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Observations on the Homogeneous Ternary Cubic Equation with Four Unknowns $3(x^3+y^3)=2zw^2$

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Abstract. The homogeneous ternary cubic equation given by $3(x^3+y^3)=2zw^2$ is analysed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramidal numbers are presented.

Keywords: homogeneous cubic, ternary cubic, integer solutions, polygonal numbers, pyramidal numbers.

AMS Mathematics Subject Classification (2010): 11D25

1. Introduction

The Diophantine equation offer an unlimited field for research due to their variety [1-3]. In particular, one may refer $[4,5]$ for cubic equations with three unknowns. In [6-8] cubic equations with four unknowns are studied for its non-trivial solutions. This communication concerns with the problem of obtaining non-zero integral solutions of cubic equation with four variables given by $3(x^3+y^3)=2zw^2$. A few properties among the solutions and special numbers are presented.

2. Notations

$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$ - Polygonal number of rank n with sides m

$Ct_{m,n}=\frac{mn(n-1)+2}{2}$ - Centered polygonal number of rank n with sides m

$S_n=6n(n-1)+1$ - Star number of rank n

$PR_n=n(n+1)$ - Pronic number of rank n

$G_n=2n-1$ - Gnomonic number of rank n

$j_n=2^n+(-1)^n$ - Jacbosthal-Lucas number of rank n

3. Method of analysis

The cubic Diophantine equation with four unknowns to be solved is given by

$$3\left(x^3 + y^3\right) = 2zw^2 \quad (1)$$

The substitution of the linear transformations

$$x = u + v\,,\ y = u - v\ ,\ z = 3u\ ,u \neq v \neq 0 \quad (2)$$

in (1) leads to

$$u^2 + 3v^2 = w^2 \quad (3)$$

(3) is solved through different approaches and the different patterns of solutions of (1) obtained are presented below.

3.1. PATTERN 1

Assume $w = a^2 + 3b^2$

Write (3) as

$$\left(u + i\sqrt{3}v\right)\left(u - i\sqrt{3}v\right) = \left[\left(a + i\sqrt{3}b\right)\left(a - i\sqrt{3}b\right)\right]^2$$

Consider the positive factor

$$u + i\sqrt{3}v = a^2 + i2\sqrt{3}ab - 3b^2$$

Equating real and imaginary parts

$$u = a^2 - 3b^2$$
$$v = 2ab$$

Substituting u,v in (2), we obtain the non-zero distinct integral solutions of (1) as

$$x(a,b) = a^2 - 3b^2 + 2ab$$
$$y(a,b) = a^2 - 3b^2 - 2ab$$
$$z(a,b) = 3a^2 - 9b^2$$
$$w(a,b) = a^2 + 3b^2$$

PROPERTIES

1. $z(a,b) + 3w(a,b) - 3t_{6,b} \equiv 0(\text{mod } 3)$
2. $z(1,n) + 18t_{3,n} - 3 \equiv 0(\text{mod } 9)$
3. $6\left[y(a,b) + 4t_{4,b}\right]$ is a nasty number
4. $z\left(2^n, 2^n\right) + 6j_{2n} + 4 = 0$
5. $w\left(2^n, 2^n\right) - 4j_{2n} + 4 = 0$

3.2. PATTERN 2

Assume $w = \left(a^2 + 3b^2\right)*1 \quad (4)$

Write '1' as

$$1 = \frac{\left(1 + i\sqrt{3}\right)\left(1 - i\sqrt{3}\right)}{4} \quad (5)$$

Substituting (4) and (5) in (1) and employing the method of factorization, we get

$$\left(u+i\sqrt{3}v\right)\left(u-i\sqrt{3}v\right)=\frac{\left(1+i\sqrt{3}\right)\left(1-i\sqrt{3}\right)}{4}*\left[\left(a+i\sqrt{3}b\right)\left(a-i\sqrt{3}b\right)\right]^2$$

Consider,

$$u+i\sqrt{3}v=\frac{\left(1+i\sqrt{3}\right)}{2}\left(a+i\sqrt{3}b\right)^2$$

Equating real and imaginary parts of the above equation, we get

$$u=\frac{a^2-3b^2-6ab}{2}$$

$$v=\frac{a^2-3b^2+2ab}{2}$$

Assume a=2A, b=2B in the above equations and in view of (2) , we obtain the non –zero distinct integral solutions of (1) as

$$x(A,B)=4A^2-12B^2-8AB$$

$$y(A,B)=-16AB$$

$$z(A,B)=6A^2-18B^2-36AB$$

$$w(A,B)=4A^2+12B^2$$

PROPERTIES

1. $2x(A,B)-y(A,B)-8t_{4,n}+12t_{6,B}\equiv 0(\text{mod } 12)$
2. $z(A,-A)-24PR_A+12G_A+12=0$
3. $w(2^n,2^n)-16j_{2n}+16=0$
4. $6[(w(A,A))]$ is a Nasty number
5. $x(A,A)-y(A,A)-32t_{4,n}=0$

3.3. PATTERN 3

Assume $w=\left(a^2+3b^2\right)*1$ (6)

'1' can also be written as

$$1=\frac{\left(1+i4\sqrt{3}\right)\left(1-i4\sqrt{3}\right)}{49} \quad (7)$$

Substituting (6) and (7) in (1) and employing the method of factorization, we get

$$\left(u+i\sqrt{3}v\right)\left(u-i\sqrt{3}v\right)=\frac{\left(1+i4\sqrt{3}\right)\left(1-i4\sqrt{3}\right)}{49}*\left[\left(a+i\sqrt{3}b\right)\left(a-i\sqrt{3}b\right)\right]^2$$

Consider the positive factor

$$u+i\sqrt{3}v=\frac{1+i4\sqrt{3}}{7}\left(a+i\sqrt{3}b\right)^2 \quad (8)$$

Equating real and imaginary parts on both sides and assume a=7A, b=7B, we get

$u = 7A^2 - 21B^2 - 168AB$

$v = 28A^2 - 84B^2 + 14AB$

Substituting u & v in (2), we obtain the non-zero distinct integral solutions of (1) as

$$x(A,B) = 35A^2 - 105B^2 - 154AB$$
$$y(A,B) = -21A^2 + 63B^2 - 182AB$$
$$z(A,B) = 21A^2 - 63B^2 - 504AB$$
$$w(A,B) = 49A^2 + 147B^2$$

PROPERTIES

1. $y(B,B) - x(B,B) - 168t_{3,n} \equiv 0 \pmod{84}$
2. $z(1,n) + 63PR_n - 21 \equiv 0 \pmod{441}$
3. $y(n,n) + w(n,n) - 2Ct_{50,n} - S_n + 3 \equiv 0 \pmod{56}$
4. $6[w(A,A)]$ is a Nasty number
5. $z(2^n,2^n) + y(2^n,2^n) - 322 j_{2n} + 322 = 0$

3.4. PATTERN 4

Consider the linear transformations

$$\left.\begin{aligned} u &= \alpha + 3T \\ v &= \alpha - T \end{aligned}\right\} \tag{9}$$

Substituting (9) in (3) we get,

$$\left.\begin{aligned} (\alpha + 3T)^2 + 3(\alpha - T)^2 &= w^2 \\ 4\alpha^2 + 12T^2 &= w^2 \end{aligned}\right\} \tag{10}$$

Take

$$w = a^2 + 12b^2 \tag{11}$$

Using (11) in (10), we get

$$\left(2\alpha + i\sqrt{12}T\right)\left(2\alpha - i\sqrt{12}T\right) = \left[\left(a + i\sqrt{12}b\right)\left(a - i\sqrt{12}b\right)\right]^2$$

Equating the positive factor, we get

$$\left(2\alpha + i\sqrt{12}T\right) = a^2 + i\sqrt{12}ab - 12b^2$$

Equating real and imaginary parts

$$\left.\begin{aligned} \alpha &= \frac{a^2 - 12b^2}{2} \\ T &= 2ab \end{aligned}\right\} \tag{12}$$

Substituting (12) in (9), we obtain

$$\left.\begin{aligned} u &= \frac{a^2-12b^2+12ab}{2} \\ v &= \frac{a^2-12b^2-4ab}{2} \end{aligned}\right\} \quad (13)$$

To get integer solutions, assume a=2A, b=B in (13) and hence the non-zero distinct integer solutions of (1) are given by,

$$x(A,B)=4A^2-12B^2+8AB$$
$$y(A,B)=16AB$$
$$z(A,B)=6A^2-18B^2+36AB$$
$$w(A,B)=4A^2+12B^2$$

PROPERTIES

1. $y(A,B)-2x(A,B)+8t_{4,A}-12t_{6,B}\equiv 0(\text{mod } 12)$
2. $z(A,B)-24PR_A+12G_A+12=0$
3. $y(A,1)+w(A,B)-9G_A-2t_{6,A}-12PR_B-9\equiv 0(\text{mod } 12)$
4. $x(A,A)+z(A,A)-24t_{2,A}=0$
5. $6[\text{w}(\text{A},\text{A})],\ 6[\text{y}(\text{A},\text{A})]$ is a Nasty number

3.5. PATTERN 5

Introducing the linear transformations

$$\left.\begin{aligned} u &= \alpha-3T \\ v &= \alpha+T \end{aligned}\right\} \quad (14)$$

Substituting (14) in (3), we get

$$(\alpha-3\text{T})^2+3(\alpha+\text{T})=w^2$$
$$4\alpha^2+12\text{T}^2=w^2 \quad (15)$$

Take

$$w=a^2+12b^2 \quad (16)$$

Using (16) in (15), we get

$$\left(2\alpha+i\sqrt{12}T\right)\left(2\alpha-i\sqrt{12}T\right)=\left[\left(a=i\sqrt{12}T\right)\left(a-i\sqrt{12}T\right)\right]^2$$

Equating the positive factor, we get

$$\left(2\alpha+i\sqrt{12}T\right)=a^2-12b^2+i2\sqrt{12}ab$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} \alpha &= \frac{a^2-12b^2}{2} \\ T &= 2ab \end{aligned}\right\} \quad (17)$$

Substituting (17) in (14), we get

$$\left.\begin{aligned} u &= \frac{a^2 - 12b^2 - 12ab}{2} \\ v &= \frac{a^2 - 12b^2 + 4ab}{2} \end{aligned}\right\} \quad (18)$$

Assume a=2A, b=2B in (18) and in view (2) the non-zero distinct integer solution of (1) are as follows

$$x(A,B) = 4A^2 - 48B^2 - 16AB$$
$$y(A,B) = -32AB$$
$$z(A,B) = 6A^2 - 72B^2 - 72AB$$
$$w(A,B) = 4A^2 + 48B^2$$

PROPERTIES

1. Each of the following expressions is a Nasty number
   i. $[z(A,-A)]$
   ii. $6[w(n,n) - x(n,n) - y(n,n)]$
2. $x(A,A) + w(A,A) - 8t_{4,n} = 0$
3. $x(A,1) + y(A,1) + w(A,1) - 8t_{4,n} \equiv 0 \pmod{48}$
4. $x(A,A) + 210\, j_{2n} - 210 = 0$
5. $w(2^n, n) - 4\, j_{2n} - 44 = 0$

3.6. PATTERN 6

Write (3) as

$$(w+u)(w-u) = 3v.v \quad (19)$$

It can be written in the form of ratio as

$$\frac{v}{w-u} = \frac{w+u}{3v} = \frac{m}{n} \quad (20)$$

which is equivalent to the system of double equations

$$\left.\begin{aligned} mu + nv - mw &= 0 \\ nu - 3mv + nw &= 0 \end{aligned}\right\} \quad (21)$$

Solving (21) by method of cross multiplication, we get

$$, \quad \left.\begin{aligned} w &= 3m^2 + n^2 \\ u &= 3m^2 - n^2 \\ v &= 2mn \end{aligned}\right\} \quad (22)$$

Substituting (22) in (2), the non-zero distinct integer solutions of (1) are given by,

$$x(m,n)=3m^2-n^2+2mn$$
$$y(m,n)=3m^2-n^2-2mn$$
$$z(m,n)=9m^2-3n^2$$
$$w(m,n)=3m^2+n^2$$

PROPERTIES

1. $x(m,n)+y(m,n)+2t_{4,n}-6t_{4,m}=0$
2. $z(1,n)+6t_{3,m}-9\equiv 0(\text{mod } 3)$
3. $\left(x\left(\frac{n(n+1)}{2},n\right)-y\left(\frac{n(n+1)}{2},n\right)\right)-4P^5{}_n=0$
4. $6[x(m,n)+y(m,n)-2w(m,n)]$ is a Nasty number
5. $z(m,n)+y(m,n)-12t_{4,n}=0$

3.7. PATTERN 7

Equation (20) can be written as

$$\frac{3v}{w-u}=\frac{w+u}{v}=\frac{m}{n} \tag{23}$$

This is equivalent to the system of double equations

$$\left.\begin{aligned} mu+3nv-mw=0 \\ nu-mv+nw=0 \end{aligned}\right\} \tag{24}$$

Solving (24) by method of cross multiplication, we get

$$\left.\begin{aligned} w=-m^2-3n^2 \\ u=3n^2-m^2 \\ v=-2mn \end{aligned}\right\} \tag{25}$$

Substituting (25) in (2), the non-zero distinct integer solutions of (1) are given by,

$$x(m,n)=3n^2-m^2-2mn$$
$$y(m,n)=3n^2-m^2+2mn$$
$$z(m,n)=9n^2-3m^2$$
$$w(m,n)=-m^2-3n^2$$

PROPERTIES :

1. $z(m,n)+3w(m,n)+12t_{4,m}+6\equiv 0(\text{mod } 12)$
2. $y(m,n)-x(m,n)-4PR_m+2G_m+2=0$
3. $z(2^n,1)-9j_{2n}+12=0$

4. $x(n,n) - y(n,n) + 4t_{4,n} = 0$
5. $[z(m,m)]$ is a Nasty number

4. Conclusion

In this paper, an attempt has been made to obtain all possible integer solutions to the homogeneous ternary cubic equation with four unknowns. $3(x^3 + y^3) = 2zw^2$. One may search for other choices of solutions and their corresponding properties.

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