Journal of Mathematics and Informatics
Vol. 1, 2013-14, 89-93
ISSN: 2349-0632 (P), 2349-0640 (online)
Published on 2 July 2014
www.researchmathsci.org

Journal of
Mathematics and Informatics

Properties of the Limit Behaviors of Markov Integrated Semigroups

Yi-jin Zhang* and *Si-dong Xian

School of Mathematics and Physics
Chongqing University of Posts and Telecommunications
Chongqing 400065, P.R.China
E-mail: zhangyj@cqupt.edu.cn

Received June 1, 2014; accepted June15, 2014

Abstract: In this paper, some conditions and properties of limit behaviors of Markov integrated semigroup are examined, such as : If $T_{(t)}$ is the corresponding Markov integrated semigroup, then $T_{(t)}^{n}x \to T_{(t)}x(t \geq 0, x \in l_{\infty})$ as $n \to \infty$. If

$$\mathrm{X}_n e^{'} = 0\ ,\ \mathrm{n} = 1, 2, 3\cdots, \tilde{Q}_n^{\ l} \text{ is weakly ergodic, then } \lim_{\mathrm{n}\to\infty} \mathrm{X}_n = 0 .$$

If $\mathrm{X}_n e^{'} = 1$, $\mathrm{n} = 1, 2, 3\cdots, \tilde{Q}_n^{\ l}$ is strongly ergodic, $\lim_{n\to\infty} \tilde{Q}_{\mathrm{m},n}^{\ l} = e^{'} \bullet \pi^{(l+1)}, m \geq 0,$ then $\lim_{\mathrm{n}\to\infty} \mathrm{X}_n = \pi^{(l+1)}$.

***Keywords*:** limit behaviors; Markov integrated semigroup; $q-$matrix; weakly ergodic ; strongly ergodic

AMS Mathematics Subject Classification (2014): 60J20

1. Introduction

A one-dimensional Markov branching process [1] is a continuous-time Markov chain (briefly, CTMC) on a countable state space $E = \{0, 1, 2, \ldots\}$ whose stochastic evolution is governed by the branching property.Markov branching process q-matrix [1] is given by

$$\tilde{q}_{ij} = \begin{cases} ib_{j-i+1} & j \geq i-1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $b_k \geq 0, k \neq 1, \text{and} \sum_{k\geq 0} b_k \leq 0.$

Consider the Dual branching q-matrix Q which is a continuous-time Markov chains on the state space $E = Z_+ = \{0, 1, 2, \cdots\}$ and the q-matrix $Q = (q_{ij}, i, j \in E)$: Dual Markov branching process (DMBP) [1] $X(t)$ is a continuous-time Markov chain on the state space $Z^+ = \{0, 1, 2, \cdots\}$, where the q-matrix $Q = \{q_{ij}; i, j \in Z^+\}$, is given by [2]

$$q_{ij} = \begin{cases} ia_{i-j+1} - (j+1)a_{i-j} & i \ge j-1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $a_k = \sum_{j=0}^{k} b_j, k \ge 0, b_j$ is the sequence defined by branching q-Matrix $\tilde{Q}$. $a_0 \le 0 \quad a_1 \ge a_2 \ge \cdots \ge a_k \ge \cdots \ge 0$.

Let us consider a measured state space (X, A), where A is the σ-algebra on X。. Our aim in this paper is to discuss some conditions and properties of limit behaviors of Markov integrated semigroup.

For more unmentioned notations and preliminary, we refer to the References of this paper.

Limit problem is a very important one on Markov process and has been widely applied in many fields. It is well known that an uniformly convergence integrated semigroup can tend to a Markov integrated semigroup by iterations.

2. Main results

Theorem 1. Let $Q = (q_{ij}, i, j \in E)$ and ${}_nQ = ({}_nq_{ij}, i, j \in E)$ are both FRR q-matrices ,let $F(t) = (f_{ij}(t))$ and ${}_nF(t) = ({}_nf_{ij}(t))$ are both FRR transition functions, $\Phi(\lambda) = (\phi_{ij}(\lambda), i, j \in E)$ and ${}_n\Phi(\lambda) = ({}_n\phi_{ij}(\lambda), i, j \in E)$ are both their corresponding resolvent functions respectively, then the following are equivalent:

(i) for any $i, j \in E$, ${}_nq_{ij} \to q_{ij}$ as $n \to \infty$;

(ii) for any $i, j \in E$ and $t \ge 0$, ${}_nf_{ij}(t) \to f_{ij}(t)$ as $n \to \infty$;

(iii) for any $i, j \in E$ and $\lambda > 0$, ${}_n\phi_{ij}(\lambda) \to \phi_{ij}(\lambda)$ as $n \to \infty$.

Proof: (i)$\Rightarrow$(ii) We known that: ${}_n\Omega_x \to \Omega_x$, ${}_nQ$ and Q are both FRR q-matrices.

Letting $x = e_j = (0, 0, \cdots, 1, 0, \cdots)^T$,for a fixed i_0, we can obtain:

$$\begin{aligned} \|{}_n\Omega e_j - \Omega e_j\|_E &= \|{}_nQe_j - Qe_j\|_E \\ &= \|({}_nq_{ij})_i - (q_{ij})_i\|_E = \|({}_nq_{ij})_i - (q_{ij})_i\| \\ &= \sup_{i \in E} |{}_nq_{ij} - q_{ij}| = |{}_nq_{i_0j} - q_{i_0j}| \to 0 \end{aligned}$$

for any $\lambda > 0$.

$$\begin{aligned} 0 &\le \int_0^{+\infty} e^{-2\lambda t} |{}_nf_{ij}(t) - f_{ij}(t)| \, dt \\ &\le \int_0^{+\infty} e^{-2\lambda t} |{}_nf_{i_0j}(t) - f_{ij}(t)| \, dt \\ &\le |{}_nq_{ij} - q_{ij}| \le |{}_nq_{i_0j} - q_{i_0j}| \to 0 \end{aligned}$$

By the squeeze theorem for limit of functions, we must get ${}_nf_{ij}(t) \to f_{ij}(t)$ as $n \to \infty$.

(ii)$\Rightarrow$(iii): It can be completed easily by Lebesgue's theorem on control and convergence.

(iii) $\Rightarrow$ (i): Because ${}_nF(t) = ({}_nf_{ij}(t))$ and $F(t) = (f_{ij}(t))$ are both FRR transition functions, ${}_nF(t)$ and $F(t)$ are both positive strong continuous contraction semigroup .

Letting their generators are ${}_nA$ and A respectively, then we know that resolvent functions $\Phi(\lambda)=(\phi_{ij}(\lambda),i,j\in E)$ and ${}_n\Phi(\lambda)=({}_n\phi_{ij}(\lambda),i,j\in E)$ are just resolvent equation of operators ${}_nA$ and A respectively, namely $0<\lambda\in\rho({}_nA)=\rho(A)$ and $R(łambda:{}_nA)={}_n\Phi(\lambda)\quad R(\lambda:A)=\Phi(\lambda)$.

We will prove that for any $\lambda>0$, ${}_n\Phi(\lambda)x\to\Phi(\lambda)x$ as $n\to\infty$. Since ${}_nf_{ij}(t)$ and $f_{ij}(t)$ are both transition functions by [1], ${}_n\phi_{ij}(\lambda)$ and $\phi_{ij}(\lambda)$ are both FRR resolvent functions, there must exist the state i_0 ,such that $\sup_{i\in E}|{}_n\phi_{ij}(\lambda)-\phi_{ij}(\lambda)|=|{}_n\phi_{i_0j}(\lambda)-\phi_{i_0j}(\lambda)|$

Because span $\{e_j,j\in E\}$ is dense and

$$\begin{aligned}\|{}_n\Phi(\lambda)-\Phi(\lambda)\|&\le\|{}_n\Phi(\lambda)\|+\|\Phi(\lambda)\|\\&=\sup_{i\in E}\sum_{j\in E}|{}_n\phi_{ij}(\lambda)|+\sup_{i\in E}\sum_{j\in E}|\phi_{ij}(\lambda)|\\&\le\sup_{i\in E}\sum_{j\in E}|{}_n\phi_{i_0j}(\lambda)|+\sup_{i\in E}\sum_{j\in E}|\phi_{i_0j}(\lambda)|\le\frac{4}{\lambda}\end{aligned}$$

so ${}_n\Phi(\lambda)-\Phi(\lambda)$ is bounded .By the corresponding theorem in [2], we obtain :

$${}_nF(t)\to F(t)\qquad t\ge0\ n\to\infty$$

$$\begin{aligned}\sup_{i\in E}|{}_nq_{ij}-q_{ij}|&=|{}_nq_{i_0j}-q_{i_0j}|\\&\le|{}_nF(t)-F(t)|\to0\end{aligned}$$

Theorem 2. Suppose that $T^n_{(t)}$ is a series of MIS, $P^n_{(t)}$ is the corresponding transition function of $T^n_{(t)}$, $n\in N$, A_n is the generator of $P^n_{(t)}$ in l_1.If there exists λ_0 such that $Re\lambda_0>0$ and the range of operator $R(\lambda_0)$ is dense in l_1,then there exists a unique operator A: A will generate a contraction c_0 semigroup $P_{(t)}$ .If $T_{(t)}$ is the corresponding Markov integrated semigroup, then $T^n_{(t)}x\to T_{(t)}x(t\ge0,x\in l_\infty)$ as $n\to\infty$.

Proof: By [3, Theorem 4.4], we know that there exists a unique operator A: generating a contraction c_0 semigroup $P_{(t)}$. $P^n_{(t)}x\to P_{(t)}x\ t\ge0,x\in l_1$.

Letting $P_{ij}(t)=\left\langle e_iP(t),e_j\right\rangle$, we obtain that $P_{(t)}=(P_{ij}(t),i,j\in E)$ is a transition function, so we have

$$\left\langle\int_0^\infty e^{-\lambda t}e_iP_{(t)}dt,t\right\rangle=\left\langle e_i,\lambda\int_0^\infty e^{-\lambda t}T_{(t)}dt\right\rangle\tag{3}$$

$$\begin{aligned}\left\langle e_iP_{(t)},e_j\right\rangle&=\int_0^\infty e^{-\lambda t}\left\langle e_iP_{(t)},e_j\right\rangle dt\\&=\int_0^\infty e^{-\lambda t}P_{ij}(t)dt=\left\langle e_i,T_{(t)}e_j\right\rangle\end{aligned}$$

So $T_{(t)}^{n} x \to T_{(t)} x (t \geq 0, x \in l_{\infty})$ as $n \to \infty$. □

Now we introduce the definitions of weakly ergodic and strongly ergodic:

Definition 1. [5,6] A sequence of stochastic q-matrices ${}_nQ$ is said to be weakly ergodic if $\forall m \geq 0, \forall i,j,k \in S$, such that $\lim_{n\to\infty}[({}_nQ_m)_{ik} - ({}_nQ_m)_{jk}] = 0$.

Definition 2. [5,6] A sequence of stochastic q-matrices ${}_nQ$ is said to be strongly ergodic if $\forall m \geq 0, \forall i,j \in S$, the limit $\lim_{n\to\infty}({}_nQ_m)_{ij} =: (\pi_m)_j$ exists and does not depend on *i*.

With the above definitions and Theorems 1, 2 we can drawn the following result：

Theorem 3. Let $Q = (q_{ij}, i, j \in E)$ and ${}_nQ = ({}_nq_{ij}, i, j \in E)$ are both FRR q-matrices, X_n and R_n, $n = 1,2,3\cdots$, be two sequences of real vectors, $R_n e' = 0$, $\sum_{n=1}^{\infty} \|R_n\|_{\infty} < \infty$, and set $\tilde{Q}_n^{l} = Q_{(n+l)}$. Then the following statements hold:

(i) If $X_n e' = 0$, $n = 1,2,3\cdots$, $\tilde{Q}_n^{l}$ is weakly ergodic, then $\lim_{n\to\infty} X_n = 0$.

(ii) If $X_n e' = 1$, $n = 1,2,3\cdots$, $\tilde{Q}_n^{l}$ is strongly ergodic, $\lim_{n\to\infty} \tilde{Q}_{m,n}^{l} = e' \bullet \pi, m \geq 0,$ then $\lim_{n\to\infty} X_n = \pi.$

(iii) If $X_n e' = 1$, $n = 1,2,3\cdots$, $\tilde{Q}_n^{l}$ is strongly ergodic, $\lim_{n\to\infty} \tilde{Q}_{m,n}^{l} = e' \bullet \pi^{(l+1)}, m \geq 0,$ then $\lim_{n\to\infty} X_n = \pi^{(l+1)}.$

Proof: Applying the recurrence relation, we have

$$X_{t+l} = X_{l+1} \tilde{Q}^{l}_{0,t} + [R_{l+1} + \sum_{k=0}^{t-1} R_{l+k} \tilde{Q}^{l}_{k+1,t+1}] \qquad t \geq 1 \tag{4}$$

If $\tilde{Q}_n^{l}$ is weakly ergodic, we will prove that

$$\lim_{t\to\infty} \sum_{k=0}^{t-1} R_{l+k} \tilde{Q}^{l}_{k+1,t+1} = 0 \tag{5}$$

We have

$$\left\| \sum_{k=0}^{t-1} R_{l+k} \tilde{Q}^{l}_{k+1,t+1} \right\|_{\infty} \leq \sum_{k=0}^{t-1} \left\| R_{l+k} \tilde{Q}^{l}_{k+1,t+1} \right\|_{\infty} \leq \sum_{k=0}^{t-1} \|R_{l+k}\|_{\infty} \delta(\tilde{Q}^{l}_{k+1,t+1})$$

Choose $a_{tk}^{l} = \|R_{l+k}\|_{\infty} \delta(\tilde{Q}^{l}_{k+1,t+1})$, $t,k \geq 1$ (take $a_{tk}^{l} = 0$ if $k>t$) ,this follows that $\lim_{t\to\infty} a_{tk}^{l} = 0$. Since $\delta(\tilde{Q}^{l}_{k,t}) \leq 1$ $0 \leq k<t$,so $\lim_{t\to\infty} \sum_{k=1}^{t} a_{tk}^{l} = \lim_{t\to\infty} \sum_{k=1}^{\infty} a_{tk}^{l} = 0$, which means (5).

(i) If $X_n e' = 0$, $n = 1,2,3\cdots$, $\tilde{Q}_n^{l}$ is weakly ergodic, there exists a sequence of stable stochastic matrices $\prod_{m,t}^{l}$, it follows that $\lim_{t\to\infty}(\tilde{Q}^{l}_{m,t} - \prod_{m,t}^{l}) = 0$.

Letting $t \to \infty$ in (4), by the weakly ergodicity , $\sum_{n=1}^{\infty}\|R_n\|_{\infty} < \infty$ and (5) , we obtain

$$\lim_{t\to\infty} X_{t+l} = \lim_{t\to\infty}(\tilde{Q}^{l}{}_{0,t+l} - \prod{}^{l}_{0,t+l}) = 0$$

Since l is arbitrary, it follows that $\lim_{t\to\infty} X_{t+l} = 0$ that is to say $\lim_{n\to\infty} X_n = 0$

(ii) Letting $t \to \infty$ in (4) , by the strongly ergodicity $\sum_{n=1}^{\infty}\|R_n\|_{\infty} < \infty$ and (5), we obtain

$$\lim_{t\to\infty} X_{t+l} = \lim_{t\to\infty}(\tilde{Q}^{l}{}_{0,t+l} - \prod{}^{l}_{0,t+l}) = \pi$$

Since l is arbitrary, it follows that $\lim_{t\to\infty} X_{t+l} = \pi$ that is to say $\lim_{n\to\infty} X_n = \pi$.

(iii) Letting $t \to \infty$ in (4) , by the strongly ergodicity , $\sum_{n=1}^{\infty}\|R_n\|_{\infty} < \infty$ and (5), we obtain

$$\lim_{t\to\infty} X_{t+l} = \lim_{t\to\infty}(\tilde{Q}^{l}{}_{0,t+l} - \prod{}^{l}_{0,t+l}) = \pi^{(l+1)}$$

Since l is arbitrary, it follows that $\lim_{t\to\infty} X_{t+l} = \pi^{(l+1)}$,that is to say $\lim_{n\to\infty} X_n = \pi^{(l+1)}$.. □

3. Remark

The Markov chain mentioned in this paper is not strongly ergodic, but we can prove that it is C-strongly ergodic by similar methods.

Acknowledgements

This work was supported by the Scientific and Technological ResearchProgram of Chongqing Municipal Education Commission (KJ1400430).

REFERENCES

1. W.J.Anderson, *Continuous-Time Markov Chains*, Springer Series in Statistics, Springer-Verlag, New York, 1991.
2. M.F.Chen, *From Markov Chains to Non-Equilibrium Particle Systems*, Singapore, World Scientific, 1992.
3. Y. R. Li, Markov Integrated Semigroups and their Applications to Continuous-Time Markov Chains, *Integr. Equ.Oper. Theory,* 60 (2008) 247-269.
4. Y.-R, Li Contraction Integrated Semigroups and Their Application to Continuous-time Markov Chains, *Acta Math Sinica*, 19(3) (2003) 605-618.
5. Chen Anyue, Phil Pollett, Zhang Hanjun, LiJunping, The Collision Branching Process, *Journal of Appl. Prob*., 41(4) (2004) 1033-1048.
6. R.Delaubenfels, *Existence Families, Functional Calculi and Evolution Equations,* Springer-Verlag, 1994.
7. Y.R.Li, Dual and feller-reuter-riley transition functions, *J Math Anal Appl*., 313(2) (2006) 461-474.
8. Y.Zhang etc. markov integrated semigroups generated by dual branching q-matrix [J]. *Journal of Southwest University* (Natural science edition) 32(4) (2010) 120-123.