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On the Binary Quadratic Diophantine Equation $y^2=80x^2-16$

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Abstract. The Binary quadratic negative pell equation $y^2 = 80x^2 - 16$ representing a hyperbola is analyzed for its non-zero integer solutions. A few interesting relations among its solutions are presented. Further, employing the solutions of the above equation, we have obtained solutions of other choices of hyperbolas, parabolas and special pythogorean triangles.

Keywords: Binary quadratic, hyperbola, parabola, negative pell equation, integral solutions.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The binary quadratic diophantine equations (both homogeneous and non-homogeneous) are rich in variety. In [1-8] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. This communication concerns with yet another binary quadratic equation given by $y^2 = 80x^2 - 16$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special pythogorean triangles.

2. Method of analysis:

The negative pell equation representing hyperbola under consideration is

$$y^2 = 80x^2 - 16 \tag{1}$$

whose smallest positive integer solution is $x_0 = 1, y_0 = 8.$

To obtain the other solutions of (1), consider the pell equation $y^2 = 80x^2 - 16$ whose solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{80}} g_n$$

$$\tilde{y}_n = \frac{1}{2} f_n$$

where,

$$f_n = \left(9+\sqrt{80}\right)^{n+1} + \left(9-\sqrt{80}\right)^{n+1}$$

$$g_n = \left(9+\sqrt{80}\right)^{n+1} - \left(9-\sqrt{80}\right)^{n+1}$$

Applying Brahmagupta lemma between the solutions (x_0, y_0) and (x_n, y_n), the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{8}{2\sqrt{80}} g_n,$$

$$y_{n+1} = 4 f_n + \frac{40}{\sqrt{80}} g_n, \text{ where } n = -1,0,1,....$$

Recurrence relations for x and y are:

$$x_{n+3} - 18x_{n+2} + x_{n+1} = 0,$$

$$y_{n+3} - 18y_{n+2} + y_{n+1} = 0, \text{ where } n = -1,0,1,....$$

Some numerical examples of x and y satisfying (1) are given in the Table1 below:

Table 1: Examples

n	x_n	y_n
-1	1	8
0	17	152
1	305	2728
2	5473	48952

From the above table, we observe some interesting relations among the solutions which are presented below.

- The values of x_n and y_n are odd and even respectively.
  - Each of the following expressions is a nasty number.
    - $\frac{1}{16}\left[192 + 1824x_{2n+2} - 96x_{2n+3}\right]$
    - $\frac{1}{288}\left[3456 + 32736x_{2n+2} - 96x_{2n+4}\right]$
    - $\frac{1}{16}\left[192 + 960x_{2n+2} - 96y_{2n+2}\right]$
    - $\frac{1}{144}\left[1728 + 16320x_{2n+2} - 96y_{2n+3}\right]$
    - $\frac{1}{2576}\left[30912 + 292800x_{2n+2} - 96y_{2n+4}\right]$
    - $\frac{1}{16}\left[192 + 32736x_{2n+3} - 1824x_{2n+4}\right]$
    - $\frac{1}{144}\left[1728 + 960x_{2n+3} - 1824y_{2n+2}\right]$

- $\frac{1}{16}\left[192+16320x_{2n+3}-1824y_{2n+3}\right]$
- $\frac{1}{144}\left[1728+292800x_{2n+3}-1824y_{2n+4}\right]$
- $\frac{1}{2576}\left[30912+960x_{2n+4}-32736y_{2n+2}\right]$
- $\frac{1}{144}\left[1728+16320x_{2n+4}-32736y_{2n+3}\right]$
- $\frac{1}{16}\left[192+292800x_{2n+4}-32736y_{2n+4}\right]$
- $\frac{1}{1280}\left[15360+960y_{2n+3}-16320y_{2n+2}\right]$
- $\frac{1}{23040}\left[276480+960y_{2n+4}-292800y_{2n+2}\right]$
- $\frac{1}{1280}\left[15360+16320y_{2n+4}-292800y_{2n+3}\right]$

➢ Each of the following expressions is a cubical integer.

- $\frac{1}{16}\left[\left(304x_{3n+3}-16x_{3n+4}\right)+3\left(304x_{n+1}-16x_{n+2}\right)\right]$
- $\frac{1}{288}\left[\left(5456x_{3n+3}-16x_{3n+5}\right)+3\left(5456x_{n+1}-16x_{n+3}\right)\right]$
- $\frac{1}{16}\left[\left(160x_{3n+3}-16y_{3n+3}\right)+3\left(160x_{n+1}-16y_{n+1}\right)\right]$
- $\frac{1}{144}\left[\left(2720x_{3n+3}-16y_{3n+4}\right)+3\left(2720x_{n+1}-16y_{n+2}\right)\right]$
- $\frac{1}{2576}\left[\left(48800x_{3n+3}-16y_{3n+5}\right)+3\left(48800x_{n+1}-16y_{n+3}\right)\right]$
- $\frac{1}{16}\left[\left(5456x_{3n+4}-304x_{3n+5}\right)+3\left(5456x_{n+2}-304x_{n+3}\right)\right]$
- $\frac{1}{144}\left[\left(160x_{3n+4}-304y_{3n+3}\right)+3\left(160x_{n+2}-304y_{n+1}\right)\right]$
- $\frac{1}{16}\left[\left(2720x_{3n+4}-304y_{3n+4}\right)+3\left(2720x_{n+2}-304y_{n+2}\right)\right]$
- $\frac{1}{144}\left[\left(48800x_{3n+4}-304y_{3n+5}\right)+3\left(48800x_{n+2}-304y_{n+3}\right)\right]$
- $\frac{1}{2576}\left[\left(160x_{3n+5}-5456y_{3n+3}\right)+3\left(160x_{n+3}-5456y_{n+1}\right)\right]$
- $\frac{1}{144}\left[\left(2720x_{3n+5}-5456y_{3n+4}\right)+3\left(2720x_{n+3}-5456y_{n+2}\right)\right]$
- $\frac{1}{16}\left[\left(48800x_{3n+5}-5456y_{3n+5}\right)+3\left(48800x_{n+3}-5456y_{n+3}\right)\right]$

- $\frac{1}{1280}\left[\left(160y_{3n+4}-2720y_{3n+3}\right)+3\left(160y_{n+2}-2720y_{n+1}\right)\right]$
- $\frac{1}{23040}\left[\left(160y_{3n+5}-48800y_{3n+3}\right)+3\left(160y_{n+3}-48800y_{n+1}\right)\right]$
- $\frac{1}{1280}\left[\left(2720y_{3n+5}-48800y_{3n+4}\right)+3\left(2720y_{n+3}-48800y_{n+2}\right)\right]$

➢ Relations among the solutions :

- $x_{n+1}-18x_{n+2}+x_{n+3}=0$
- $9x_{n+1}-x_{n+2}+y_{n+1}=0$
- $9x_{n+3}-161x_{n+2}-y_{n+1}=0$
- $x_{n+1}-9x_{n+2}+y_{n+2}=0$
- $y_{n+1}+80x_{n+2}-9y_{n+2}=0$
- $9x_{n+1}-161x_{n+2}+y_{n+3}=0$
- $9x_{n+3}-x_{n+2}-y_{n+3}=0$
- $80x_{n+1}-y_{n+2}+9y_{n+1}=0$
- $1440x_{n+1}-y_{n+3}+161y_{n+1}=0$
- $160x_{n+2}-y_{n+3}+y_{n+1}=0$
- $80x_{n+2}-y_{n+3}+9y_{n+2}=0$
- $y_{n+1}-18y_{n+2}+y_{n+3}=0$
- $80x_{n+1}-9y_{n+3}+161y_{n+2}=0$
- $161x_{n+1}-x_{n+3}+18y_{n+1}=0$
- $x_{n+1}-x_{n+3}-2y_{n+2}=0$
- $9x_{n+2}-x_{n+3}+y_{n+2}=0$
- $288y_{n+1}+2560x_{n+3}-5152y_{n+2}=0$
- $x_{n+1}-161x_{n+3}+18y_{n+3}=0$
- $y_{n+1}+1440x_{n+3}-161y_{n+3}=0$
- $y_{n+2}+80x_{n+3}-9y_{n+3}=0$

3. Remarkable observations

3.1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in Table 2 below.

Table 2: Hyperbola

S. No	Hyperbola	(X_n, Y_n)
1	$80X_n^2 - Y_n^2 = 81920$	$[(304x_{n+1} - 16x_{n+2}), (160x_{n+2} - 2720x_{n+1})]$
2	$80X_n^2 - Y_n^2 = 26542080$	$[(5456x_{n+1} - 16x_{n+3}), (160x_{n+3} - 48800x_{n+1})]$
3	$80X_n^2 - Y_n^2 = 81920$	$[(160x_{n+1} - 16y_{n+1}), (160y_{n+1} - 1280x_{n+1})]$
4	$80X_n^2 - Y_n^2 = 6635520$	$[(2720x_{n+1} - 16y_{n+2}), (160y_{n+2} - 24320x_{n+1})]$
5	$80X_n^2 - Y_n^2 = 2123448320$	$[(48800x_{n+1} - 16y_{n+3}), (160y_{n+3} - 436480x_{n+1})]$
6	$80X_n^2 - Y_n^2 = 81920$	$[(5456x_{n+2} - 304x_{n+3}), (2720x_{n+3} - 48800x_{n+2})]$
7	$80X_n^2 - Y_n^2 = 6635520$	$[(160x_{n+2} - 304y_{n+1}), (2720y_{n+1} - 1280x_{n+2})]$
8	$80X_n^2 - Y_n^2 = 81920$	$[(2720x_{n+2} - 304y_{n+2}), (2720y_{n+2} - 24320x_{n+2})]$
9	$80X_n^2 - Y_n^2 = 6635520$	$[(48800x_{n+2} - 304y_{n+3}), (2720y_{n+3} - 436480x_{n+2})]$
10	$80X_n^2 - Y_n^2 = 2123448320$	$[(160x_{n+3} - 5456y_{n+1}), (48800y_{n+1} - 1280x_{n+3})]$
11	$80X_n^2 - Y_n^2 = 6635520$	$[(2720x_{n+3} - 5456y_{n+2}), (48800y_{n+2} - 24320x_{n+3})]$
12	$80X_n^2 - Y_n^2 = 81920$	$[(48800x_{n+3} - 5456y_{n+3}), (48800y_{n+3} - 436480x_{n+3})]$
13	$80X_n^2 - Y_n^2 = 52428800$	$[(160y_{n+2} - 2720y_{n+1}), (24320y_{n+1} - 1280y_{n+2})]$
14	$80X_n^2 - Y_n^2 = 1698693120\ 00$	$[(160y_{n+3} - 48800y_{n+1}), (436480y_{n+1} - 1280y_{n+3})]$
15	$80X_n^2 - Y_n^2 = 524288000$	$[(2720y_{n+3} - 48800y_{n+2}), (436480y_{n+2} - 24320y_{n+3})]$

3.2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in Table 3 below.

Table 3: Parabola

S. No	Parabola	(X_n, Y_n)
1	$1280\,X_n - Y_n^2 = 81920$	$[(32+304\,x_{2n+2}-16x_{2n+3}),(160\,x_{n+2}-2720\,x_{n+1})]$
2	$23040\,X_n - Y_n^2 = 26542080$	$[(576+5456\,x_{2n+2}-16x_{2n+4}),(160\,x_{n+3}-48800\,x_{n+1})]$
3	$1280\,X_n - Y_n^2 = 81920$	$[(32+160\,x_{2n+2}-16\,y_{2n+2}),(160\,y_{n+1}-1280\,x_{n+1})]$
4	$11520\,X_n - Y_n^2 = 6635520$	$[(288+2720\,x_{2n+2}-16\,y_{2n+3}),(160\,y_{n+2}-24320\,x_{n+1})]$
5	$206080\,X_n - Y_n^2 = 2123448320$	$[(5152+48800\,x_{2n+2}-16\,y_{2n+4}),(160\,y_{n+3}-436480\,x_{n+1})]$
6	$1280\,X_n - Y_n^2 = 81920$	$[(32+5456\,x_{2n+3}-304\,x_{2n+4}),(2720\,x_{n+3}-48800\,x_{n+2})]$
7	$11520\,X_n - Y_n^2 = 6635520$	$[(288+160\,x_{2n+3}-304\,y_{2n+2}),(2720\,y_{n+1}-1280\,x_{n+2})]$
8	$1280\,X_n - Y_n^2 = 81920$	$[(32+2720\,x_{2n+3}-304\,y_{2n+3}),(2720\,y_{n+2}-24320\,x_{n+2})]$
9	$11520\,X_n - Y_n^2 = 6635520$	$[(288+48800\,x_{2n+3}-304\,y_{2n+4}),(2720\,y_{n+3}-436480\,x_{n+2})]$
10	$206080\,X_n - Y_n^2 = 2123448320$	$[(2254+160\,x_{2n+4}-5456\,y_{2n+2}),(48800\,y_{n+1}-1280\,x_{n+3})]$
11	$11520\,X_n - Y_n^2 = 6635520$	$[(230+2720\,x_{2n+4}-5456\,y_{2n+3}),(48800\,y_{n+2}-24320\,x_{n+3})]$
12	$1280\,X_n - Y_n^2 = 81920$	$[(32+48800\,x_{2n+4}-5456\,y_{2n+4}),(48800\,y_{n+3}-436480\,x_{n+3})]$
13	$102400\,X_n - Y_n^2 = 52428800$	$[2560+(160\,y_{2n+3}-2720\,y_{2n+2}),(24320\,y_{n+1}-1280\,y_{n+2})]$
14	$1843200\,X_n - Y_n^2 = 1698693120\ 00$	$[(46080+160\,y_{2n+4}-48800\,y_{2n+2}),(436480\,y_{n+1}-1280\,y_{n+3})]$
15	$102400\,X_n - Y_n^2 = 524288000$	$[(2560+2720\,y_{2n+4}-48800\,y_{2n+3}),(436480\,y_{n+2}-24320\,y_{n+3}$

3.3. Consider $m = x_{n+1} + y_{n+1}$, $n = x_{n+1}$, observe that $m > n > 0$.
Treat m, n as the generators of the pythogorean triangle $T(\alpha, \beta, \gamma)$,

$$\alpha = 2mn,\ \beta = m^2 - n^2,\ \gamma = m^2 + n^2.$$

Then the following interesting relations are observed.

1) $\alpha - 40\beta + 39\gamma = 16$
2) $41\alpha - \gamma - \frac{160A}{P} = 16$
3) $21\alpha - 20\beta + 19\gamma - \frac{80A}{P} = 16$
4) $\frac{2A}{P} = x_{n+1}y_{n+1}$

4. Conclusion

In this paper, we have presented infinitely many integer solutions for all hyperbola represented by the negative pellequation $y^2 = 80x^2 - 16$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of equations and determine their integer solutions along with suitable properties.

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