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On the Non-Homogeneous Cubic Equation with Four Unknowns x^2-y^2 = z^3+w

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Abstract. An attempt has been made to determine four non-zero distinct integers x, y, z and w such that the difference of squares of any two integers equals the sum of the cubes of other two integers. A few relations among x, y, z and w are presented. A general formula for generating sequence of integer solutions based on the given solution is also presented.

Keywords: non-homogeneous cubic, cubic with four unknowns, integer solutions

AMS Mathematics Subject Classification (2010): 11D25

1. Introduction

Integral solutions for the non-homogeneous Diophantine cubic equation is an interesting concept as it can be seen from [1,2,3]. In [4-8], a few special cases of cubic Diophantine equations with three and four unknowns are studied. In this communication, we present the integral solutions of an interesting cubic equation with four unknowns $x^2 - y^2 = z^3 + w^3$. A few remarkable relations between the solutions are presented.

2. Notations

$t_{3,n} = \frac{n(n+1)}{2}$ = Triangular number of rank n

$t_{4,n} = n^2$ = Square number of rank n

$t_{6,n} = n(2n-1)$ = Hexagonal number of rank n

$PR_n = n(n+1)$ = Pronic number of rank n

$G_n = 2n-1$ = Gnomonic number of rank n

$Ct_{m,n} = \frac{mn(n-1)+2}{2}$ = Centered polygonal number of rank n with m sides.

$CP_{n,6} = n^3$ = Centered hexagonal pyramidal number of rank n.

$CP_{n,5} = \frac{n^3+n}{2}$ = Centered pentagonal pyramidal number of rank n

3. Method of analysis

The non-homogeneous cubic equation with four unknowns to be solved is,

$$x^2 - y^2 = z^3 + w^3 \tag{1}$$

Applying the method of factorization, (1) is written as the system of double equations represented by

$$x + y = z^2 - zw + w^2 \tag{2}$$

$$x - y = z + w \tag{3}$$

Solving (2) and (3) for x and y, we have

$$x = \frac{1}{2}\left(z^2 - zw + w^2 + z + w\right) \tag{4}$$

$$y = \frac{1}{2}\left(z^2 - zw + w^2 - z - w\right) \tag{5}$$

As our interest is on finding integer solutions, we have to choose z and w suitably so that, x and y are integers.

3.1. Choice

$$\text{Let } \left.\begin{aligned} z(k) &= 2k \\ w(l) &= 2l \end{aligned}\right\} \tag{6}$$

Using (6) in (4) and (5)
we have,

$$x(k,l) = 2k^2 - 2kl + 2l^2 + k + l \tag{7}$$

$$y(k,l) = 2k^2 - 2kl + 2l^2 - k - l \tag{8}$$

Thus, (6),(7) and (8) are represent integer solutions to (1).

PEOPERTIES

1. $x(k,l) - y(k,l)$ is always even.
2. $6[x(k,k) + y(k,k)]$ is a Nasty number.
3. $2x(k,1) + z(k) - 8t_{3,k} - 6 \equiv 0 \pmod 4$
4. $x(k,-1) - 2PR_k + t_{6,k} - 2t_{4,k} - 1 = 0$
5. $y(1,l) + w(l) - t_{6,l} - G_l \equiv 0 \pmod 2$
6. $x(k-1,1) + 3z(k-1) - t_{6,k} \equiv \pmod 2$
7. $x(k,k) + w(k) - 4t_{3,k} - G_k - 1 = 0$

3.2. Choice

Introducing the linear transformations

$$\left.\begin{aligned} x &= 2u + v \\ y &= 2u - v \end{aligned}\right\} \tag{9}$$

and taking

$$\left.\begin{aligned} z(\mathrm{P}) &= 2\mathrm{P} \\ w(Q) &= 2Q \end{aligned}\right\} \tag{10}$$

in (1), it is written as

$$uv = P^3 + Q^3$$

which is satisfied by

$$u = P + Q, v = P^2 - PQ + Q^2 \tag{11}$$

Substituting (11) in (9), we have

$$x(P,Q) = 2(P+Q) + P^2 - PQ + Q^2 \tag{12}$$

$$y(P,Q) = 2(P+Q) - \left(P^2 - PQ + Q^2\right) \tag{13}$$

Thus, (10),(12) and (13) represent integer solution to (1).

PROPERTIES

1. $x(P,Q) + y(P,Q)$ is always even.
2. $x(P,P) + 4t_{6,P} - 9t_{4,n} = 0$
3. $x(P,1) - w(P) - PR_P - 2Ct_{3,P} + 3t_{4,P} - P - 1 = 0$
4. $x(2,P) - y(2,P) - 2PR_P + 2G_P - 6 \equiv 0 (\mathrm{mod}\, 2)$
5. $y(-1,P) - z(P) + PR_P + 3 = 0$
6. $z(P-1) + w(P) - 4PR_P + 4t_{4,P} + 2 = 0$
7. $x(P,P) + z(P) - PR_P - 10CP_{P,5} + 5CP_{P,6} = 0$

3.3. Choice

Let

$$\left.\begin{aligned} z(k) &= 2k+1 \\ w(l) &= 2l \end{aligned}\right\} \tag{14}$$

Using (14) in (4) and (5), we have

$$x(k,l) = 2k^2 + 2l^2 - 2kl + 3k + 1 \tag{15}$$

$$y(k,l) = 2k^2 + 2l^2 - 2kl + k - 2l \tag{16}$$

Thus, (14),(15) and (16) represent integer solutions to (1).

PROPERTIES

1. $x(k,l) - y(k,l)$ is always odd.
2. $[x(k,l) - y(k,l) - w(l)]^2 = 8t_{3,k} + 1$
3. $x(k,k) + y(k,k) - 4t_{3,k} - t_{6,k} - k - 1 = 0$
4. $x(k,1) - z(k) - 2PR_k - 2 \equiv 0 (\mathrm{mod}\, 3)$
5. $y(k-1,1) + 3w(k-1) - t_{6,k} + 3 \equiv 0 (\mathrm{mod}\, 2)$
6. $6[x(2,k) + y(2,k) - 10k]$ is a Nasty number.
7. $y(k,k) + z(k) - 4t_{3,k} - G_k + k = 0$

3.4. Choice

Let

$$\left.\begin{aligned} z(k) &= (4k+4)k-5 \\ w(l) &= 4l-3 \end{aligned}\right\} \tag{17}$$

Using (17) in (1), we have

$$x(k,l) = 16k^6 + 48k^5 - 12k^4 - 104k^3 + 16l^3 + 15k^2 - 36l^2 + 75k + 27l - 37 \tag{18}$$

$$y(k,l) = 16k^6 + 48k^5 - 12k^4 - 104k^3 + 16l^3 + 15k^2 - 36l^2 + 75k + 27l - 39 \tag{19}$$

Thus, (17), (18) and (19) represent integer solutions to (1).

PROPERTIES

1. $x(k,l) - y(k,l)$ is always two.
2. $z(k) + w(k) - 4PR_k - 2G_k + 6 = 0$
3. $y(2,k) + z(k) - 16CP_{k,6} + 16t_{6,n} - 1702 \equiv 0 \pmod{15}$
4. $z(k-1) - 4PR_k + 4G_k + 9 = 0$
5. $w(k) - 4PR_k + 4t_{4,k} + 3 = 0$
6. $y(1,-k) + 32CP_{k,5} + 36PR_n + 1 \equiv 0 \pmod{25}$
7. $z(k) - w(2k) - 4PR_k + 16CP_{k,5} - 8CP_{k,6} + 2 = 0$

4. Generation of solutions

Let (x_0, y_0, z_0, w_0) be the given initial integer solution of (1).

Let $x_1 = 2h - 3^3 x_0, y_1 = h + 3^3 y_0, z_1 = 3^2 z_0, w_1 = 3^2 w_0$ (20)

be the second solution of (1), where h is a non-zero integer to be determined.
Substituting (20) in (1) and simplifying, we get

$$h = 36x_0 + 18y_0$$

Therefore, the second solution of (1) expressed in the matrix form is,

$$(x_1, y_1)^t = \mathrm{M}(x_0, y_0)^t, \qquad z_1 = 3^2 z_0, w_1 = 3^2 w_0$$

where, $\mathrm{M} = \begin{bmatrix} 45 & 36 \\ 36 & 45 \end{bmatrix}$

Repeating the above process, we have, in general

$$(x_n, y_n)^t = \mathrm{M}^n (x_0, y_0)^t, \qquad z_n = 3^{2n} z_0, w_n = 3^{2n} w_0 \tag{21}$$

where, $\mathrm{M}^n = \dfrac{9^n}{2}\begin{bmatrix} 9^n + 1 & 9^n - 1 \\ 9^n - 1 & 9^n + 1 \end{bmatrix}$

Giving $n = 1,2,3,\ldots$ inturn in (21), one obtains sequence of integer solutions to (1) based on the given solution (x_0, y_0, z_0, w_0).

5. Conclusion

In this paper, we have presented infinitely many non-zero distinct solutions to the non-homogeneous cubic equation with four unknowns given by $x^2 - y^2 = z^3 + w^3$. In other words, this problem under consideration is equivalent to finding non-zero distinct integer quadruples such that the difference of squares of any two members in a quadruple equals the sum of the cubes of other two member of the quadruple. In conclusion, one may search for quadruples with different relations among its members.

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