(B), h(B)) = 1\), i.e., \(hh(B) \leq h(B)\).

\(\Leftarrow.\) At first, we prove that \((U2)-(U5)\) implies \((HOM^-)\). Indeed, for any \(A, B \in L^X\),

\[ S_\gamma(A, h(B))^{(U_2)+(U_5)=(HOD)} \leq S_\gamma(h(A), hh(B))^{(U_4)} \leq S_\gamma(h(A), h(B))^{(L_3)} \leq S_\gamma(A, h(B)). \]

Hence, \(S_\gamma(A, h(B)) = S_\gamma(h(A), h(B)),\) i.e., \((HOM^-)\) holds. Then together \((HOM^-)\) and \((U1)\) we obtain \((HOM)\). □
A Further Study on L-Fuzzy Covering Rough Sets

4. The relationships between L-fuzzy covering approximation operators and L-fuzzy relation approximation operators

In this section, we shall prove that some special L-fuzzy covering approximation operators and L-fuzzy relation approximation operators can be mutually induced.

An L-fuzzy relation \( R \) on \( X \) is called reflexive whenever \( \forall x \in X, R(x, x) = 1 \); and \( * \)-transitive whenever \( \forall x, z \in X, \bigvee_{y \in X} (R(x, y) \ast R(y, z)) \leq R(x, z) \).

**Definition 4.1.** Let \( R \) be a reflexive L-fuzzy relation on \( X \). Then the family 
\[
C^R = \{ R(x) \in L^X \mid x \in X \}, \text{where } \forall y \in X, R(x)(y) = R(x, y)
\]
forms an L-fuzzy covering on \( X \), called the L-fuzzy covering induced by \( R \).

The following theorem shows that \( R \) and \( C^R \) yield the same L-fuzzy approximation operators if \( R \) is reflexive and \( * \)-transitive.

**Theorem 4.1.** Let \( R \) be a reflexive and \( * \)-transitive L-fuzzy relation on \( X \). Then 
\[
C^R = R \quad \text{and} \quad \overline{C^R} = \overline{R}.
\]

**Proof:** Let \( A \in L^X \). We prove below that \( C^R(A) = R(A) \) and \( \overline{C^R}(A) = \overline{R}(A) \).

For any \( x \in X \), note that 
\[
C^R(A)(x) = \bigvee_{y \in X} (R(x, y) \ast S_{\rightarrow}(R(x), A)) = \bigvee_{y \in X} (R(x, y) \ast S_{\rightarrow}(R(y), A)) \geq R(x, x) \ast S_{\rightarrow}(R(x), A), \text{by reflexivity}
\]
\[
= 1 \ast S_{\rightarrow}(R(x), A) = R(A)(x).
\]

Conversely, 
\[
C^R(A)(x) = \bigvee_{y \in X} (R(y, x) \ast S_{\rightarrow}(R(y), A)) = \bigvee_{y \in X} (R(y, x) \ast \bigwedge_{z \in X} (R(y, z) \rightarrow A(z))) \leq \bigvee_{y \in X} \bigwedge_{z \in X} (R(y, x) \ast (R(y, z) \rightarrow A(z))), \text{by } *-\text{transitivity}
\]
\[
\leq \bigvee_{y \in X} \bigwedge_{z \in X} (R(y, z) \rightarrow A(z)) \ast (R(y, z) \rightarrow A(z)) \leq \bigwedge_{z \in X} (R(x, z) \rightarrow A(z)) = R(A)(x).
\]

Hence \( C^R(A) = R(A) \).

For any \( x \in X \), note that 
\[
\overline{C^R}(A)(x) = \bigwedge_{k \in C^R} (K(x) \rightarrow I_*(K, A)) = \bigwedge_{y \in X} (R(y, x) \rightarrow I_*(R(y), A)) \leq R(x, x) \rightarrow I_*(R(x), A), \text{by reflexivity}
\]
\[
= I_*(R(x), A) = \overline{R}(A)(x).
\]

Conversely,
Yao-liang Xu, Dan-dan Zou and Ling-qiang Li

\[ \overline{C}_R(A)(x) = \bigwedge_{y \in X} \left( R(y,x) \Rightarrow \bigvee_{z \in X} \left( R(y,z) \ast A(z) \right) \right) \]

\[ \geq \bigwedge_{y \in X} \bigvee_{z \in X} \left( R(y,x) \Rightarrow \left( R(y,z) \ast A(z) \right) \right), \text{by \ast-transitivity} \]

\[ \geq \bigvee_{z \in X} \left( R(x,z) \Rightarrow R(y,z) \Rightarrow \left( R(y,z) \ast A(z) \right) \right) \]

\[ \geq \bigvee_{z \in X} \left( R(x,z) \ast A(z) \right) = \overline{R}_r(A)(x). \]

Hence \( \overline{C}_R(A) = \overline{R}_r(A) \). \( \square \)

Let \( C \) be an \( L \)-fuzzy covering on \( X \). It is well known that the \( L \)-fuzzy relation \( R^C \) on \( X \) defined by \( \forall x, y \in X, R^C(x,y) = \bigwedge_{K \in C} (K(x) \rightarrow K(y)) \) is reflexive and \( \ast \)-transitive, and called the \( L \)-fuzzy relation induced by \( L \)-fuzzy covering \( C \).

**Definition 4.2.** An \( L \)-fuzzy covering \( C \) on \( X \) is called unary if \( R^C(x) \subseteq C \) for any \( x \in X \).

**Remark 4.1.** When \( L = \{0,1\} \), an \( L \)-fuzzy covering \( C \) degenerates into a crisp covering, and the \( L \)-fuzzy set \( R^C(x) \) degenerates into a crisp set \( \bigwedge \{ K \in C : x \in K \} \). Hence from Definition 2.4, we know that the notion of unary \( L \)-fuzzy covering \( C \) is a generalization of the corresponding crisp notion.

The next theorem shows that \( C \) and \( R^C \) yield the same \( L \)-fuzzy approximation operators if \( C \) is unary.

**Theorem 4.2.** Let \( C \) be an unary \( L \)-fuzzy covering on \( X \). Then \( R^C_x = C_x \) and \( \overline{C}_R = \overline{R}_r \).

**Proof:** Let \( A \in L_X \). We prove that \( R^C_x(A) = C_x(A) \) and \( \overline{C}_R(A) = \overline{R}_r(A) \).

Note that for any \( x \in X \),

\[ R^C_x(A)(x) = \bigwedge_{y \in X} \left( R^C(x,y) \Rightarrow A(y) \right) = \bigwedge_{y \in X} \left( K(x) \Rightarrow K(y) \Rightarrow A(y) \right) \]

\[ \geq \bigwedge_{y \in X} \bigvee_{K \in C} \left( K(x) \Rightarrow K(y) \Rightarrow A(y) \right) \geq \bigwedge_{y \in X} \bigvee_{K \in C} \left( K(x) \ast (K(y) \Rightarrow A(y)) \right) \]

\[ \geq \bigvee_{K \in C} \left( K(x) \ast \bigvee_{y \in X} \left( K(y) \Rightarrow A(y) \right) \right) = C_x(A)(x), \]

which means \( R^C_x(A) \geq C_x(A) \). On the other hand,
A Further Study on $L$-Fuzzy Covering Rough Sets

$$C_x(A)(x) = \bigvee_{y \in X} \left( K(x) \ast \bigwedge_{y \in X} \left( K(y) \rightarrow A(y) \right) \right), \text{by unary condition}$$

$$K=K_x$$

$$= 1 \ast \bigwedge_{y \in X} \left( R^C(x, y) \rightarrow A(y) \right)$$

which means $C_\rightarrow(A) \geq R^C_\rightarrow(A)$. Hence $C_\rightarrow = R^C_\rightarrow$.

Note that for any $x \in X$,

$$
\bar{R}_\rightarrow(A)(x) = \bigvee_{y \in X} \left( R^C(x, y) \ast A(y) \right) = \bigvee_{y \in X} \left( \bigwedge_{k \in C} \left( K(x) \rightarrow K(y) \right) \ast A(y) \right)
$$

$$\geq \bigvee_{y \in X} \bigwedge_{k \in C} \left( \left( K(x) \rightarrow K(y) \right) \ast A(y) \right) \leq \bigvee_{y \in X} \left( \bigwedge_{k \in C} \left( K(x) \rightarrow K(y) \right) \ast A(y) \right)$$

which means $\bar{C}_\rightarrow(A) \geq \bar{R}_\rightarrow(A)$. On the other hand,

$$\bar{C}_\rightarrow(A)(x) = \bigwedge_{y \in X} \left( K(x) \rightarrow \bigvee_{y \in X} \left( K(y) \ast A(y) \right) \right), \text{by unary condition}$$

$$K=K_x$$

$$= 1 \rightarrow \bigvee_{y \in X} \left( R^C(x, y) \ast A(y) \right)$$

which means $\bar{C}_\rightarrow(A) \leq \bar{R}_\rightarrow(A)$. Hence $\bar{C}_\rightarrow = \bar{R}_\rightarrow$. □

5. Concluding remarks

In this paper, a further study on $L$-fuzzy covering rough sets was given. The single axiom characterization on $L$-fuzzy covering approximation operators was presented, and the connections between $L$-fuzzy covering approximation operators and the $L$-fuzzy relation approximation operators were constructed.

Acknowledgments. This work is supported by the National Natural Science Foundation of China (No. 12171220) and the Natural Science Foundation of Shandong Province (No. ZR2020MA042).

The authors are also thankful to the reviewers for their critical comments on the improvement of the paper.

Conflict of interest. The authors declare that they have no conflict of interest.

Authors’ Contributions. All the authors contributed equally to this work.

REFERENCES


A Further Study on $L$-Fuzzy Covering Rough Sets