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On the Ternary Quadratic Diophantine Equation $3(X^2+Y^2)-5XY=75\,Z^2$

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Abstract. The ternary quadratic equation given by $3(X^2+Y^2)-5XY=75Z^2$ is considered and searched for its many different integer solutions. Four different choices of integer solutions to the above equation are presented. A few interesting relations between the solutions and special polygonal numbers are presented.

Keywords: Ternary quadratic, integer solutions.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The Diophantine equations offer an unlimited field for research due to their variety (1-3). In particular, one may refer (4-15) for quadratic equations with three unknowns.

This communication concerns with yet another interesting equation $3(X^2+Y^2)-5XY=75Z^2$ representing non-homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, few interesting relations among the solutions are presented.

2. Notations

- $t_{m,n} = n^{th}$ *term of a regular polygon with m sides*

$$= n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

- Triangular number of rank n, $T_{3,n} = \frac{n(n+1)}{2}$

3. Method of analysis

The ternary quadratic Diophantine equation to be solved for its non-zero distinct integral solution is

$$3(X^2+Y^2)-5XY=75Z^2 \tag{1}$$

The solution of linear transformations $(u \neq v \neq 0)$

$$X = u + v, Y = u - v \tag{2}$$

In (1) leads to

$$u^2 + 11v^2 = 75Z^2 \tag{3}$$

Different patterns of solutions of (1) are presented below:

3.1. PATTERN-1

Write 75 as

$$75 = \left(8 + i\sqrt{11}\right)\left(8 - i\sqrt{11}\right) \tag{4}$$

Assume $Z = a^2 + 11b^2$ (5)

Where a, b are non-zero distinct integers.

Use (4) and (5) in (3), we get

$$u^2 + 11v^2 = \left(8 + i\sqrt{11}\right)\left(8 - i\sqrt{11}\right)\left(a^2 + 11b^2\right)$$

Equating the positive and negative factors, we get

$$u + i\sqrt{11}v = \left(8 + i\sqrt{11}\right)\left(a + i\sqrt{11}b\right)^2 \tag{5a}$$

$$u - i\sqrt{11}v = \left(8 - i\sqrt{11}\right)\left(a - i\sqrt{11}b\right)^2 \tag{5b}$$

Equating real and imaginary parts either in (5a) or (5b) we get

$$u = u(a,b) = 8a^2 - 88b^2 - 22ab$$

$$v = v(a,b) = a^2 - 11b^2 + 16ab$$

Substituting the values of u and v in (2) we get

$$X = X(a,b) = 9a^2 - 99b^2 - 6ab \tag{6}$$

$$Y = Y(a,b) = 7a^2 - 77b^2 - 38ab \tag{7}$$

Thus (6), (7) and (5) represents non-zero distinct integral solutions of (1) in two parameters.

Properties observed are as follows:

1. $X(a,1) - t_{20,a} + 99 \equiv 0 \pmod 2$
2. $7Z(a,1) + Y(a,1) - t_{30,a} \equiv 0 \pmod 5$
3. $9Z(a,1) + X(a,1) - t_{38,a} \equiv 0 \pmod{11}$
4. $6\left(t_{30,\alpha^2} - 7Z\left(\alpha^2,1\right) - Y\left(\alpha^2,1\right)\right) = 6 * 5\left(\alpha^2\right)$ is a nasty number
5. $Z(a,a+1) - t_{26,a} - 11 \equiv 0 \pmod 3$
6. $Z(a,a+2) - t_{26,a} - 44 \equiv 0 \pmod 5$

3.2. PATTERN-2

Write 75 as

$$75 = \frac{\left(17 + i\sqrt{11}\right)\left(17 - i\sqrt{11}\right)}{4} \tag{8}$$

Assume $Z=a^2+11b^2$

where a, b are non-zero distinct integers.

Use (8) and (5) in (3), we get

$$u^2+11v^2=\frac{(17+i\sqrt{11})(17-i\sqrt{11})}{4}(a^2+11b^2) \tag{9}$$

Equating the positive and negative factors, we get

$$u+i\sqrt{11}v=\left(\frac{17+i\sqrt{11}}{2}\right)(a+i\sqrt{11}b)^2 \tag{9a}$$

$$u-i\sqrt{11}v=\left(\frac{17-i\sqrt{11}}{2}\right)(a-i\sqrt{11}b)^2 \tag{9b}$$

Equating real and imaginary parts either in (9a) or (9b) we get

$$u=u(a,b)=\tfrac{1}{2}\left\{17a^2-187b^2-22ab\right\}$$

$$v=v(a,b)=\tfrac{1}{2}\left\{a^2-11b^2+34ab\right\}$$

Substituting the values of u and v in (2) we get

$$X=X(a,b)=9a^2-99b^2+6ab \tag{10}$$

$$Y=Y(a,b)=8a^2-88b^2-28ab \tag{11}$$

Thus (10), (11) and (5) represents non-zero distinct integral solutions of (1) in two parameters.

Properties observed are as follows:

1. $X(a,1)-t_{20,a}+99\equiv 0 \pmod{14}$
2. $8Z(a,1)+Y(a,1)-t_{34,a}\equiv 0 \pmod{13}$
3. $9Z(a,1)+X(a,1)-t_{38,a}\equiv 0 \pmod{23}$

3.3. PATTERN-3

(3) is written as

$$u^2+11v^2=64Z^2+11Z^2$$

$$u^2-64Z^2=11(Z^2-v^2)$$

$$(u+8Z)(u-8Z)=11(Z+v)(Z-v) \tag{12}$$

Write (12) as,

$$\frac{u+8Z}{Z+v}=\frac{11(Z-v)}{u-8Z}=\frac{\alpha}{\beta},\beta\neq 0 \tag{12a}$$

we have $\frac{u+8Z}{Z+v}=\frac{\alpha}{\beta} \quad \Rightarrow u\beta+v\alpha+(8\beta-\alpha)Z=0$ (13)

From the 1st and 3rd factors of (12a), we get

$$\frac{11(Z-v)}{u-8Z}=\frac{\alpha}{\beta} \quad \Rightarrow u\alpha+11v\beta-(8\alpha+11\beta)Z=0 \tag{14}$$

Applying the method of cross multiplication for solving (13) and (14), we have

$u = u(\alpha, \beta) = 8\alpha^2 - 88\beta^2 + 22\alpha\beta$

$v = v(\alpha, \beta) = 11\beta^2 - \alpha^2 + 16\alpha\beta$

$Z = Z(\alpha, \beta) = \alpha^2 + 11\beta^2$

Substituting the values of u and v in (2)

$X = X(\alpha, \beta) = 7\alpha^2 - 77\beta^2 + 38\alpha\beta$

$Y = Y(\alpha, \beta) = 9\alpha^2 - 99\beta^2 + 6\alpha\beta$

$Z = Z(\alpha, \beta) = \alpha^2 + 11\beta^2$

Properties observed are as follows:

1. $X(\alpha,1) - t_{16,\alpha} + 77 \equiv 0(\bmod 11)$
2. $9Z(\alpha,1) + Y(\alpha,1) - t_{38,\alpha} \equiv 0(\bmod 23)$
3. $7Z(\alpha,1) + X(\alpha,1) - t_{30,\alpha} \equiv 0(\bmod 51)$
4. $Y(\alpha,1) - t_{20,\alpha} + 99 \equiv 0(\bmod 2)$

3.4. PATTERN-4

Write (12) as,

$$\frac{u+8Z}{11(Z+v)} = \frac{Z-v}{u-8Z} = \frac{\alpha}{\beta}, \beta \neq 0 \qquad (12b)$$

From the 1st and 2nd factors of (12b), we get

$$\frac{u+8Z}{11(Z+v)} = \frac{\alpha}{\beta} \quad \Rightarrow u\beta - 11v\alpha - (11\alpha - 8\beta)Z = 0 \qquad (15)$$

From the 1st and 3rd factors of (12b), we get

$$\frac{Z-v}{u-8Z} = \frac{\alpha}{\beta} \quad \Rightarrow u\alpha + v\beta - (8\alpha + \beta)Z = 0 \qquad (16)$$

Applying the method of cross multiplication for solving (15) and (16), we have

$u = u(\alpha, \beta) = 88\alpha^2 - 8\beta^2 + 22\alpha\beta$

$v = v(\alpha, \beta) = \beta^2 - 11\alpha^2 + 16\alpha\beta$

$Z = Z(\alpha, \beta) = 11\alpha^2 + \beta^2$

Substituting the values of u and v in (2)

$X = X(\alpha, \beta) = 77\alpha^2 - 7\beta^2 + 38\alpha\beta$

$Y = Y(\alpha, \beta) = 99\alpha^2 - 9\beta^2 + 6\alpha\beta$

$Z = Z(\alpha, \beta) = 11\alpha^2 + \beta^2$

Properties observed are as follows:

1. $X(\alpha,1) - t_{156,\alpha} - 7 \equiv 0(\bmod 6)$
2. $15[7Z(\alpha,1) + X(\alpha,1) - t_{60,\alpha} - t_{52,\alpha} - 180t_{3,\alpha}]$ is nasty number

3. $2\lfloor 9Z(\alpha,1)+Y(\alpha,1)-12t_{3,\alpha}\rfloor$ is a nasty number
4. $Z(\alpha,\alpha+1)-t_{26,\alpha}-1\equiv 0(\text{mod}13)$

4. Remarkable observations

4.1. Triple 1

Let

$$u_1 = u_0 + 9h$$
$$v_1 = v_0 \qquad (17)$$
$$Z_1 = Z_0 + h$$

Substituting in (3) we get

$$u_1^2 + 11v_1^2 = 75Z_1^2$$

$$(u_0+9h)^2 + 11v_0^2 = 75(Z_0+h)^2$$

$$h = 25Z_0 - 3u_0$$

Substituting h value in (17) we get

$$u_1 = 225Z_0 - 26u_0$$
$$v_1 = v_0$$
$$Z_1 = 26Z_0 - 3u_0$$

Hence the matrix representation of the above solution is

$$\begin{bmatrix} u_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} -26 & 225 \\ -3 & 26 \end{bmatrix}\begin{bmatrix} u_0 \\ Z_0 \end{bmatrix}$$

Similarly, to find the n^{th} solution of (3)

let $A = \begin{bmatrix} -26 & 225 \\ -3 & 26 \end{bmatrix}$

To find eigen values of A.

Consider $|A-\lambda I| = 0$

$$\begin{vmatrix} -26-\lambda & 225 \\ -3 & 26-\lambda \end{vmatrix} = 0 \Rightarrow (-26-\lambda)(26-\lambda)+675 = 0$$

$$\lambda = \pm 1$$

The eigen values are $\alpha = 1$ and $\beta = -1$

Substituting α,β and A values in the below equation

$$M^n = \frac{\alpha^n}{(\alpha-\beta)}(A-\beta I) + \frac{\beta^n}{(\beta-\alpha)}(A-\alpha I)$$

We get

$$M^n = \frac{1}{2}\begin{pmatrix} -25 & 225 \\ -3 & 27 \end{pmatrix} + \frac{(-1)^n}{2}\begin{pmatrix} 27 & -225 \\ 3 & -25 \end{pmatrix}$$

$$\therefore \begin{bmatrix} u_n \\ Z_n \end{bmatrix} = \frac{1}{2}\begin{pmatrix} -25+(-1)^n 27 & 225-225(-1)^n \\ -3+(-1)^n 3 & 27-(-1)^n 25 \end{pmatrix}\begin{bmatrix} u_0 \\ Z_0 \end{bmatrix}$$

Then we get n^{th} solution,

$$X_n = u_n + v_n = \frac{1}{2}\left\{\left(-25+(-1)^n 27\right)u_0 + \left(225-(-1)^n 225\right)Z_0\right\} + v_0$$

$$Y_n = u_n - v_n = \frac{1}{2}\left\{\left(-25+(-1)^n 27\right)u_0 + \left(225-(-1)^n 225\right)Z_0\right\} - v_0$$

$$Z_n = \frac{1}{2}\left\{\left(-3+(-1)^n 3\right)u_0 + \left(27-(-1)^n 25\right)Z_0\right\}$$

4.2. Triple 2

Let

$$\begin{aligned} u_1 &= 8u_0 \\ v_1 &= -8v_0 + 3h \\ Z_1 &= 8Z_0 + h \end{aligned} \tag{18}$$

Substituting in (3) we get

$$u_1^2 + 11v_1^2 = 75Z_1^2$$

$$(8u_0)^2 + 11(-8v_0 + 3h)^2 = 75(8Z_0 + h)^2$$

$$h = 50Z_0 + 22v_0$$

Substituting h value in (18) we get

$$v_1 = 58v_0 + 150Z_0$$

$$Z_1 = 22v_0 + 58Z_0$$

Hence the matrix representation of the above solution is

$$\begin{bmatrix} v_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 58 & 150 \\ 22 & 58 \end{bmatrix}\begin{bmatrix} v_0 \\ Z_0 \end{bmatrix}$$

Similarly, to find the n^{th} solution of (3)

let $A = \begin{bmatrix} 58 & 150 \\ 22 & 58 \end{bmatrix}$

To find eigen values of *A*.

Consider $|A - \lambda I| = 0$

$$\begin{vmatrix} 58-\lambda & 150 \\ 22 & 58-\lambda \end{vmatrix} = 0 \Rightarrow (58-\lambda)(58-\lambda) - 3300 = 0$$

$$\lambda = 58 \pm 10\sqrt{33}$$

The eigen values are $\alpha = 58 + 10\sqrt{33}$ and $\beta = 58 - 10\sqrt{33}$

$$\therefore M^n = \frac{\left(58+10\sqrt{33}\right)^n}{20\sqrt{33}}\begin{pmatrix} 10\sqrt{33} & 150 \\ 22 & 10\sqrt{33} \end{pmatrix} + \frac{\left(58-10\sqrt{33}\right)^n}{20\sqrt{33}}\begin{pmatrix} 10\sqrt{33} & -150 \\ -22 & 10\sqrt{33} \end{pmatrix}$$

$$= \frac{1}{20\sqrt{33}}\begin{pmatrix} 10\sqrt{33}A_n & 150B_n \\ 22B_n & 10\sqrt{33}A_n \end{pmatrix}$$

where $A_n = \left[\left(58+10\sqrt{33}\right)^n + \left(58-10\sqrt{33}\right)^n\right]$

$$B_n = \left[\left(58+10\sqrt{33}\right)^n - \left(58-10\sqrt{33}\right)^n\right]$$

Thus,

$$\begin{bmatrix} v_n \\ Z_n \end{bmatrix} = \frac{1}{20\sqrt{33}}\begin{pmatrix} 10\sqrt{33}A_n & 150B_n \\ 22B_n & 10\sqrt{33}A_n \end{pmatrix}\begin{bmatrix} v_0 \\ Z_0 \end{bmatrix}$$

Then we get n^th solution,

$$X_n = u_n + v_n = 8^n u_0 + \frac{1}{20\sqrt{33}}\left\{10\sqrt{33}A_n v_0 + 150 B_n Z_0\right\}$$

$$Y_n = u_n - v_n = 8^n u_0 - \frac{1}{20\sqrt{33}}\left\{10\sqrt{33}A_n v_0 + 150 B_n Z_0\right\}$$

$$Z_n = \frac{1}{20\sqrt{33}}\left\{22B_n v_0 + 10\sqrt{33}A_n Z_0\right\}$$

4.3. Triple 3

Let

$$u_1 = h - 6u_0$$
$$v_1 = h - 6v_0 \qquad (19)$$
$$Z_1 = 6Z_0$$

Substituting in (3) we get

$$u_1^2 + 11v_1^2 = 75Z_1^2$$

$$(h-6u_0)^2 + 11(h-6v_0)^2 = 75(6Z_0)^2$$

$$h = u_0 + 11v_0$$

Substituting h value in (19) we get

$$u_1 = -5u_0 + 11v_0$$
$$v_1 = u_0 + 5v_0$$

Hence the matrix representation of the above solution is

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} -5 & 11 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

Similarly, to find the n^th solution of (3) let $A = \begin{bmatrix} -5 & 11 \\ 1 & 5 \end{bmatrix}$

To find eigen values of A:

Consider $|A - \lambda I| = 0$

$$\begin{vmatrix} -5-\lambda & 11 \\ 1 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (-5-\lambda)(5-\lambda)+11 = 0$$

$\lambda = \pm 6$

The eigen values are $\alpha = 6$ and $\beta = -6$

$$\therefore M^n = \frac{6^n}{12}\begin{pmatrix} 1 & 11 \\ 1 & 11 \end{pmatrix} + \frac{(-6)^n}{12}\begin{pmatrix} 11 & -11 \\ -1 & 1 \end{pmatrix}$$

Then

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \frac{1}{12}\begin{pmatrix} 6^n + 11(-6)^n & 11 * 6^n - 11(-6)^n \\ 6^n - (-6)^n & 11 * 6^n + (-6)^n \end{pmatrix}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

Then we get n[th] solution,

$$X_n = u_n + v_n = \frac{1}{12}\left\{\left(2 * 6^n + 10(-6)^n\right)u_0 + \left(22 * 6^n - 10(-6)^n\right)v_0\right\}$$

$$Y_n = u_n - v_n = \frac{1}{12}\left\{\left(2 * 6^n + 10(-6)^n\right)u_0 - \left(22 * 6^n - 10(-6)^n\right)v_0\right\}$$

$$Z_n = 6^n Z_0$$

5. Conclusion

In this paper, we have presented infinitely many non-zero distinct integer solutions to the ternary quadratic equation

$$3\left(X^2 + Y^2\right) - 5XY = 75Z^2$$

representing a homogeneous cone. As Diophantine equations are rich in variety.

To conclude, one may search for other forms of three dimensional surfaces, namely, non-homogeneous cone, paraboloid, ellipsoid, hyperboloid, hyperbolic paraboloid and so on for finding integral points on them and corresponding properties

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