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Minimal Equitable Dominating Symmetric *n*-Sigraphs

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Abstract. An n-tuple $(a_1, a_2, \ldots, a_n)$ is symmetric, if $a_k = a_{n-k+1}, 1 \le k \le n$. Let $H_n = \{(a_1, a_2, \ldots, a_n): a_k \in \{+,-\},\ a_k = a_{n-k+1},\ 1 \le k \le n\}$ be the set of all symmetric n-tuples. A symmetric n-sigraph (symmetric n-marked graph) is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the underlying graph of S_n and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function. In this paper, we introduced a new notion minimal equitable dominating symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Also, we obtained the structural characterization of minimal equitable dominating symmetric n-signed graphs.

Keywords: Symmetric n-sigraphs, Symmetric n-marked graphs, Balance, Switching, Minimal equitable dominating symmetric n-sigraphs, Complementation.

AMS Mathematics Subject Classification (2010): 05C22

1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [1]. We consider only finite, simple graphs free from self-loops.

Let $n \ge 1$ be an integer. An *n-tuple* $(a_1, a_2, \ldots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \le k \le n$. Let $H_n = \{(a_1, a_2, \ldots, a_n) : a_k \in \{+,-\}, a_k = a_{n-k+1}, 1 \le k \le n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G,\sigma)$ ($S_n = (G,\mu)$), where $G = (V,E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \rightarrow H_n$ ($\mu : V \rightarrow H_n$) is a function.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric *n*-tuple/symmetric *n*-sigraph/symmetric *n*-marked graph.

An *n*-tuple $(a_1, a_2, ... ,a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an *n*-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity *n*-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an *n*-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the *n*-tuples on the edges of *A*.

In [7], the authors defined two notions of balance in *n*-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P. S. K. Reddy [3]):

Definition 1.1. Let $S_n = (G, \sigma)$ be an *n*-sigraph. Then,

(i) S_n is *identity balanced* (or *i-balanced*), if product of *n*-tuples on each cycle of S_n is the identity *n*-tuple, and

(ii) S_n is *balanced*, if every cycle in S_n contains an even number of non-identity edges.

Note: An *i*-balanced *n*-sigraph need not be balanced and conversely.

The following characterization of *i*-balanced *n*-sigraphs is obtained in [7].

Theorem 1.1. (**Sampathkumar et al. [7]**) An *n*-sigraph $S_n = (G, \sigma)$ is *i*-balanced if, and only if, it is possible to assign *n*-tuples to its vertices such that the *n*-tuple of each edge uv is equal to the product of the *n*-tuples of u and v.

Let $S_n = (G, \sigma)$ be an *n*-sigraph. Consider the *n*-marking μ on vertices of S_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the *n*-tuple which is the product of the *n*-tuples on the edges incident with v. *Complement* of S_n is an *n-sigraph* $\overline{S_n} = (\bar{G}, \sigma^c)$, where for any edge $e = uv \in \bar{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ is defined here is an *i*-balanced *n-sigraph due to Theorem 1.1.*

In [7], the authors also have defined switching and cycle isomorphism of an *n*-sigraph $S_n = (G, \sigma)$ as follows: (See also [2, 4-6, 9–19])

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ be two *n*-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \rightarrow G'$ such that if *uv* is an edge in S_n with label $(a_1, a_2,... ,a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2,... ,a_n)$.

Given an *n*-marking μ of an *n*-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the *n*-tuple of every edge *uv* of S_n by $\mu(u)\sigma(uv)\mu(v)$. The *n*-sigraph obtained in this way is denoted by $S_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*.

Further, an *n*-sigraph S_n *switches* to *n*-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an *n*-marking of S_n such that $S_\mu(S_n) \cong S'_n$.

Two *n*-sigraphs $S_n = (G,\sigma)$ and $S'_n = (G',\sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the *n*-tuple $\sigma(C)$ of every cycle C in S_n equals to the *n*-tuple $\boldsymbol{\sigma}(\Phi(C))$ in S'_n.
We make use of the following known result (see [7]).

Theorem 1.2. (**Sampathkumar et al. [7]**) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

2. Minimal equitable dominating *n*-sigraph of an *n*-sigraph

A subset D of $V(G)$ is called an equitable dominating set of a graph G, if for every $v \in V - D$ there exists a vertex $v \in D$ such that $uv \in E(G)$ and $|d(u) - d(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ_e and is called equitable domination number of G. An equitable dominating set D is minimal, if for any vertex $u \in D$, $D - \{u\}$ is not an equitable dominating set of G. A subset S of V is called an equitable independent set, if for any $u \in S$, $v \notin N_e(u)$, for all $v \in S - \{u\}$. If If a vertex $u \in V$ be such that $|d(u) - d(v)| \leq 2$, for all $v \in N(u)$ then u is in equitable dominating set. Such vertices are called equitable isolates. An equitable dominating set D is minimal, if for any vertex $u \in D$, $D - \{u\}$ is not an equitable dominating set of G.

Let $\mathcal{F}$ be a finite set and $F = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n\}$ be a partition of $\mathcal{F}$. Then the intersection graph $\omega(F)$ is the graph whose vertices are the subsets in F and in which two vertices $\mathcal{F}_i$ and $\mathcal{F}_j$ are adjacent if and only if $\mathcal{F}_i \cap \mathcal{F}_j \neq \phi, i \neq j$.

Motivated by the existing definition of complement of an *n*-sigraph, we extend the notion of minimal equitable dominating graphs to *n*-sigraphs as follows: The minimal equitable dominating *n*-sigraph $MED(S_n)$ of an *n*-sigraph $S_n = (G, \sigma)$ is an *n*-sigraph whose underlying graph is $MED(G)$ and the *n*-tuple of any edge uv is $MED(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical *n*-marking of S_n. Further, an *n*-sigraph $S_n = (G, \sigma)$ is called minimal equitable dominating *n*-sigraph, if $S_n \cong MED(S'_n)$ for some *n*-sigraph S'_n. The following result indicates the limitations of the notion $MED(S_n)$ as introduced above, since the entire class of *i*-unbalanced *n*-sigraphs is forbidden to be minimal equitable dominating *n*-sigraphs.

Theorem 2.1. *For any n-sigraph $S_n = (G, \sigma)$, its minimal equitable dominating n-sigraph $MED(S_n)$ is i-balanced.*

Proof: Since the *n*-tuple of any edge uv in $MED(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical *n*-marking of S_n, by Theorem 1.1, $MED(S_n)$ is *i*-balanced.

For any positive integer k, the k^{th} iterated minimal equitable dominating *n*-sigraph, $MED^k(S_n)$ of S_n is defined as follows:

$$MED^0(S_n) = S,\ MED^k(S_n) = MED(MED^{k-1}(S_n)).$$

Corollary 2.2. *For any n-sigraph $S_n = (G, \sigma)$ and for any positive integer k, $MED^k(S_n)$ is i-balanced.*

The following result characterizes n-sigraphs which are minimal equitable dominating n-sigraphs.

Theorem 2.3. *An n-sigraph $S_n = (G, \sigma)$ is a* minimal equitable dominating *n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is a* minimal equitable dominating *graph.*

Proof: Suppose that S_n is i-balanced and G is a minimal equitable dominating graph. Then there exists a graph H such that $MED(H) \cong G$. Since S_n is i-balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge $e = uv$ in S_n satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the n-sigraph $S_n{'} = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $MED(S_n{'}) \cong S_n$. Hence S_n is a minimal equitable dominating n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a is a minimal equitable dominating n-sigraph. Then there exists an n-sigraph $S_n{'} = (H, \sigma')$ such that $MED(S_n{'}) \cong S_n$. Hence G is the minimal equitable dominating graph of H and by Theorem 2.1, S_n is i-balanced.

In [20], the authors characterized graphs for which $MED(G) \cong \bar{G}$.

Theorem 2.4. (**Sumathi & Soner [20]**)

For any graph $G=(V, E)$, $MED(G) \cong \bar{G}$ if and only if G is a complete with p vertices.

We now characterize n-sigraphs whose minimal equitable dominating n-sigraphs and complementary n-sigraphs are switching equivalent.

Theorem 2.5. *For any n-sigraph $S_n = (G, \sigma)$, $\overline{S_n} \sim MED(S_n)$ if, and only if, G is K_p.*

Proof: Suppose $\overline{S_n} \sim MED(S_n)$. This implies, $MED(G) \cong \bar{G}$ and hence by Theorem 2.4, G is K_p.

Conversely, suppose that G is K_p. Then $MED(G) \cong \bar{G}$ by Thorem 2.4. Now, if S_n is an n-sigraph with underlying graph G is K_p, by the definition of complementary n-sigraph and Theorem 2.1, $\overline{S_n}$ and $MED(S_n)$ are i-balanced and hence, the result follows from Theorem 1.2.

Theorem 2.6. *For any two n-sigraphs S_n and $S_n{'}$ with the same underlying graph, their minimal equitable dominating n-sigraphs are switching equivalent.*

Proof: Suppose $S_n = (G, \sigma)$ and $S_n{'} = (G', \sigma')$) be two n-sigraphs with $G \cong G'$. By Theorem 2.1, $MED(S_n)$ and $MED(S_n{'})$ are i-balanced and hence, the result follows from Theorem 1.2.

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2,.., a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the *m- complement* of an n-sigraph$S_n = (G, \sigma)$, written $({S_n}^m)$, is the same graph but with each edge label $a = (a_1, a_2,\ldots ,a_n)$ replaced by a^m.

For an n-sigraph $S_n = (G, \sigma)$, the $MED(S_n)$ is i-balanced. We now examine, the condition under which m-complement of $MED(S_n)$ is i-balanced, where for any $m \in H_n$.

Theorem 2.7. *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $MED(G)$ is bipartite then $(MED(S_n))^m$ is i-balanced.*

Proof: Since, by Theorem 2.1, $MED(S_n)$ is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $MED(S_n)$ whose k^{th} co-ordinate are − is even. Also, since $MED(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $MED(S_n)$ whose k^{th} co-ordinate are + is also even. This implies that the same thing is true in any m-complement, where for any $m \in H_n$. Hence $(MED(S_n))^t$ is i-balanced.

Theorem 2.6 provides easy solutions to other n-sigraph switching equivalence relations, which are given in the following results.

Corollary 2.8. *For any two n-sigraphs S_n and ${S_n}'$ with the same underlying graph, $MED(S_n)$ and $MED(({S_n}')^m)$ are switching equivalent.*

Corollary 2.9. *For any two n-sigraphs S_n and ${S_n}'$ with the same underlying graph, $MED((S_n)^m)$ and $MED({S_n}')$ are switching equivalent.*

Corollary 2.10. *For any two n-sigraphs S_n and ${S_n}'$ with the same underlying graph, $MED((S_n)^m)$ and $MED(({S_n}')^m)$ are switching equivalent.*

Corollary 2.11. *For any two n-sigraphs $S_n = (G, \sigma)$ and ${S_n}' = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $(MED(S_n))^m$ and $MED({S_n}')$ are switching equivalent.*

Corollary 2.12. *For any two n-sigraphs $S_n = (G,\sigma)$ and ${S_n}' = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $MED(S_n)$ and $MED(({S_n}')^m)$ are switching equivalent.*

Corollary 2.13. *For any two n-sigraphs $S_n = (G, \sigma)$ and ${S_n}' = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $(MED(S_1))^m$ and $(MED(S_2))^m$ are switching equivalent.*

3. Conclusion

We have introduced a new notion for n-signed graphs called minimal equitable dominating n-sigraph of an n-signed graph. We have proved some results and presented the structural characterization of minimal equitable dominating n-signed graph. There is no structural characterization of minimal equitable dominating graph, but we have obtained the structural characterization of minimal equitable dominating n-signed graph.

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