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On the Hyperbola $2x^2$-$3y^2$=23

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Abstract. The hyperbola represented by the binary quadratic equation $2x^2 - 3y^2 = 23$ is analyzed for finding its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also, employing the solutions of the given equation, special Pythagorean triangle is constructed.

Keywords: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The binary quadratic Diophantine equations of the form $ax^2 - by^2 = N, (a, b, N \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. In this context, one may refer [1-13].

This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by $2x^2 - 3y^2 = 23$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also, employing the solutions of the given equation, special Pythagorean triangle is constructed.

2. Method of analysis

The binary quadratic equation representing hyperbola is given by

$$2x^2 - 3y^2 = 23 \tag{1}$$

Taking $x = X + 3T$, $y = X + 2T$ (2)

in (1), it simplifies to the equation

$$X^2 = 6T^2 - 23 \tag{3}$$

The smallest positive integer solution (T_0, X_0) of (3) is

$T_0 = 2,\ X_0 = 1$

To obtain, the other solutions of (3), consider the pellian equation

$$X^2 = 6T^2 + 1 \tag{4}$$

whose smallest positive integer solution is

$\tilde{T}_0 = 2,\ \tilde{X}_0 = 5$

The general solution $(\tilde{T}_n, \tilde{X}_n)$ of (4) is given by

$$\tilde{X}_n + \sqrt{6}\tilde{T}_n = \left(5 + 2\sqrt{6}\right)^{n+1}, n = 0,1,2.... \tag{5}$$

Since, irrational roots occur in pairs, we have

$$\tilde{X}_n - \sqrt{6}\tilde{T}_n = \left(5 - 2\sqrt{6}\right)^{n+1}, n = 0,1,2.... \tag{6}$$

From (5) and (6), solving for $\tilde{X}_n, \tilde{T}_n$,we have

$$\tilde{X}_n = \frac{1}{2}\left[\left(5 + 2\sqrt{6}\right)^{n+1} + \left(5 - 2\sqrt{6}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{T}_n = \frac{1}{2\sqrt{6}}\left[\left(5 + 2\sqrt{6}\right)^{n+1} - \left(5 - 2\sqrt{6}\right)^{n+1}\right] = \frac{1}{2\sqrt{6}} g_n$$

Applying Brahmagupta lemma between the solutions (T_0, X_0) and $(\tilde{T}_n, \tilde{X}_n)$, the general solution (T_{n+1}, X_{n+1}) of (3) is found to be

$$T_{n+1} = X_0\tilde{T}_n + T_0\tilde{X}_n$$

$$X_{n+1} = X_0\tilde{X}_n + 6T_0\tilde{T}_n$$

$$\Rightarrow T_{n+1} = \frac{1}{2\sqrt{6}} g_n + f_n \tag{7}$$

$$X_{n+1} = \frac{1}{2} f_n + \sqrt{6} g_n \tag{8}$$

Using (7) and (8) in (2) we have

$$x_{n+1} = X_{n+1} + 3T_{n+1} = \frac{7}{2} f_n + \frac{15}{2\sqrt{6}} g_n \tag{9}$$

$$y_{n+1} = X_{n+1} + 2T_{n+1} = \frac{5}{2} f_n + \frac{7}{\sqrt{6}} g_n \tag{10}$$

Thus, (9) and (10) represent the integer solutions of the hyperbola (1).
A few numerical examples are given in the following table 1.

Table 1: Examples

n	x_{n+1}	y_{n+1}
-1	7	5
0	65	53
1	643	525
2	6365	5197

On the Hyperbola $2x^2$-$3y^2$=23

Recurrence relations for x and y are:

$$x_{n+3} - 10x_{n+2} + x_{n+1} = 0, n = -1,0,1.....$$
$$y_{n+3} - 10y_{n+2} + y_{n+1} = 0, n = -1,0,1.....$$

A few interesting relations among the solutions are given below:

- $x_{n+2} - 5x_{n+1} - 6y_{n+1} = 0$
- $5x_{n+2} - x_{n+1} - 6y_{n+2} = 0$
- $49x_{n+2} - 5x_{n+1} - 6y_{n+3} = 0$
- $x_{n+3} - 49x_{n+1} - 60y_{n+1} = 0$
- $x_{n+3} - x_{n+1} - 12y_{n+2} = 0$
- $49x_{n+3} - x_{n+1} - 60y_{n+3} = 0$
- $49x_{n+1} + 60y_{n+1} - x_{n+3} = 0$
- $4x_{n+1} + 5y_{n+1} - y_{n+2} = 0$
- $40x_{n+1} + 49y_{n+1} - y_{n+3} = 0$
- $x_{n+1} + 6y_{n+2} - 5x_{n+2} = 0$
- $4x_{n+1} + 49y_{n+2} - 5y_{n+3} = 0$
- $6y_{n+3} + 5x_{n+1} - 49x_{n+2} = 0$
- $y_{n+3} - 40x_{n+1} - 49y_{n+1} = 0$
- $5x_{n+3} - 49x_{n+2} - 6y_{n+1} = 0$
- $x_{n+3} - 5x_{n+2} - 6y_{n+2} = 0$
- $5x_{n+3} - x_{n+2} - 6y_{n+3} = 0$
- $49x_{n+2} + 6y_{n+1} - 5x_{n+3} = 0$
- $4x_{n+2} + y_{n+1} - 5y_{n+2} = 0$
- $y_{n+3} - 8x_{n+2} - y_{n+1} = 0$
- $y_{n+1} + 8x_{n+2} - y_{n+3} = 0$
- $4x_{n+2} + 5y_{n+2} - y_{n+3} = 0$
- $4x_{n+3} + 5y_{n+1} - 49y_{n+2} = 0$
- $40x_{n+3} + y_{n+1} - 49y_{n+3} = 0$
- $4x_{n+3} + y_{n+2} - 5y_{n+3} = 0$
- $5y_{n+3} - y_{n+2} - 4x_{n+3} = 0$

Each of the following expressions represents a cubical integer.

- $\frac{1}{23}\left[\left(53x_{3n+3}-5x_{3n+4}\right)+3\left(53x_{n+1}-5x_{n+2}\right)\right]$
- $\frac{1}{300}\left[\left(525x_{3n+3}-5x_{3n+5}\right)+3\left(525x_{n+1}-5x_{n+3}\right)\right]$
- $\frac{1}{23}\left[\left(28x_{3n+3}-30y_{3n+3}\right)+3\left(28x_{n+1}-30y_{n+1}\right)\right]$
- $\frac{1}{115}\left[\left(260x_{3n+3}-30y_{3n+4}\right)+3\left(260x_{n+1}-30y_{n+2}\right)\right]$
- $\frac{1}{1127}\left[\left(2572x_{3n+3}-30y_{3n+5}\right)+3\left(2572x_{n+1}-30y_{n+3}\right)\right]$
- $\frac{1}{138}\left[\left(3150x_{3n+4}-318x_{3n+5}\right)+3\left(3150x_{n+2}-318x_{n+3}\right)\right]$
- $\frac{1}{115}\left[\left(28x_{3n+4}-318y_{3n+3}\right)+3\left(28x_{n+2}-318y_{n+1}\right)\right]$
- $\frac{1}{23}\left[\left(260x_{3n+4}-318y_{3n+4}\right)+3\left(260x_{n+2}-318y_{n+2}\right)\right]$
- $\frac{1}{115}\left[\left(2572x_{3n+4}-318y_{3n+5}\right)+3\left(2572x_{n+2}-318y_{n+3}\right)\right]$
- $\frac{1}{1127}\left[\left(28x_{3n+5}-3150y_{3n+3}\right)+3\left(28x_{n+3}-3150y_{n+1}\right)\right]$
- $\frac{1}{115}\left[\left(260x_{3n+5}-3150y_{3n+4}\right)+3\left(260x_{n+3}-3150y_{n+2}\right)\right]$
- $\frac{1}{23}\left[\left(2572x_{3n+5}-3150y_{3n+5}\right)+3\left(2572x_{n+3}-3150y_{n+3}\right)\right]$
- $\frac{1}{92}\left[\left(28y_{3n+4}-260y_{3n+3}\right)+3\left(28y_{n+2}-260y_{n+1}\right)\right]$
- $\frac{1}{920}\left[\left(28y_{3n+5}-2572y_{3n+3}\right)+3\left(28y_{n+3}-2572y_{n+1}\right)\right]$
- $\frac{1}{92}\left[\left(260y_{3n+5}-2572y_{3n+4}\right)+3\left(260y_{n+3}-2572y_{n+2}\right)\right]$

Each of the following expressions represents bi-quadratic integer:

a) $\frac{1}{23^2}\left[\left(1219x_{4n+4}-115x_{4n+5}\right)+4\left(53x_{n+1}-5x_{n+2}\right)^2-1058\right]$

b) $\frac{1}{300^2}\left[\left(157500x_{4n+4}-1500x_{4n+6}\right)+4\left(525x_{n+1}-5x_{n+3}\right)^2-180000\right]$

c) $\frac{1}{23^2}\left[(694x_{4n+4}-690y_{4n+4})+4(28x_{n+1}-30y_{n+1})^2-1058\right]$

d) $\frac{1}{115^2}\left[(29900\,x_{4n+4}-3450y_{4n+5})+4(260x_{n+1}-30y_{n+2})^2-26450\right]$

e) $\frac{1}{1127^2}\left[(2898644\ x_{4n+4}-33810y_{4n+6})+4(2572x_{n+1}-30y_{n+3})^2-2540258\right]$

f) $\frac{1}{138^2}\left[(434700\,x_{4n+5}-43884\,x_{4n+6})+4(3150\,x_{n+2}-318x_{n+3})^2-38088\right]$

g) $\frac{1}{115^2}\left[(3220\,x_{4n+5}-36570y_{4n+4})+4(28x_{n+2}-318y_{n+1})^2-26450\ \right]$

h) $\frac{1}{23^2}\left[(5980x_{4n+5}-7314y_{4n+5})+4(260x_{n+2}-318y_{n+2})^2-1058\right]$

i) $\frac{1}{115^2}\left[(295780\ x_{4n+5}-36570\,y_{4n+6})+4(2572x_{n+2}-318y_{n+3})^2-26450\right]$

j) $\frac{1}{1127^2}\left[(31556\,x_{4n+6}-3550050\,y_{4n+4})+4(28x_{n+3}-3150y_{n+1})^2-2540258\ \right]$

k) $\frac{1}{115^2}\left[(29900\ x_{4n+6}-362250\ y_{4n+5})+4(260\ x_{n+3}-3150\ y_{n+2})^2-26450\right]$

l) $\frac{1}{23^2}\left[(59156\ x_{4n+6}-75450\ y_{4n+6})+4(2572\ x_{n+3}-3150\ y_{n+3})^2-1058\right]$

m) $\frac{1}{92^2}\left[(2576y_{4n+5}-23920y_{4n+4})+4(28y_{n+2}-260y_{n+1})^2-16928\right]$

n) $\frac{1}{920^2}\left[(25760y_{4n+6}-2366240y_{4n+4})+4(28y_{n+3}-2572y_{n+1})^2-1692800\right]$

o) $\frac{1}{92^2}\left[(23920y_{4n+6}-236624y_{4n+5})+4(260y_{n+3}-2572y_{n+2})^2-16928\right]$

Each of the following expressions represents Nasty number:

a) $\frac{1}{23}\left[276+318x_{2n+2}-30x_{2n+3}\right]$

b) $\frac{1}{230}\left[2760+3150x_{2n+2}-30x_{2n+4}\right]$

c) $\frac{1}{23}\left[276+168x_{2n+2}-180y_{2n+2}\right]$

d) $\frac{1}{115}\left[1380+1560x_{2n+2}-180y_{2n+3}\right]$

e) $\frac{1}{1127}\left[13524+15432x_{2n+2}-180y_{2n+4}\right]$

f) $\frac{1}{138}\left[1656+18900x_{2n+3}-1908x_{2n+4}\right]$

g) $\frac{1}{115}\left[1380+168x_{2n+3}-1908y_{2n+2}\right]$

h) $\frac{1}{23}\left[276+1560x_{2n+3}-1908y_{2n+3}\right]$

i) $\frac{1}{115}\left[1380+15432x_{2n+3}-1908y_{2n+4}\right]$

j) $\frac{1}{1127}\left[13524+168x_{2n+4}-18900y_{2n+2}\right]$

k) $\frac{1}{115}\left[1380+1560x_{2n+4}-18900y_{2n+3}\right]$

l) $\frac{1}{23}\left[276+15432x_{2n+4}-18900y_{2n+4}\right]$

m) $\frac{1}{92}\left[1104+168y_{2n+3}-1560y_{2n+2}\right]$

n) $\frac{1}{920}\left[11040+168y_{2n+4}-15432y_{2n+2}\right]$

o) $\frac{1}{92}\left[1104+1560y_{2n+4}-15432y_{2n+3}\right]$

3. Remarkable observations

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in table 2 below

Table 2: Hyperbola

Sl.no	Hyperbola	(X_n, Y_n)
1	$6X_n^2-Y_n^2=12696$	$\left[(53x_{n+1}-5x_{n+2}),(14x_{n+2}-130x_{n+1})\right]$
2	$6X_n^2-Y_n^2=1269600$	$\left[(525x_{n+1}-5x_{n+3}),(14x_{n+3}-1286x_{n+1})\right]$
3	$6X_n^2-Y_n^2=12696$	$\left[(28x_{n+1}-30y_{n+1}),(84y_{n+1}-60x_{n+1})\right]$
4	$6X_n^2-Y_n^2=3174400$	$\left[(260x_{n+1}-30y_{n+2}),(84y_{n+2}-636x_{n+1})\right]$
5	$6X_n^2-Y_n^2=30483096$	$\left[(2572x_{n+1}-30y_{n+3}),(84y_{n+3}-6300x_{n+1})\right]$
6	$6X_n^2-Y_n^2=457056$	$\left[(3150x_{n+2}-318x_{n+3}),(780x_{n+3}-7716x_{n+2})\right]$
7	$6X_n^2-Y_n^2=317400$	$\left[(28x_{n+2}-318y_{n+1}),(780y_{n+1}-60x_{n+2})\right]$

8	$6X_n^2 - Y_n^2 = 12696$	$[(260x_{n+2} - 318y_{n+2}),(780y_{n+2} - 636x_{n+2})]$
9	$6X_n^2 - Y_n^2 = 317400$	$[(2572x_{n+2} - 318y_{n+3}),(780y_{n+3} - 6300x_{n+2})]$
10	$6X_n^2 - Y_n^2 = 30483096$	$[(28x_{n+3} - 3150y_{n+1}),(7716y_{n+1} - 60x_{n+3})]$
11	$6X_n^2 - Y_n^2 = 317400$	$[(260x_{n+3} - 3150y_{n+2}),(7716y_{n+2} - 636x_{n+3})]$
12	$6X_n^2 - Y_n^2 = 12696$	$[(2572x_{n+3} - 3150y_{n+3}),(7716y_{n+3} - 6300x_{n+3})]$
13	$6X_n^2 - Y_n^2 = 203136$	$[(28y_{n+2} - 260y_{n+1}),(636y_{n+1} - 60y_{n+2})]$
14	$6X_n^2 - Y_n^2 = 20313600$	$[(28y_{n+3} - 2572y_{n+1}),(6300y_{n+1} - 60y_{n+3})]$
15	$6X_n^2 - Y_n^2 = 203136$	$[(260y_{n+3} - 2572y_{n+2}),(6300y_{n+2} - 636y_{n+3})]$

Employing linear combination among the solutions for other choices of parabola which are presented in table 3 below

Table 3: Parabola

Sl. no	Parabola	(X_n, Y_n)
1	$138X_n - Y_n^2 = 12696$	$[(46 + 53x_{2n+2} - 5x_{2n+3}),(14x_{n+2} - 130x_{n+1})]$
2	$1380X_n - Y_n^2 = 1269600$	$[(460 + 525x_{2n+2} - 5x_{2n+4}),(14x_{n+3} - 1286x_{n+1})]$
3	$138X_n - Y_n^2 = 12696$	$[(46 + 28x_{2n+2} - 30y_{2n+2}),(84y_{n+1} - 60x_{n+1})]$
4	$690X_n - Y_n^2 = 317400$	$[(230 + 260x_{2n+2} - 30y_{2n+3}),(84y_{n+2} - 636x_{n+1})]$
5	$6762X_n - Y_n^2 = 30483096$	$[(2254 + 2572x_{2n+2} - 30y_{2n+4}),(84y_{n+3} - 6300x_{n+1})]$
6	$828X_n - Y_n^2 = 457056$	$[(276 + 3150x_{2n+3} - 318x_{2n+4}),(780x_{n+3} - 7716x_{n+2})]$
7	$690X_n - Y_n^2 = 317400$	$[(230 + 28x_{2n+3} - 318y_{2n+2}),(780y_{n+1} - 60x_{n+2})]$
8	$138X_n - Y_n^2 = 12696$	$[(46 + 260x_{2n+3} - 318y_{2n+3}),(780y_{n+2} - 636x_{n+2})]$
9	$690X_n - Y_n^2 = 317400$	$[(230 + 2572x_{2n+3} - 318y_{2n+4}),(780y_{n+3} - 6300x_{n+2})]$
10	$6762X_n - Y_n^2 = 30483096$	$[(2254 + 28x_{2n+4} - 3150y_{2n+2}),(7716y_{n+1} - 60x_{n+3})]$
11	$690X_n - Y_n^2 = 317400$	$[(230 + 260x_{2n+4} - 3150y_{2n+3}),(7716y_{n+2} - 636x_{n+3})]$
12	$138X_n - Y_n^2 = 12696$	$[(46 + 2572x_{2n+4} - 3150y_{2n+4}),(7716y_{n+3} - 6300x_{n+3})]$
13	$552X_n - Y_n^2 = 203136$	$[(184 + 28y_{2n+3} - 260y_{2n+2}),(636y_{n+1} - 60y_{n+2})]$
14	$5520X_n - Y_n^2 = 20313600$	$[(1840 + 28y_{2n+4} - 2572y_{2n+2}),(6300y_{n+1} - 60y_{n+3})]$
15	$552X_n - Y_n^2 = 203136$	$[(184 + 260y_{2n+4} - 2572y_{2n+3}),(6300y_{n+2} - 636y_{n+3})]$

3.3. Properties

(i) Let p, q be two non-zero distinct integers such that $p > q > 0$, treat p, q as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$

where $\alpha = 2pq, \beta = p^2 - q^2, \gamma = p^2 + q^2, p > q > 0$

Treating $p = x_{n+1} + y_{n+1}, q = x_{n+1}$, it is observed that $T(\alpha, \beta, \gamma)$ is satisfied by the following relations:

$3\alpha - \beta - 2\gamma = 23$

$4\dfrac{A}{P} = \alpha + \beta - \gamma$

$2\dfrac{A}{P} = x_{n+1} y_{n+1}$

where A, P represents the area and perimeter of $T(\alpha, \beta, \gamma)$.

(ii) Let M, N be two non-zero distinct positive integers.

Consider $N = \dfrac{x_{n+1} - 1}{2}, M = \dfrac{y_{n+1} - 1}{2}$

It is observed that

$2t_{3,N} - 3t_{3,M} = 3$, $t_{3,M} =$ Triangular number of rank M.

4. Conclusion

In this paper, we have presented infinitely many integer solutions for the Diophantine equation, represented by hyperbola is given by $2x^2 - 3y^2 = 23$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of equations and determine their integer solutions along with suitable properties.

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