Journal of Mathematics and Informatics
Vol. 1, 2013-14, 25-33
ISSN: 2349-0632 (P), 2349-0640 (online)
Published on 1 October 2013
www.researchmathsci.org

Journal of
Mathematics and Informatics

On the solutions of a rational recursive sequence

Chang-you Wang [1, 2] **and *Min Hu*** [1, 3]

[1]Key Laboratory of Industrial Internet of Things & Networked Control of Ministry of Education, Chongqing University of Posts and Telecommunications, Chongqing 400065 P.R. China.

[2]College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065 P. R. China.
E-mail: wangchangyou417@163.com

[3]College of automatic, Chongqing University of Posts and Telecommunications, Chongqing 400065 P. R. China. E-mail: humin_0503@163.com

Received 1 August 2013; accepted 6 September 2013

***Abstract*.** In this paper, we study the global asymptotic behavior of the solutions for the rational recursive sequence

$$x_{n+1} = \frac{x_{n-p}}{A + x_{n-q}x_{n-r}}\ , \quad n = 0,1,\cdots,$$

where the p,q,r are nonnegative integers, and A is positive constants. The initial conditions $x_{-s}, x_{-s+1}, \cdots, x_{-1}, x_0$ are arbitrary nonnegative real numbers where $s = \max\{p,q,r\}$. Moreover, some numerical simulations are given to illustrate our results.

***Keywords*:** recursive sequence; equilibrium point; asymptotical stability; positive solutions

1. Introduction

As a discrete analogue and a numerical solution of differential or delay differential equation, difference equations have applications in various scientific branches such as biology, ecology, physiology, physics, engineering and economics, etc[1-5]. So recently there has been an increasing interest in the study of qualitative analysis of rational difference equations. And the present cardinal problem is about the globally asymptotic behavior of solutions for a rational difference equations [6-15].

Elabbasy et al. [16] deal with the behavior of the solution for the following recursive sequence:

$$x_{n+1} = ax_{n-1} - \frac{bx_n x_{n-1}}{cx_n + dx_{n-2}} \quad , \quad n = 0,1,\cdots \tag{1.1}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers and a,b,c,d are positive constants. In particular, Cinar[17, 18] studied respectively the properties of positive solutions for the following equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}} \quad , \quad n = 0,1,\cdots \tag{1.2}$$

$$x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}} \quad , \quad n = 0,1,\cdots \tag{1.3}$$

where the initial values x_{-1} and x_0 are the real numbers such that $x_{-1}x_0 \neq -1$ or $x_{-1}x_0 \neq 1$.

The main theorem in this note is motivated by the above studies. The essential problem we consider in this paper is the asymptotic behavior of the solution for the difference equation

$$x_{n+1} = \frac{x_{n-p}}{A + x_{n-q}x_{n-r}} \quad , \quad n = 0,1,\cdots, \tag{1.4}$$

where the initial conditions $x_{-s}, x_{-s+1}, \cdots, x_{-1}, x_0$ are arbitrary positive real numbers, p,q,r is nonnegative integer, $s = \max\{p,q,r\}$ and A are positive constants.

This paper proceeds as follows. In Section 2, we introduce some definitions and preliminary results. The main results and their proofs are given in Section 3. Finally, some numerical simulations are given in Section 4 to illustrate our theoretical results.

2. Preliminaries

In this section, some definitions and preliminary results [19, 20] which are used throughout this paper are given to prove the main results.

Lemma 2.1. Let I be some interval of real numbers and let

$$f : I^{k+1} \to I \tag{2.1}$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \cdots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}), \quad n = 0,1,\cdots \tag{2.2}$$

has a unique solution $\{x_n\}_{n=-k}^{+\infty}$.

Definition 2.1. If there exits a point $\bar{x} \in I$ such that $\bar{x} = f(\bar{x}, \bar{x}, \cdots, \bar{x})$, then $\bar{x}$ is called an equilibrium point of Eq.(2.2). That is, $x_n = \bar{x}$ for $n \geq 0$ is a solution of (2.2), or equivalently, $\bar{x}$ is a fixed point of f.

Definition 2.2. Let $\bar{x}$ be an equilibrium point of Eq. (2.2).

(i) The equilibrium $\bar{x}$ of Eq.(2.2) is locally stable if for every $\varepsilon > 0$, there exits $\delta > 0$ such that for any initial data $(x_{-m}, x_{-m+1}, \cdots, x_{-1}, x_0) \in I^{m+1}$ satisfying max $\{|x_{-m} - \bar{x}|, |x_{-m+1} - \bar{x}|, \cdots, |x_0 - \bar{x}|\} < \delta,\ |x_n - \bar{x}| < \varepsilon$ holds for all $n \geq -m$.

(ii) The equilibrium $\bar{x}$ of Eq.(2.2) is a local attractor if there exits $\delta > 0$ such that $\lim_{n\to\infty} x_n = \bar{x}$ for any $(x_{-m}, x_{-m+1}, \cdots, x_{-1}, x_0) \in I^{m+1}$ satisfying max $\{|x_{-m} - \bar{x}|,$ $|x_{-m+1} - \bar{x}|, \cdots, |x_0 - \bar{x}|\} < \delta$.

(iii) The equilibrium $\bar{x}$ of Eq.(2.2) is locally asymptotically stable if it is stable and is a local attractor.

(iv) The equilibrium $\bar{x}$ of Eq.(2.2) is a global attractor if for all $x_{-m}, x_{-m+1}, \cdots,$ $x_{-1}, x_0 \in I$, $\lim_{n\to\infty} x_n = \bar{x}$ holds.

(v) $\bar{x}$ is globally asymptotically stable if it is stable and is a global attractor.

(vi) $\bar{x}$ is unstable if it is not locally stable.

Definition 2.3. Let p, q be two nonnegative integers such that $p + q = n$. Splitting $x = (x_1, x_2, \cdots, x_n)$ into $x = ([x]_p, [x]_q)$, where $[x]_\sigma$ denotes a vector with σ-compo-nents of x, we say that the function $f(x_1, x_2, \cdots, x_n)$ possesses a mixed monotone property in subsets I^n of R^n if $f([x]_p, [x]_q)$ is monotone nondecreasing in each component of $[x]_p$ and is monotone nonincreasing in each component of $[x]_q$ for $x \in I^n$. In particular, if $q = 0$, then it is said to be monotone nondecreasing in I^n.

The linearized equation of (2.2) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, \cdots, \bar{x})}{\partial x_{n-i}} y_{n-i}. \tag{2.3}$$

Now assume that the characteristic equation associated with (2.3) is

$$P(\lambda) = a_0\lambda^k + a_1\lambda^{k-1} + \cdots + a_{k-1}\lambda + a_k = 0, \tag{2.4}$$

where $a_i = \partial f(\bar{x}, \bar{x}, \cdots, \bar{x}) / \partial x_{n-i}, i = 0, 1, 2, \cdots, k.$

Lemma 2.2. Assume that f is a C^1 function and let $\bar{x}$ be an equilibrium point of Eq.(2.2). Then the following statements are true.

(i) If all roots of the polynomial equation (2.4) lie in the open unite disk $|\lambda| < 1$, then the equilibrium point $\bar{x}$ of (2.2) is locally asymptotically stable.

(ii) If at least one root of (2.4) has absolute value greater than one, then the equilibrium point $\bar{x}$ of (2.2) is unstable.

3. The Main Results

Consider the system (1.4), if $A \geq 1$, system (1.4) has a unique equilibrium point $\bar{x} = 0$. In addition, if $A < 1$, then system (1.4) has two equilibrium points $\bar{x} = 0$ and $\bar{\bar{x}} = \sqrt{1 - A}$.

Let $f:(0,\infty)^3 \to (0,\infty)$ be a function defined by

$$f(u,v,w)=\frac{u}{A+vw} \tag{3.1}$$

then it follows that

$$f_u=\frac{1}{A+vw},\quad f_v=-\frac{uw}{(A+vw)^2},\quad f_w=-\frac{uv}{(A+vw)^2}. \tag{3.2}$$

As $\bar{x}$ and $\bar{\bar{x}}$ are the equilibrium points of (1.4), then we have

$$f_u(\bar{x},\bar{x},\bar{x})=\frac{1}{A},\quad f_v(\bar{x},\bar{x},\bar{x})=f_w(\bar{x},\bar{x},\bar{x})=0.$$

$$f_u(\bar{\bar{x}},\bar{\bar{x}},\bar{\bar{x}})=1,\quad f_v(\bar{\bar{x}},\bar{\bar{x}},\bar{\bar{x}})=f_w(\bar{\bar{x}},\bar{\bar{x}},\bar{\bar{x}})=A-1.$$

Thus, the linearized equations of (1.4) about equilibrium points $\bar{x}$ and $\bar{\bar{x}}$ are as follows

$$z_{n+1}=\frac{1}{A}z_{n-p} \tag{3.3}$$

$$z_{n+1}=z_{n-p}+(A-1)z_{n-q}+(A-1)z_{n-r} \tag{3.4}$$

where p,q,r is nonnegative integer.

The characteristic equations associated with (3.3) and (3.4) are

$$P(\lambda)=\lambda^{p+1}-\frac{1}{A}=0, \tag{3.5}$$

$$P(\lambda)=\lambda^{s-p}+(A-1)\lambda^{s-q}+(A-1)\lambda^{s-r}=0, \tag{3.6}$$

where $s=\max\{p,q,r\}$.

By Lemmas 2.2, we have the following results.

Theorem 3.1. If $A<1$, the equilibrium point $\bar{x}=0$ of (1.4) is unstable. Moreover, we have the following results.

(i) If all roots of the characteristic equation (3.6) lie in the open unite disk $|\lambda|<1$, then the equilibrium point $\bar{\bar{x}}$ of (1.4) is locally asymptotically stable.

(ii) If at least one root of (3.6) has absolute value greater than one, then the equilibrium point $\bar{\bar{x}}$ of (1.4) is unstable.

Theorem 3.2. If $A>1$, the equilibrium point $\bar{x}=0$ of (1.4) is locally asymptotically stable.

Proof. The linearized equations of (1.4) about equilibrium points $\bar{x}=0$ is

$$z_{n+1}=\frac{1}{A}z_{n-p}$$

where $A>1$, then $0<\frac{1}{A}<1$. By Lemmas 2.2, the equilibrium point $\bar{x}=0$ of (1.4) is locally asymptotically stable. And then the proof is complete.

Theorem 3.3. Let $[a,b]$ be an interval of real numbers and assume that $f:[a,b]^{k+1}\to[a,b]$ is a continuous function satisfying the mixed monotone property. If there exits

$$m_0\le\min\{x_{-k},x_{-k+1},\cdots,x_0\}\le\max\{x_{-k},x_{-k+1},\cdots,x_0\}\le M_0 \qquad (3.7)$$

such that

$$m_0\le f([m_0]_p,[M_0]_q)\le f([M_0]_p,[m_0]_q)\le M_0, \qquad (3.8)$$

then there exit $(m,M)\in[m_0,M_0]^2$ satisfying

$$M=f([M]_p,[m]_q),\quad m=f([m]_p,[M]_q). \qquad (3.9)$$

Moreover, if $m=M$, then (2.2) has a unique equilibrium point $\bar{x}\in[m_0,M_0]$ and every solution of (2.2) converges to $\bar{x}$.

Proof. Using m_0 and M_0 as a couple of initial iteration, we construct two sequences $\{m_i\}$ and $\{M_i\}$ $(i=1,2,\cdots)$ from the following equations

$$m_i=f([m_{i-1}]_p,[M_{i-1}]_q),\quad M_i=f([M_{i-1}]_p,[m_{i-1}]_q). \qquad (3.10)$$

It is obvious from the mixed monotone property of f that the sequences $\{m_i\}$ and $\{M_i\}$ possess the following monotone property

$$m_0\le m_1\le\cdots\le m_i\le\cdots\le M_i\le\cdots\le M_1\le M_0, \qquad (3.11)$$

where i=0,1,2,$\cdots$, and

$$m_i\le x_l\le M_i \quad \text{for } l\ge(k+1)i+1,\ i=0,1,2,\cdots. \qquad (3.12)$$

Set

$$m=\lim_{i\to\infty}m_i,\qquad M=\lim_{i\to\infty}M_i, \qquad (3.13)$$

then

$$m\le\liminf_{i\to\infty}x_i\le\limsup_{i\to\infty}x_i\le M. \qquad (3.14)$$

By the continuity of f , we have

$$M=f([M]_p,[m]_q),\qquad m=f([m]_p,[M]_q). \qquad (3.15)$$

Moreover, if $m=M$, then $m=M=\lim_{i\to\infty}x_i=\bar{x}$, and then the proof is complete.

Theorem 3.4. If $A>1$, the equilibrium point $\bar{x}=0$ of (1.4) is a global attractor for any initial conditions

$$(x_{-s},x_{-s+1},\cdots,x_{-1},x_0)\in(0,\sqrt{A-1})^{s+1}. \qquad (3.16)$$

Proof. Let $f:(0,\infty)^3\to(0,\infty)$ be a function defined by

$$f(u,v,w)=\frac{u}{A+vw} \qquad (3.17)$$

We can easily see that the function $f(u,v,w)$ is increasing in u and decreasing in v,w .

Let

$$M_0 = \max\{x_{-s}, x_{-s+1}, \cdots, x_{-1}, x_0\}, \qquad m_0 < 0, \tag{3.18}$$

we have

$$m_0 \le \frac{m_0}{A+M_0^2} \le \frac{M_0}{A+m_0^2} \le M_0. \tag{3.19}$$

Then from (1.4) and Theorem 3.3, there exit $m, M \in [m_0, M_0]$ satisfying

$$m = \frac{m}{A+M^2}, \qquad M = \frac{M}{A+m^2}, \tag{3.20}$$

thus

$$(A - Mm - 1)(M - m) = 0. \tag{3.21}$$

In view of $M_0 < \sqrt{A-1}$, we have

$$A - Mm - 1 > 0. \tag{3.22}$$

Then

$$M = m. \tag{3.23}$$

It follows by Theorem 3.3 that the equilibrium point $\overline{x} = 0$ of (1.4) is a global attractor. The proof is complete.

Theorem 3.5. If $A > 1$, the equilibrium point $\overline{x} = 0$ of (1.4) is global asymptotically stability for any initial conditions

$$(x_{-s}, x_{-s+1}, \cdots, x_{-1}, x_0) \in (0, \sqrt{A-1})^{s+1}. \tag{3.24}$$

Proof. The result follows from Theorems 3.2 and 3.4.

4. Numerical Simulations

In this section, some numerical simulations are given to support our theoretical analysis with the software package Matlab7.0. Such as, we consider the following difference equations:

$$x_{n+1} = \frac{x_{n-1}}{5 + x_{n-2}x_{n-3}}, \quad n = 0, 1, \cdots, \tag{4.1}$$

$$x_{n+1} = \frac{x_{n-1}}{2 + x_{n-1}x_{n-2}}, \quad n = 0, 1, \cdots, \tag{4.2}$$

$$x_{n+1} = \frac{x_{n-1}}{0.5 + x_{n-2}x_{n-3}}, \quad n = 0, 1, \cdots, \tag{4.3}$$

where the initial conditions of (4.1) $(x_{-3}, x_{-2}, x_{-1}, x_0) \in (0, 2)$, the initial conditions of (4.2) $(x_{-2}, x_{-1}, x_0) \in (0, 1)$ and the initial conditions of (4.3) $(x_{-3}, x_{-2}, x_{-1}, x_0) \in (0, +\infty)$. Let $m_0 = -0.5, M_0 = 2$, it is obvious that equations (4.1) and (4.2) satisfy the conditions of Theorems 3.5 and equation (4.3) satisfies the condition of Theorem 3.1.

By employing the software package MATLAB7.0, we can solve the numerical solutions of equations (4.1), (4.2) and (4.3) which are shown respectively in Figure 4.1, Figure 4.2 and 4.3. More precisely, Figure 4.1 shows the numerical solution of equation

(4.1) with $x(-3)=0.2, x(-2)=1.2,\ x(-1)=0.5, x(0)=0.8$ and the relations that $m_i \le x_l \le M_i$ when $l \ge 4i+1,\ i=0,1,2,\cdots$, Figure 4.2 shows the numerical solution of (4.2) with $x(-2)=0.2,\ x(-1)=0.6,\ x(0)=0.4$ and the relations that $m_i \le x_l \le M_i$ when $l \ge 3i+1,\ i=0,1,2,\cdots$ and Figure 4.3 shows the numerical solution of (4.3) with $x(-3)=0.2, x(-2)=0.6,\ x(-1)=0.4, x(0)=0.8$.

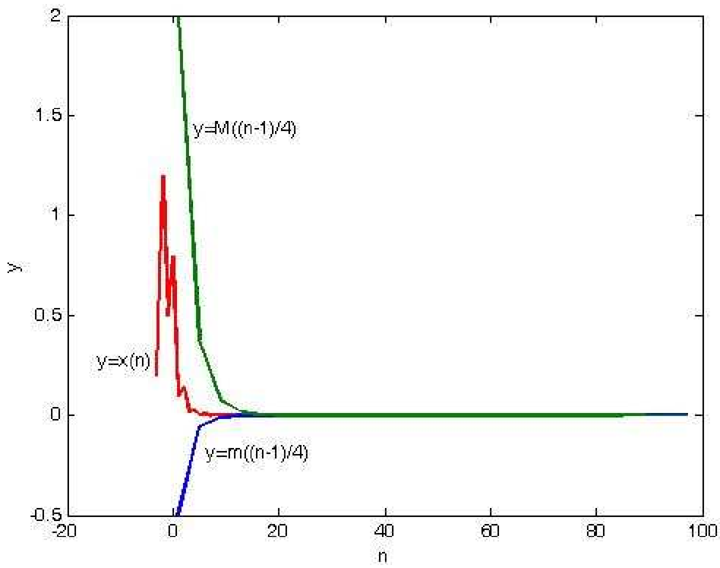

Figure 4.1: Chart of (4.1) with $x(-3)=0.2, x(-2)=1.2,\ x(-1)=0.5, x(0)=0.8$

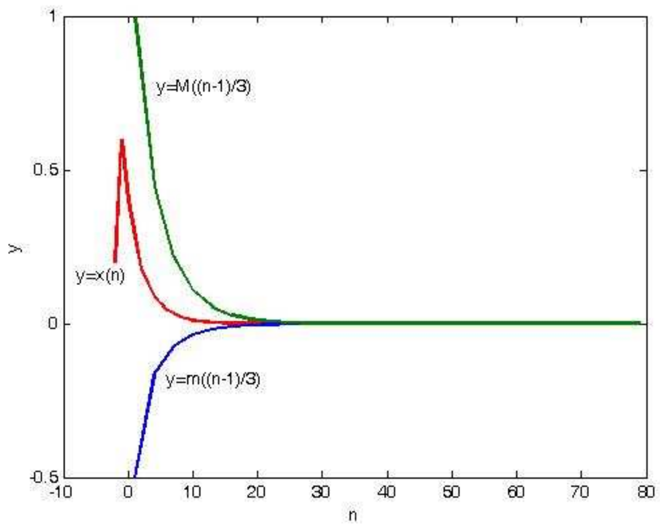

Figure 4.2: Chart of (4.2) with $x(-2)=0.2, x(-1)=0.6, x(0)=0.4$

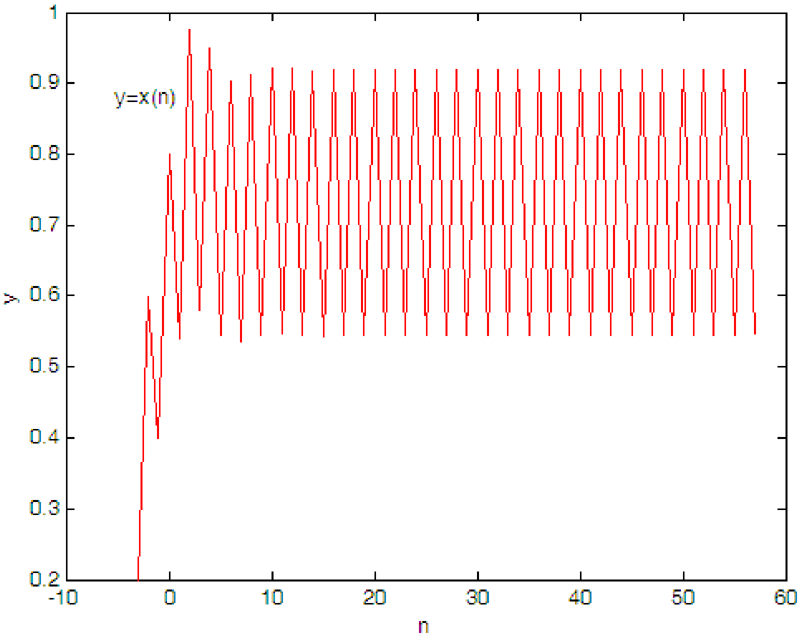

Figure 4.3: Chart of (4.3) with $x(-3)=0.2, x(-2)=0.6,\ x(-1)=0.4, x(0)=0.8$

5. Conclusions

This paper presents the use of a variational iteration method for systems of nonlinear difference equations. This technique is a powerful tool for solving various difference equations and can also be applied to other nonlinear differential equations in mathematical physics. The numerical simulations show that this method is an effective and convenient one. The variational iteration method provides an efficient method to handle the nonlinear structure. Computations are performed using the software package MATLAB7.0.

We have dealt with the problem of global asymptotic stability analysis for a class of nonlinear high order difference equation. The general sufficient conditions have been obtained to ensure the existence, global asymptotic stability and unstability of the equilibrium point for the nonlinear difference equation. These criteria generalize and improve some known results. In particular, some illustrate examples are given to show the effectiveness of the obtained results. In addition, the sufficient conditions that we obtained are very simple, which provide flexibility for the application and analysis of nonlinear difference equation.

Acknowledgments

This work is supported by the Natural Science Foundation Project of CQ CSTC (Grant nos. cstc2012jjA20016 and cstc2012jjA40035) of China, and the NSFC (Grant nos. 51005264 and 11101298) of China.

REFERENCES

1. R.P.Agarwal, Difference equations and inequalities, Marcel Eekker, New York, (2000).
2. L.Erbe, J.Baoguoa, A.Peterson, Nonoscillation for second order sublinear dynamic equations on time scales, J. Comput. Appl. Math, 232 (2009) 594-599.
3. Shojaei M, Saadati R, Adibi H: Stability and periodic character of a rational third order difference equation, Chaos, Solitons and Fractals, 39(3) (2009) 1203-1209.
4. C.Wang, S.Wang, X.Yan, Oscillation of a class of partial functional population model, J. Math. Anal. Appl., 368(1) (2010) 32-42.
5. W.T.Li, H.Sun, Global attractivity in a rational recursive sequence, Dyn. Syst. Appl., 11 (2002) 339–346.
6. L.X.Hu, W.S.He, H.M.Xia, Global asymptotic behavior of a rational difference equation, Appl. Math. Comput., 218(15) (2012) 7818-7828.
7. E.Camouzis, G.Ladas, Dynamics of third-order rational difference equations with open problems and conjectures, Chapman & Hall/CRC, Boca Raton, (2007).
8. J.Migda, Asymptotically polynomial solutions of difference equations, Advances in Difference Equations, 2013, 92.
9. M.Hasanbulli and Y.V.Rogovchenko, Asymptotic behavior nonoscillatory solutions to n-th order nonlinear neutral differential equations, Nonlinear Anal., 69 (2008) 1208-1218.
10. R.P.Elsayed, On the solutions of a rational system of difference equations, Polytechnica Posnaniensis, 45 (2010) 25–36.
11. C.Hengkrawit, V.Laohakosol, W.Pimsert, Solutions of a class of nonlinear recursive equations and applications, Science Asia, 37 (2011) 136–144.
12. G.Papaschinopoulos, M.Radin, C.J.Schinas, Study of the asymptotic behavior of the solutions of three systems of difference equations of exponential form, Appl. Math. Comput, 218(9) (2012) 5310-5318.
13. M.R.S.Kulenović, G.Ladas, N.R.Prokup, A rational difference equation, Comput. Math. Appl., 41(5) (2001) 671-678.
14. X.X.Yan XX, W.T.Li, H.R.Sun, Global attractivity in a higher order nonlinear difference equation, Applied Mathematics E-Notes, 2 (2002) 51-58.
15. Y.Muroya, E.Ishiwata and N.Guglielmi, Global stability for nonlinear difference equations with variable coefficients, J. Math. Anal. Appl., 334(1) (2007) 232-247.
16. E.M.Elsayed, Solution and attractivity for a rational recursive sequence, Discrete Dynamics in Nature and Society, 2011, Article ID 982309, 17 pages (2011).
17. C.Cinar, On the positive solutions of difference equation $x_{n+1} = x_{n-1} / (1 + x_n x_{n-1})$, Appl. Math. Comput, 158(3) (2004) 303-305.
18. C.Cinar, On the difference equation $x_{n+1} = x_{n-1} / (-1 + x_n x_{n-1})$, Appl. Math. Comput, 158(3) (2004) 813-816.
19. M.R.S.Kulenonvic and G.Ladas, Dynamics of second order rational difference equations with open problem and conjectures, Chapman &Hall/CRC, Boca Raton, 2001.
20. V.L.Kocic and G.Ladas, Global behavior of nonlinear difference equations of higher order with applications, Kluwer Academic, Dordrecht, 1993.