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Some properties of fuzzy star-shaped sets

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Abstract. In this paper, we study the star-shapedness for fuzzy sets. Particularly, we clarify the exact relationships among the concepts of star-shaped fuzzy sets, quasi-star-shaped fuzzy sets, pseudo-star-shaped fuzzy sets and generalized star-shaped fuzzy star-shaped sets, and we obtain some important properties of these different types of star-shapedness.

Keywords: Fuzzy sets; Fuzzy star-shapedness; Fuzzy convexity

AMS Mathematics Subject Classification (2010): 03E72; 06G65

1. Introduction

The fuzzy set theory was introduced initially by Zadeh [32] in 1965. In the theory and applications of fuzzy sets, convexity plays a most useful role. From the very first, Zadeh [32] recognised its importance, and the property has been exploited in many ways involving convex fuzzy set. For example, convexity is central to the metric definitions of Klement, Puri and Ralescu [14] and Diamond and Kloeden [6, 7, 8], and to the topological properties of the corresponding metric spaces of fuzzy convex sets [10, 13]. Following the seminal work of Zadeh on the definition of a convex fuzzy set, Ammar and Metz defined another type of convex fuzzy sets in [1]. To avoid misunderstanding, Zadeh's convex fuzzy sets were called quasi-convex fuzzy sets. A lot of scholars have discussed various aspects of the theory and applications of fuzzy convex analysis [4, 5, 11, 12, 17, 18, 19, 20, 23, 24, 26, 27, 31].

However, Nature is not convex and, apart from possible applications, it is of independent interest to see how far the supposition of convexity can be weakened without losing too much structure. Star-shaped sets are a fairly natural extension and this note defines the notion of fuzzy star-shaped sets and explores some of their properties. In [2], Brown introduced the concept of star-shaped fuzzy sets, in [9] Diamond defined another type of star-shaped fuzzy sets (f.s., for short), and in [22] Qiu given a new type of star-shaped fuzzy sets is different with the other two and established some of the basic properties of this family of fuzzy sets. In order to distinguish between these three star-shaped fuzzy sets, Brown's star-shaped fuzzy sets were called quasi-star-shaped fuzzy sets (f.q-s., for short) and Qiu's star-shaped fuzzy sets were called pseudo-star-shaped fuzzy sets (f.p-s., for short). Recently, the research of fuzzy star-shaped (f.s.) sets have been again attracting the deserving attention [3, 28, 33], motivated both by Diamond's research and by the importance of the concept of fuzzy convexity [15, 16, 25, 29, 30].

In this paper, for simplicity, we consider only the star-shaped fuzzy sets defined on the Euclidean space. But it is not difficult to generalize most of the results obtained in the paper to the case that starshaped fuzzy sets are defined in linear space over real field or complex field. In Section 2, we will recall some basic concepts related to this paper and generalize star-shapedness of the normal fuzzy sets to general fuzzy sets. In Section 3, we clarify the exact relationships among the concepts of f.s. sets, f.q-s. sets and f.p-s. sets, and we will study some important properties of these different types of star-shapedness.

2. Preliminaries

Let $x, y \in \mathbb{R}^n$, the line segment $\overline{xy}$ joining x and y is the set of all points of the form $\alpha x + \beta y$ where $\alpha \geq 0, \beta \geq 0, \alpha + \beta = 1$.

(1) A set $S \subseteq \mathbb{R}^n$ is said to be convex if for each pair points $x, y \in S$, it is true that $\overline{xy} \in S$.

(2) A set $S \subseteq \mathbb{R}^n$ is said to be star-shaped with respect to a point $x \in \mathbb{R}^n$ if for each point $y \in S$, it is true that $\overline{xy} \subseteq S$. A set S is simply said to be star-shaped, which means that there is some point x in $\mathbb{R}^n$ such that S is star-shaped with respect to it. The kernel *kerS* of S is the set of all points $x \in S$ such that $\overline{xy} \subseteq S$ for each $y \in S$. For convenience, we assume the empty set $\phi \subseteq \mathbb{R}^n$ is also a convex set and a star-shaped set.

Most results on ordinary convex sets which are used in this paper can be found in [11,24]. A fuzzy set μ on X is defined by its membership function $\mu(x)$ which is a mapping from X into $[0,1]$. An α-cut set of μ is

$$[\mu]^\alpha = \{x \in X : \mu(x) \geq \alpha\},$$

where $\alpha \in (0,1]$, and we separately specify the support set $[\mu]^0$ of μ to be the closure of the union of $[\mu]^\alpha$ for $\alpha \in (0,1]$, i.e., $[\mu]^0 = \overline{\bigcup_{\alpha \in (0,1]} [\mu]^\alpha}$.

Denote by $F(X)$, the family of all fuzzy subsets of X. Let $\mu_1, \mu_2 \in F(X)$, then μ_1 is said to be included in μ_2, denoted by $\mu_1 \subseteq \mu_2$, if and only if $\mu_1(x) \leq \mu_2(x)$ for each $x \in X$. Thus we have that $\mu_1 \subseteq \mu_2$ if and only if $[\mu_1]^\alpha \subseteq [\mu]^\alpha$ for all $\alpha \in [0,1]$.

Definition 2.1. [1] A fuzzy set $\mu \in F(\mathbb{R}^n)$ is quasi-convex if

$$\mu(\lambda x + (1-\lambda)y) \geq \mu(x) \wedge \mu(y)$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$, and a fuzzy set $\mu \in F(\mathbb{R}^n)$ is convex if

$$\mu(\lambda x + (1-\lambda)y) \geq \lambda\mu(x) + (1-\lambda)\mu(y)$$

for all $x, y \in [\mu]^0$ and $\lambda \in [0,1]$.

It is obvious that every convex fuzzy set $\mu \in F(\mathbb{R}^n)$ is quasi-convex.

Definition 2.2. [28] A fuzzy set $\mu \in F(\mathbb{R}^n)$ is said to be fuzzy star-shaped (f.s. for short) with respect to $y \in \mathbb{R}^n$, if

$$\mu(\lambda x + (1-\lambda)y) \geq \mu(x)$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$.

The following describes two more other types of star-shapedness are derived from the concepts quasi-convex fuzzy sets and convex fuzzy sets.

Definition 2.3. [2] A fuzzy set $\mu \in F(\mathbb{R}^n)$ is said to be fuzzy quasi-star-shaped (f.q-s. for short) with respect to $y \in \mathbb{R}^n$, if

$$\mu(\lambda x + (1-\lambda)y) \geq \mu(x) \wedge \mu(y)$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$.

Definition 2.4. [22] A fuzzy set $\mu \in F(\mathbb{R}^n)$ is said to be fuzzy pseudo-star-shaped sets (f.p-s. for short) with respect to $y \in [\mu]^0$, if

$$\mu(\lambda x + (1-\lambda)y) \geq \lambda\mu(x) + (1-\lambda)\mu(y)$$

for for all $x \in [\mu]^0$ and $\lambda \in [0,1]$.

Definition 2.5. [22] Let $\ker(\mu)$ (respectively, p - $\ker(\mu)$, q - $\ker(\mu)$) be the totality of $y \in \mathbb{R}^n$ such that μ is f.s. (respectively, f.p-s., f.q-s.) with respect to y.

Definition 2.6. [22] The fuzzy hypograph of μ denoted by $f.hpy(\mu)$, is defined as

$$f.hpy(\mu) = \{(x,t) : x \in [\mu]^0, t \in [0, \mu(x)]\}.$$

3. Main results

Now we will establish our main theorems.

Lemma 3.1. [21] Let $\mu \in F(\mathbb{R}^n)$ be f.s. with respect to $y \in \mathbb{R}^n$. Then $\mu(y) = \sup_{x\in\mathbb{R}^n} \mu(x)$.

Lemma 3.2. [21] If a set $A \subseteq \mathbb{R}^n$ is star-shaped with respect to $y \in \mathbb{R}^n$, then $\overline{A}$ is star-shaped with respect to y.

Lemma 3.3. [21] A fuzzy set $\mu \in F(\mathbb{R}^n)$ is f.s. with respect to y if and only if its each non- empty α-cut set is star-shaped with respect to y for all $\alpha \in [0,1]$ or $[\mu]^0 = \phi$.

Lemma 3.4. [21] Let a fuzzy set $\mu \in F(\mathbb{R}^n)$ be f.s. with respect to some points in $\mathbb{R}^n$ and $\beta = \sup_{x\in\mathbb{R}^n} \mu(x)$. If $[\mu]^0 \neq \phi$, then $\ker(\mu) \subseteq \ker[\mu]^\alpha$ for all $\alpha \in [0, \beta]$.

Theorem 3.1. For a fuzzy set $\mu \in F(\mathbb{R}^n)$, the following statements hold:

(1) if μ is f.s. with respect to y, then it is f.q-s. with respect to the same point;

(2) if μ is f.p-s. with respect to y, then it is f.q-s. with respect to the same point;

(3) μ is f.q-s. with respect to y and $\mu(y) = \sup_{x\in\mathbb{R}^n} \mu(x)$ if and only if it is f.s. with respect to the same point;

(4) if μ is f.p-s. with respect to y and $\mu(y) = \sup_{x\in\mathbb{R}^n} \mu(x)$, then it is f.s. with respect to the same point.

Proof. (1) Let μ is f.s. with respect to y. By Lemma 3.1, we have $\mu(y) \geq \mu(x)$ for all $x \in \mathbb{R}^n$. Thus, we get that

$$\mu(\lambda x + (1-\lambda)y) \geq \mu(x) = \mu(x) \wedge \mu(y)$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$.

(2) If $x \in \mathbb{R}^n - [\mu]^0$, then $\mu(x) = 0$, that is, $\mu(x) \wedge \mu(y) = 0$. Thus, we get that

$$\mu(\lambda x + (1-\lambda)y) \geq 0 = \mu(x) \wedge \mu(y)$$

for all $\lambda \in [0,1]$. If $x \in [\mu]^0$, then

$$\mu(\lambda x+(1-\lambda)y)\geq \lambda\mu(x)+(1-\lambda)\mu(y)\geq \mu(x)\wedge\mu(y)$$

for all $\lambda\in[0,1]$.

(3) *Sufficiency*. It is obvious from (1) and Lemma 3.1.

Necessity. If μ is f.q-s. with respect to y and $\mu(y)=\sup_{x\in\mathbb{R}^n}\mu(x)$, then we have

$$\mu(\lambda x+(1-\lambda)y)\geq \mu(x)\wedge\mu(y)=\mu(x)$$

for all $x\in\mathbb{R}^n$ and $\lambda\in[0,1]$.

(4) It is obvious from (2) and (3).

Remark 3.1. The inverse statements of (1), (2) and (4) do not hold in general as shown in the following examples.

Example 3.1. The fuzzy set $\mu\in F(\mathbb{R})$ with

$$\mu(x)=\begin{cases}\dfrac{|\sin(x)|}{2}, & if\ x\in[-\pi,\pi],\\ 0, & otherwise\end{cases}$$

is f.p-s. and f.q-s with respect to $y=0$. But it is not f.s. with respect to $y=0$, because of $y\notin[\mu]^\alpha$ and $[\mu]^\alpha\neq\phi$ for all $\alpha\in(0,1/2]$.

Example 3.2. The fuzzy set $\mu\in F(\mathbb{R})$ with $\mu(x)=e^{-|x|}/2$ is f.s. and f.q-s. with respect to $y=0$. But it is not f.p-s. with respect to $y=0$.

Theorem 3.2. A fuzzy set $\mu\in F(\mathbb{R}^n)$ is f.q-s. with respect to $y\in[\mu]^0$ and $\mu(y)\neq 0$ if and only if its α-cut sets are star-shaped with respect to y for all $\alpha\in(0,\mu(y)]$.

Proof: *Necessity*. If μ is f.q-s. with respect to is f.q-s. with respect to $y\in[\mu]^0$, then

$$\mu(\lambda x+(1-\lambda)y)\geq \mu(x)\wedge\mu(y)$$

for all $x\in\mathbb{R}^n$ and $\lambda\in[0,1]$. For any $\alpha\in(0,\mu(y)]$, let $x\in[\mu]^\alpha$. Then we have that $x,y\in[\mu]^\alpha$. From the above inequality we get that

$$\mu(\lambda x+(1-\lambda)y)\geq\alpha,\ \lambda\in[0,1]$$

that is, $\overline{xy}\subseteq[\mu]^\alpha$.

Sufficiency. For any $x\in\mathbb{R}^n$, if $\mu(x)=0$, then

$$\mu(\lambda x+(1-\lambda)y)\geq 0=\mu(x)=\mu(x)\wedge\mu(y)$$

for all $\lambda\in[0,1]$. If $0<\mu(x)\leq\mu(y)$, then let $\alpha=\mu(x)\in(0,\mu(y)]$. Thus, we have that x, $y\in[\mu]^\alpha$. Since α-cut set is star-shaped with respect to y, that is,

$$\mu(\lambda x+(1-\lambda)y)\geq\alpha=\mu(x)\wedge\mu(y)$$

for all $\lambda\in[0,1]$. If $\mu(x)>\mu(y)$, let $\alpha=\mu(y)\in(0,\mu(y)]$. Thus, we have that $x,y\in[\mu]^\alpha$. Since α-cut set is star-shaped with respect to y, that is,

$$\mu(\lambda x+(1-\lambda)y)\geq\alpha=\mu(x)\wedge\mu(y)$$

for all $\lambda\in[0,1]$.

Theorem 3.3. If a fuzzy set $\mu \in F(\mathbb{R}^n)$ is f.q-s. with respect to $y \in [\mu]^0$ and $\mu(y) \neq 0$, then $[\mu]^0$ is star-shaped with respect to y.

Proof: Let

$$A = \bigcup_{\alpha \in (0,1]} [\mu]^\alpha = \bigcup_{\alpha \in (0,\mu(y)]} [\mu]^\alpha .$$

By Theorme 3.2, we have that A is star-shaped with respect to y. By Lemma 3.2, we get that $\overline{A} = [\mu]^0$ is star-shaped with respect to y.

Remark 3.2. In Theorem 3.3, the condition $\mu(y) \neq 0$ is necessary.

Example 3.3. The fuzzy set $\mu \in F(\mathbb{R})$ with

$$\mu(x) = \begin{cases} 0, & if \ x \in [-1,1], \\ \dfrac{2}{\pi}\arctan(|x|-1) & if \ x \in \mathbb{R} - [-1,1] \end{cases}$$

is f.q-s. with respect to $y = 1 \in [\mu]^0$. But $[\mu]^0$ is not star-shaped with respect to $y = 1$, because of $[\mu]^0 = \mathbb{R} - [-1,1]$.

Corollary 3.1. Let a fuzzy set $\mu \in F(\mathbb{R}^n)$ be f.q-s. with respect to some points in $\mathbb{R}^n$ and $A = \{x \in \mathbb{R}^n \mid \mu(x) = 0\}$. If $q\text{-}\ker(\mu) \cap A = \phi$, then $q\text{-}\ker(\mu) \subseteq \ker[\mu]^\alpha$ for all $\alpha \in [0, \beta]$, where $\beta = \inf\{\mu(y) \mid y \in q\text{-}\ker(\mu)\}$.

Proof: By Thoerem 3.2 and 3.3, we have that $q\text{-}\ker(\mu) \subseteq \bigcap_{\alpha \in [0,\beta]} \ker[\mu]^\alpha \subseteq \ker[\mu]^\alpha$ for all $\alpha \in [0, \beta]$.

Theorem 3.4. If for a fuzzy set $\mu \in F(\mathbb{R}^n)$, the point $y \in \mathbb{R}^n$ satisfies that

$$\mu(y) = \inf_{x \in \mathbb{R}^n} \mu(x).$$

Then μ is f.q-s. with respect to y, that is, $y \in q\text{-}\ker(\mu)$.

Proof: By Definition 2.4, this statement is true, because of $\mu(y) = \inf_{x \in \mathbb{R}^n} \mu(x)$.

Remark 3.3 The inverse statements of Theorem 3.4 do not hold. In Example 3.2, we can get that μ is f.q-s. with respect to $y = 0$. But $\mu(y) = 1/2 \geq \mu(x)$ for all $x \in \mathbb{R}$.

Theorem 3.5. If a fuzzy set $\mu \in F(\mathbb{R}^n)$ is f.p-s. with respect to $y \in [\mu]^0$, then its α-cut sets are star-shaped with respect to y for all $\alpha \in [0, \mu(y)]$.

Proof: Suppose $\mu(y) \neq 0$. By (2) of Theorem 3.1, Theorem3.2 and 3.3, we have that α-cut sets of μ are star-shaped with respect to y for all $\alpha \in [0, \mu(y)]$. Suppose $\mu(y) = 0$. Let

$$A = \bigcup_{\alpha \in (0,1]} [\mu]^\alpha, \text{ i.e., } \overline{A} = [\mu]^0.$$

If $x \in A$, that is, $\mu(x) > 0$, then we have

$$\mu(\lambda x + (1-\lambda)y) \geq \lambda\mu(x) > 0$$

for all $\lambda \in (0,1]$, which implies that $\lambda x + (1-\lambda)y \in A \subseteq \overline{A}$. Since $\lambda = 0$, we have $\lambda x + (1-\lambda)y = y \in \overline{A} = [\mu]^0$. Thus, we get that $\overline{xy} \subseteq \overline{A} = [\mu]^0$.

If $x_0 \in [\mu]^0 - A$, that is, $\mu(x_0) = 0$, then for each neighborhood V of x_0, we have $V \bigcap A \neq \phi$. Choose $\lambda x_0 + (1-\lambda)y$ with $\lambda \in (0,1]$. Since the mapping

$$f(x) = \lambda x + (1-\lambda)y$$

is continuous in x_0, for each neighborhood U of $\lambda x_0 + (1-\lambda)y$ in $\mathbb{R}^n$, there exists a neighborhood V of x_0 such that if $x \in V$, then $\lambda x + (1-\lambda)y \in U$, for all $\lambda \in (0,1]$. Since $V \bigcap A \neq \phi$, let $x \in V \bigcap A$. Then, we have that $\lambda x + (1-\lambda)y \in A \bigcap U$, which implies that

$$\lambda x_0 + (1-\lambda)y \in \overline{A} = [\mu]^0,$$

for all $\lambda \in (0,1]$. Since $\lambda = 0$, we have $\lambda x_0 + (1-\lambda)y = y = [\mu]^0$. Thus, we get that $\overline{x_0 y} \subseteq [\mu]^0$. Consequently, we obtain that α-cut sets of μ are star-shaped with respect to y for all $\alpha \in [0, \mu(y)]$.

Corollary 3.2. Let a fuzzy set $\mu \in F(\mathbb{R}^n)$ be f.p-s. with respect to some points in $[\mu]^0$ and $\beta = \inf\{\mu(y) \mid y \in p\text{-}\ker(\mu)\}$. Then $p\text{-}\ker(\mu) \subseteq \ker[\mu]^\alpha$ for all $\alpha \in [0, \beta]$.

Proof: By Theorem 3.5, we have that

$$p\text{-}\ker(\mu) \subseteq \bigcap\nolimits_{\alpha \in [0,\beta]} \ker[\mu]^\alpha \subseteq \ker[\mu]^\alpha$$

for all $\alpha \in [0, \beta]$.

Theorem 3.6. A fuzzy set $\mu \in F(\mathbb{R}^n)$ is f.p-s. with respect to $y \in [\mu]^0$ if and only if its $f.hpy(\mu)$ is star-shaped with respect to $(y, \mu(y))$.

Proof: *Necessity*. Let μ be f.p-s. with respect to y and $(x,t) \in f.hyp(\mu)$, that is, $x \in [\mu]^0$, $t \in [0, \mu(x)]$. Since μ is f.p-s. with respect to y, for each $\lambda \in [0,1]$, we have

$$\mu(\lambda x + (1-\lambda)y) \geq \lambda\mu(x) + (1-\lambda)\mu(y) \geq \lambda t + (1-\lambda)\mu(y),$$

And by Theorem 3.5, we have $\lambda x + (1-\lambda)y \in [\mu]^0$. Hence, we get that

$$\lambda(x,t) + (1-\lambda)(y, \mu(y)) \in f.hyp(\mu).$$

Sufficiency. Let $x \in [\mu]^0$ and $(x, \mu(x)) \in f.hyp(\mu)$. By the star-shapedness of $f.hyp\ (\mu)$, for each $\lambda \in [0,1]$, we have

$$(\lambda x + (1-\lambda)y, \lambda\mu(x) + (1-\lambda)\mu(y)) \in f.hyp(\mu).$$

Hence, we get that

$$\mu(\lambda x + (1-\lambda)y) \geq \lambda\mu(x) + (1-\lambda)\mu(y)$$

for all $\lambda \in [0,1]$.

Corollary 3.3. Let μ be a fuzzy set. If $f.hpy(\mu)$ of μ is star-shaped with respect to $(y, \mu(y))$ and $\mu(y) = \sup_{x \in \mathbb{R}^n} \mu(x)$, then μ is f.s. with respect to y.

Proof: By Theorem 3.6, we get that μ is f.p-s. with respect to y. By (4) of Theorem 3.1, we obtain that μ is f.s. with respect to y, because of $\mu(y) = \sup_{x \in \mathbb{R}^n} \mu(x)$.

Corollary 3.4. Let μ be a fuzzy set. If $f.hpy(\mu)$ of μ is star-shaped with respect to $(y, \mu(y))$, then μ is f.q-s. with respect to y.

Proof: By Theorem 3.6, we get that μ is f.p-s. with respect to y. By (2) of Theorem 3.1, we obtain that μ is f.q-s. with respect to y.

Theorem 3.7. Let $\mu_1, \mu_2 \in F(\mathbb{R}^n)$ be f.q-s. (respectively, f.p-s.) with respect to $y \in \mathbb{R}^n$. Then $\mu_1 \cap \mu_2$ is f.q-s. (respectively, f.p-s.) with respect to y.

Proof: Let $\mu_1, \mu_2 \in F(\mathbb{R}^n)$ be f.q-s. with respect to $y \in \mathbb{R}^n$. For all $x \in \mathbb{R}^n$, we have

$$\mu_i(\lambda x + (1-\lambda)y) \ge \mu_i(x) \wedge \mu_i(y), \ i = 1, 2,$$

for all $\lambda \in [0,1]$. Hence, we get that

$$\begin{aligned}(\mu_1 \cap \mu_2)(\lambda x + (1-\lambda)y) &= \mu_1(\lambda x + (1-\lambda)y) \wedge \mu_2(\lambda x + (1-\lambda)y) \\ &\ge (\mu_1(x) \wedge \mu_1(y)) \wedge (\mu_2(x) \wedge \mu_2(y)) \\ &= (\mu_1(x) \wedge \mu_2(x)) \wedge (\mu_1(y) \wedge \mu_2(y)) \\ &= (\mu_1 \cap \mu_2)(x) \wedge (\mu_1 \cap \mu_2)(y)\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$.

Let $\mu_1, \mu_2 \in F(\mathbb{R}^n)$ be f.p-s. with respect to $y \in [\mu]^0$. For all $x \in [\mu_1 \cap \mu_2]^0 \subseteq [\mu_1]^0 \cap [\mu_2]^0$, we have

$$\mu_i(\lambda x + (1-\lambda)y) \ge \lambda\mu_i(x) + (1-\lambda)\mu_i(y), \ i = 1, 2,$$

for all $\lambda \in [0,1]$. Hence, we get that

$$\begin{aligned}(\mu_1 \cap \mu_2)(\lambda x + (1-\lambda)y) &= \mu_1(\lambda x + (1-\lambda)y) \wedge \mu_2(\lambda x + (1-\lambda)y) \\ &\ge (\lambda\mu_1(x) + (1-\lambda)\mu_1(y)) \wedge (\lambda\mu_2(x) + (1-\lambda)\mu_2(y)) \\ &= \lambda(\mu_1(x) \wedge \mu_2(x)) + (1-\lambda)(\mu_1(y) \wedge \mu_2(y)) \\ &= \lambda(\mu_1 \cap \mu_2)(x) + (1-\lambda)(\mu_1 \cap \mu_2)(y)\end{aligned}$$

for all $x \in [\mu_1 \cap \mu_2]^0$ and $\lambda \in [0,1]$.

Theorem 3.8. Let $\mu_1, \mu_2 \in F(\mathbb{R}^n)$ be f.q-s. (respectively, f.p-s.) with respect to $y \in \mathbb{R}^n$ and $\mu_1(y) = \mu_2(y)$. Then $\mu_1 \cup \mu_2$ is f.q-s. (respectively, f.p-s.) with respect to y.

Proof: Let $\mu_1, \mu_2 \in F(\mathbb{R}^n)$ be f.q-s. with respect to $y \in \mathbb{R}^n$. For all $x \in \mathbb{R}^n$, we have

$$\mu_i(\lambda x + (1-\lambda)y) \ge \mu_i(x) \wedge \mu_i(y), \ i = 1, 2,$$

for all $\lambda \in [0,1]$. Thus, we get that

$$\begin{aligned}(\mu_1 \cup \mu_2)(\lambda x + (1-\lambda)y) &= \mu_1(\lambda x + (1-\lambda)y) \vee \mu_2(\lambda x + (1-\lambda)y) \\ &\ge (\mu_1(x) \wedge \mu_1(y)) \vee (\mu_2(x) \wedge \mu_2(y))\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Since $\mu_1(y) = \mu_2(y)$, we have

$$\begin{aligned}&(\mu_1(x) \wedge \mu_1(y)) \vee (\mu_2(x) \wedge \mu_2(y)) \\ &= (\mu_1(x) \vee \mu_2(x)) \wedge (\mu_1(y) \vee \mu_2(y)) \\ &= (\mu_1 \cup \mu_2)(x) \wedge (\mu_1 \cup \mu_2)(y)\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Hence, we obtain that

$$(\mu_1 \cup \mu_2)(\lambda x + (1-\lambda)y) \geq (\mu_1 \cup \mu_2)(x) \wedge (\mu_1 \cup \mu_2)(y)$$

for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$

Let $\mu_1, \mu_2 \in F(\mathbb{R}^n)$ be f.p-s. with respect to $y \in [\mu]^0$. For all $x \in [\mu_1 \cup \mu_2]^0 = [\mu_1]^0 \cup [\mu_2]^0$, we have

$$\mu_i(\lambda x + (1-\lambda)y) \geq \lambda \mu_i(x) + (1-\lambda)\mu_i(y), \; i = 1, 2,$$

for all $\lambda \in [0,1]$. Hence, we get that

$$\begin{aligned}(\mu_1 \cup \mu_2)(\lambda x + (1-\lambda)y) &= \mu_1(\lambda x + (1-\lambda)y) \vee \mu_2(\lambda x + (1-\lambda)y) \\ &\geq (\lambda \mu_1(x) + (1-\lambda)\mu_1(y)) \vee (\lambda \mu_2(x) + (1-\lambda)\mu_2(y))\end{aligned}$$

for all $x \in [\mu_1 \cup \mu_2]^0$ and $\lambda \in [0,1]$. Since $\mu_1(y) = \mu_2(y)$, we have

$$\begin{aligned}&(\lambda \mu_1(x) + (1-\lambda)\mu_1(y)) \vee (\lambda \mu_2(x) + (1-\lambda)\mu_2(y)) \\ &= \lambda(\mu_1(x) \vee \mu_2(x)) + (1-\lambda)(\mu_1(y) \vee \mu_2(y)) \\ &= \lambda(\mu_1 \cup \mu_2)(x) + (1-\lambda)(\mu_1 \cup \mu_2)(y)\end{aligned}$$

for all $x \in [\mu_1 \cup \mu_2]^0$ and $\lambda \in [0,1]$. Hence, we obtain that

$$(\mu_1 \cup \mu_2)(\lambda x + (1-\lambda)y) \geq (\mu_1 \cup \mu_2)(x) \wedge (\mu_1 \cup \mu_2)(y)$$

for all $x \in [\mu_1 \cup \mu_2]^0$ and $\lambda \in [0,1]$.

Let x_0 be some point in $\mathbb{R}^n$ and μ some fuzzy set then the translation of μ by x_0 is the fuzzy set $x_0 + \mu$ defined as $(x_0 + \mu)(x) = \mu(x - x_0)$ [13].

Theorem 3.9. Let $\mu \in F(\mathbb{R}^n)$ be f.q-s. (respectively, f.p-s.) with respect to $y \in \mathbb{R}^n$ and there is a point $x_0 \in \mathbb{R}^n$. Then $x_0 + \mu$ is f.q-s. (respectively, f.p-s.) with respect to $y + x_0$.

Proof: Suppose μ is f.q-s. with respect to $y \in \mathbb{R}^n$. For any $x \in \mathbb{R}^n$, we get that

$$\begin{aligned}(x_0 + \mu)(\lambda x + (1-\lambda)(x_0 + y)) &= \mu(\lambda(x - x_0) + (1-\lambda)y) \\ &\geq \mu(x - x_0) \wedge \mu(y) \\ &= (x_0 + \mu)(x) \wedge (x_0 + \mu)(x_0 + y)\end{aligned}$$

for all $\lambda \in [0,1]$. Hence, $x_0 + \mu$ is f.q-s. with respect to $x_0 + y$.

Suppose μ is f.p-s. with respect to $y \in [\mu]^0$. Let

$$A = \bigcup_{\alpha \in (0,1]} [\mu]^\alpha \text{ and } B = \bigcup_{\alpha \in (0,1]} [x_0 + \mu]^\alpha.$$

Now we show that $x \in \overline{A} = [\mu]^0$ if and only if $x + x_0 \in \overline{B} = [x_0 + \mu]^0$. Let $x \in \overline{A} = [\mu]^0$. If $x \in A$, then there exists $\alpha \in (0,1]$ such that $x \in [\mu]^\alpha$, that is, $\mu(x) \geq \alpha$. Thus, we have

$$(x_0 + \mu)(x + x_0) = \mu(x + x_0 - x_0) = \mu(x) \geq \alpha,$$

that is, $x + x_0 \in [x_0 + \mu]^\alpha \subseteq B$. If $x \in [\mu]^0 - A$, that is, $\mu(x) = 0$, then for each neighborhood V of x, we have $V \cap A \neq \phi$. Since the mapping $f(x) = x + x_0$ is continuous in x, for each neighborhood U of $x + x_0$ in $\mathbb{R}^n$, there exists a neighborhood V of x such that if $z \in V$, then $z + x_0 \in U$. Since $V \cap A \neq \phi$, let $z \in V \cap A$. Then, we have $z + x_0 \in U \cap B$, which implies that $x+x_0 \in \overline{B}=[x_0+\mu]^0$. Similarly, we can get that if $x+x_0 \in \overline{B}=[x_0+\mu]^0$, then $x\in \overline{A}=[\mu]^0$. Hence, $y+x_0 \in [x_0+\mu]^0$, and for any $x\in [x_0+\mu]^0$, we have $x-x_0 \in [\mu]^0$. Since μ is f.p-s. with respect to y, we get that

$$\begin{aligned}(x_0+\mu)(\lambda x+(1-\lambda)(x_0+y)) &= \mu(\lambda(x-x_0)+(1-\lambda)y)\\ &\geq \lambda\mu(x-x_0)+(1-\lambda)\mu(y)\\ &= \lambda(x_0+\mu)(x)+(1-\lambda)(x_0+\mu)(y+x_0)\end{aligned}$$

for all $x\in [x_0+\mu]^0$ and $\lambda\in[0,1]$. Hence, we obtain that $x_0+\mu$ is f.p-s. with respect to x_0+y.

Remark 3.4. Since the property of being star-shaped is translation invariant in $\mathbb{R}^n$, this proposition also holds for fuzzy general star-shapedness.

Let T be a linear invertible transformation on $\mathbb{R}^n$ and μ a fuzzy set. Then by the Extension Principle we have that $(T(\mu))(x)=\mu(T^{-1}(x))$ [22].

Theorem 3.10. Let $\mu\in F(\mathbb{R}^n)$ be f.q-s. (respectively, f.p-s.) with respect to $y\in\mathbb{R}^n$ and T a linear invertible transformation on $\mathbb{R}^n$. Then $T(\mu)$ is f.q-s. (respectively, f.p-s.) with respect to $T(y)$.

Proof: Suppose μ is f.q-s. with respect to $y\in\mathbb{R}^n$. For any $x\in\mathbb{R}^n$, we get that

$$\begin{aligned}(T(\mu))(\lambda x+(1-\lambda)(x_0+y)) &= \mu(\lambda T^{-1}(x)+(1-\lambda)y)\\ &\geq \mu(T^{-1}(x))\wedge\mu(y)\\ &= (T^{-1}(\mu))(x)\wedge(T(\mu))(T(y))\end{aligned}$$

for all $\lambda\in[0,1]$. Hence, $T(\mu)$ is f.q-s. with respect to $T(y)$.

Suppose μ is f.p-s. with respect to $y\in[\mu]^0$. Let

$$A=\bigcup\nolimits_{\alpha\in(0,1]}[\mu]^\alpha \text{ and } B=\bigcup\nolimits_{\alpha\in(0,1]}[T(\mu)]^\alpha .$$

Now we show that $x\in\overline{A}=[\mu]^0$ if and only if $T(x)\in\overline{B}=[T(\mu)]^0$. Let $x\in\overline{A}=[\mu]^0$. If $x\in A$, then there exists $\alpha\in(0,1]$ such that $x\in[\mu]^\alpha$, that is, $\mu(x)\geq\alpha$. Thus, we have

$$(T(\mu))(T(x))=\mu(x)\geq\alpha,$$

that is, $T(x)\in[T(\mu)]^\alpha\subseteq B$. If $x_0\in[\mu]^0-A$, that is, $\mu(x_0)=0$, then for each neighborhood V of x_0, we have $V\cap A\neq\phi$. Since T a linear invertible transformation on $\mathbb{R}^n$, for each neighborhood U of $T(x_0)$ in $\mathbb{R}^n$, there exists a neighborhood V of x_0 such that if $x\in V$, then $T(x)\in U$. Since $V\cap A\neq\phi$, let $x\in V\cap A$. Then, we have $T(x)\in U\cap B$, which implies that $T(x_0)\in\overline{B}=[T(\mu)]^0$. Similarly, we can get that if $T(x)\in\overline{B}=[T(\mu)]^0$, then $x\in\overline{A}=[\mu]^0$. Hence, $T(y)\in[T(\mu)]^0$, and for any $x\in[T(\mu)]^0$, we have $T^{-1}(x)\in[\mu]^0$. Since μ is f.p-s. with respect to y, we get that

$$\begin{aligned}(T(\mu))(\lambda x+(1-\lambda)T(y)) &= \mu(\lambda(T^{-1}(x))+(1-\lambda)y)\\ &\geq \lambda\mu(T^{-1}(x))+(1-\lambda)\mu(y)\\ &= \lambda(T(\mu))(x)+(1-\lambda)(T(\mu))(T(y))\end{aligned}$$

for all $x\in[T(\mu)]^0$ and $\lambda\in[0,1]$. Hence, we obtain that $T(\mu)$ is f.p-s. with respect to $T(y)$.

Remark 3.5. Since rotation and mirror image transformation are linear invertible transformations, this proposition holds for them. Moreover, the linear invertible transformation invariant of the star-shapedness of the ordinary star-shaped set in $\mathbb{R}^n$, this proposition also holds for fuzzy general star-shapedness.

For a star-shaped fuzzy set $\mu\in F(\mathbb{R}^n)$, if for any $\alpha,\beta\in[0,1]$, $\alpha\leq\beta$, we have that $\ker[\mu]^\beta\subseteq\ker[\mu]^\alpha$. Define a fuzzy set $f\ker(\mu)$ by $[f\ker(\mu)]^\alpha=\ker[\mu]^\alpha$ [22].

Theorem 3.11. If a fuzzy set $\mu\in F(\mathbb{R}^n)$ is f.s. with respect to $y\in\mathbb{R}^n$, then $f\ker(\mu)$ is a quasi-convex fuzzy set.

Proof: Since the kernel $\ker[\mu]^\alpha$ of $[\mu]^\alpha$ is convex set for all $\alpha\in[0,1]$, we have that $f\ker(\mu)$ is a fuzzy quasi-convex set, because of a fuzzy set is quasi-convex if and only if its nonempty cut sets are convex sets.

Thoerem 3.12. Let μ be a fuzzy set. Then $\ker(\mu)$ is a convex set in $\mathbb{R}^n$.

Proof: If $[\mu]^0=\phi$, then $\mu(x)=0$ for all $x\in\mathbb{R}^n$, which implies that $\ker(\mu)=\mathbb{R}^n$. Thus, we have $\ker(\mu)$ is a convex set. If $\ker(\mu)=\phi$, it is obvious that $\ker(\mu)$ is a convex set. Suppose $[\mu]^0\neq\phi$ and $\ker(\mu)\neq\phi$. Let $\beta=\sup_{x\in\mathbb{R}^n}\mu(x)$ and $y_1,y_2\in\ker(\mu)$. By Lemma 3.4, we have $y_1,y_2\in\ker[\mu]^\alpha$ for all $\alpha\in[0,\beta]$. Since $\ker[\mu]^\alpha$ is convex set for all $\alpha\in[0,\beta]$, we have $\overline{y_1y_2}\subseteq\ker[\mu]^\alpha$ for all $\alpha\in[0,\beta]$, which implies that $\overline{y_1y_2}\subseteq\bigcap_{\alpha\in[0,\beta]}\ker[\mu]^\alpha$. By Lemma 3.3, we have $\ker(\mu)=\bigcap_{\alpha\in[0,\beta]}\ker[\mu]^\alpha$. Thus, we get that $\overline{y_1y_2}\subseteq\ker(\mu)$. Hence, we obtain that $\ker(\mu)$ is a convex set in $\mathbb{R}^n$.

Remark 3.6. The conclusion of Theorem 3.12 does not hold for f.q-s. set.

Example 3.4. Let $S^n=\{(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n : x_1^2+x_2^2+\cdots+x_n^2=1\}$. Define the fuzzy set $\mu\in F(\mathbb{R}^n)$ as

$$\mu(x_1,x_2,\cdots,x_n)=\begin{cases}0.5, & if\ (x_1,x_2,\cdots,x_n)\in S^n,\\ 0, & otherwise.\end{cases}$$

Then μ is f.q-s. set and f.p-s. set but its q-$\ker(\mu)=\mathbb{R}^n-S^n$ and p-$\ker(\mu)=\mathbb{R}^n-S^n$ are not convex.

Theorem 3.13. If a fuzzy set $\mu \in F(\mathbb{R}^n)$ is f.s. with respect to y and is upper semi-continu- ous, then $\ker(\mu)$ is closed in $\mathbb{R}^n$.

Proof: If $[\mu]^0 = \phi$, then $\mu(x) = 0$ for all $x \in \mathbb{R}^n$, which implies that $\ker(\mu) = \mathbb{R}^n$, it is obvious that $\ker(\mu)$ is closed. Suppose $[\mu]^0 \neq \phi$. Let $\beta = \sup_{x \in \mathbb{R}^n} \mu(x)$. For arbitrary given $\alpha \in (0, \beta]$, if $y_0 \in \overline{\ker[\mu]^\alpha}$, then for each neighborhood V of y_0, we have $V \bigcap \ker[\mu]^\alpha \neq \phi$. Choose $\lambda x + (1-\lambda) y_0$ with $\lambda \in [0,1]$, where $x \in [\mu]^\alpha$. Since the mapping

$$f(y) = \lambda x + (1-\lambda) y$$

Is continuous in y_0, for each neighborhood U of $\lambda x + (1-\lambda) y_0$ in $\mathbb{R}^n$, there exists a neighborhood V of y_0 such that if $y \in V$, then $\lambda x + (1-\lambda) y \in U$. Since $V \bigcap \ker[\mu]^\alpha \neq \phi$, let $y \in V \bigcap \ker[\mu]^\alpha$. Then we have $\lambda x + (1-\lambda) y \in U \bigcap [\mu]^\alpha$, which implies that $\mu(\lambda x + (1-\lambda) y) \geq \alpha$. Since μ is upper semi-continuous, we get that

$$\mu(\lambda x + (1-\lambda) y_0) \geq \alpha,$$

That is, $\lambda x + (1-\lambda) y_0 \in [\mu]^\alpha$. Thus, we have that $[\mu]^\alpha$ is star-shaped with respect to y_0, i.e., $y_0 \in \ker[\mu]^\alpha$. Hence, we get that $\ker[\mu]^\alpha$ is closed in $\mathbb{R}^n$ for all $\alpha \in (0, \beta]$, that is, $\bigcap_{\alpha \in (0,\beta]} \ker[\mu]^\alpha$ is closed in $\mathbb{R}^n$. For any $y \in \bigcap_{\alpha \in (0,\beta]} \ker[\mu]^\alpha$, we get that

$$y \in \bigcup_{\alpha \in (0,\beta]} [\mu]^\alpha$$

and $\bigcup_{\alpha \in (0,\beta]} [\mu]^\alpha$ is star-shaped with respect to y. By Lemma 3.2, we have that

$$[\mu]^0 = \overline{\bigcup_{\alpha \in (0,1]} [\mu]^\alpha} = \overline{\bigcup_{\alpha \in (0,\beta]} [\mu]^\alpha}$$

is star-shaped with respect to y, that is, $y \in \ker[\mu]^0$. Thus, we have

$$\bigcap_{\alpha \in (0,\beta]} \ker[\mu]^\alpha \subseteq \ker[\mu]^0.$$

Hence, by Lemma 3.3, we obtain that

$$\ker(\mu) = \bigcap_{\alpha \in [0,\beta]} \ker[\mu]^\alpha = \bigcap_{\alpha \in (0,\beta]} \ker[\mu]^\alpha,$$

Is closed in $\mathbb{R}^n$.

4. Conclusions

In this paper, we have considered the star-shaped fuzzy sets. We have discuss relationships among the concepts of star-shaped fuzzy sets, quasi-star-shaped fuzzy sets, pseudo-starshaped fuzzy sets and generalized star-shaped fuzzy star-shaped sets. Fuzzy star-shaped sets were defined above as a generalisation of fuzzy convex fuzzy sets. In [21], Qiu have extended convexity in important ways and have got that some propositions are untrue for fuzzy convex sets for f.s. set. For f.p-s. set and f.q-s. set, we also have obtained that some corresponding propositions, for example, the intersection of two f.q-s. (respectively, f.p-s.) sets is also f.q-s. (respectively, f.p-s.) set, if the intersection of their kernel is not empty set, and the unions of two f.q-s. (respectively, f.p-s.) sets is also f.q-s. (respectively, f.p-s.) set, if the intersection of their kernel is nonempty set and they have the same membership function values for the common kernel points.

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