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On Homogeneous Cubic Diophantine equation with Four Unknowns $3(x^3+y^3) = 8zp^2$

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Abstract. The homogeneous cubic Diophantine equation with four unknowns represented by $3(x^3+y^3)=8zp^2$ is analyzed for finding its non-zero distinct integral solutions. Different patterns of solutions of the equation under consideration are obtained the relations between the integer solutions and special numbers namely polygonal number and pyramidal number are exhibited.

Keywords: Homogeneous equation with four unknowns, integral solutions

AMS Mathematics Subject Classification (2010): 11D25

1. Introduction

Integral solutions for the homogeneous Diophantine cubic equation is an interesting concept as it can be seen from [1-3]. In [4-8], a few special cases of cubic Diophantine equations with three and four unknowns are studied. In this communication, we present the integral solutions of an interesting cubic equation with four unknowns $3(x^3+y^3)=8zp^2$. A few remarkable relations between the solutions are presented.

2. Notations

1. $t_{3,n} = \dfrac{n(n+1)}{2}$ = Triangular number of rank n
2. $t_{4,n} = n^2$ = Square number of rank n
3. $CP_{k,6} = n^3$ = Centered hexagonal pyramidal number of rank n.

3. Method of analysis

The homogeneous cubic equation with four unknowns to be solved is

$$3(x^3+y^3)=8zp^2 \qquad (1)$$

Introducing the linear transformations

$$x=u+v, y=u-v, z=3u, (u \neq v \neq 0) \qquad (2)$$

In (1) leads to

$$u^2 + 3v^2 = 4p^2 \quad (3)$$

We present below different methods of solving (3) and thus, in view of (2), different patterns of solutions to (1) are obtained.

3.1. Pattern-1

Consider

$$p = X \pm 3T, v = X \pm 4T \quad (4)$$

Using (4) in (3)

$$X^2 = u^2 + 12T^2 \quad (5)$$

This satisfied by

$$\left.\begin{aligned} X &= 12R^2 + S^2 \\ T &= 2RS \\ u &= 12R^2 - S^2 \end{aligned}\right\} \quad (6)$$

In view of (2), the corresponding non zero distinct integer solutions to (1) are given by

$$x = 24R^2 \pm 8RS$$

$$y = -2S^2 \mp 8RS$$

$$z = 36R^2 - 3S^2$$

$$p = 12R^2 + S^2 \pm 6RS$$

Properties:

1. $x(R,1) + y(R,1) + 24t_{3,R} - 2t_{4,R} = 0$
2. $x(R,R) - 32t_{4,R} = 0$
3. $z(R,1) + p(R,1) - 25t_{4,R} + 2t_{3,R} + 4 = 0$ **3.2.**

3.2. Pattern -2

Let $v = 2ab, u = 3a^2 - b^2, 2p = 3a^2 + b^2$ (7)

Case 1:

Assume $a = 2a + 1, b = 2b + 1$ (8)

Substituting (8) and in (7)

$$\left.\begin{aligned} u &= 12a^2 - 4b^2 + 12a - 4b + 2 \\ v &= 4a + 4b + 8ab + 2 \end{aligned}\right\} \quad (9)$$

In view of (2) we have

$$\left.\begin{aligned} x &= 12a^2 - 4b^2 + 16a + 8ab + 4 \\ y &= 12a^2 - 4b^2 - 8a + 8b - 8ab \\ z &= 36a^2 - 12b^2 + 36a - 12b + 6 \\ p &= 3a^2 + 2b^2 + 2b + 6a + 2 \end{aligned}\right\} \quad (10)$$

Thus (10) represent non-zero distinct integral solutions to (1).

Properties:

1. $x(a,b) - y(a,b) - 80t_{3,a} + 40t_{4,b} + 4 = 0$
2. $z(a,b) - p(a,b) + 28t_{3,b} - 60t_{4,a} - 4 = 0$
3. $x(a,1) - 72t_{3,a} - 36t_{4,a} = 0$

Case 2:

Assume $a = 2a, b = 2b$ (11)

Using (11) in (7)

$$\left.\begin{aligned} u &= 12a^2 - 4b^2 \\ v &= 8ab \end{aligned}\right\} \quad (12)$$

In view of (2), we have

$$\left.\begin{aligned} x &= (12a^2 - 4b^2 + 8ab) \\ y &= (12a^2 - 4b^2 - 8ab) \\ z &= (36a^2 - 12b^2) \\ p &= 6a^2 + 2b^2 \end{aligned}\right\} \quad (13)$$

Thus (13) represent non-zero distinct integral solutions to (1).

Properties:

1. $x(a,1) - 16t_{3,a} - 4t_{4,a} + 4 = 0$
2 $x(a,1) + y(a,1) + z(a,1) - 50t_{4,a} - 20 = 0$.
3. $x(a,1) + y(a,1) + z(a,1) - 4(t_{3,a} + t_{4,a}) = 8$

3.3. Pattern-3

Let

$$p = a^2 + 3b^2 \quad (14)$$

Write 4 as

$$4 = (1 + i\sqrt{3})(1 - i\sqrt{3}) \quad (15)$$

Substituting (14) and (15) in (3) and applying the method of factorization we define

$$(u + i\sqrt{3}v) = (1 + i\sqrt{3})(a + ib\sqrt{3})^2 \quad (16)$$

Equating the real and imaginary part

$$\left.\begin{aligned} u &= a^2 - 3b^2 - 6ab \\ v &= 2ab + a^2 - 3b^2 \end{aligned}\right\} \tag{17}$$

Using (2) in (17), the corresponding non-zero distinct integral solutions to (1) are obtained as

$$\left.\begin{aligned} x &= 2a^2 - 6b^2 - 4ab \\ y &= -2b^2 - 8ab \\ z &= 3a^2 - 9b^2 - 18ab \end{aligned}\right\} \tag{18}$$

Thus (14) and (18) represent non zero distinct integral solutions to (1).

Properties:

1. $x(a,1) - y(a,1) - 8t_{3,a} + 6 = 0$
2. $z(1,b) + p(1,b) + 6t_{4,a} + 14 = 0$
3. $x(a,1) + p(a,1) + 4t_{3,a} - 2t_{4,a} - t_{8,a} + 3 = 0$

3.4. Pattern-4

Write 4 as

$$4 = \frac{(2 + i8\sqrt{3})(2 - i8\sqrt{3})}{49} \tag{19}$$

Substituting (14) and (19) in (3) and applying the method of factorization we define

$$(u + i\sqrt{3}v) = \frac{(2 + i8\sqrt{3})}{7}(a + ib\sqrt{3})^2 \tag{20}$$

Equating the real and imaginary part

$$\left.\begin{aligned} 7u &= 2a^2 - 7b^2 - 48ab \\ 7v &= 4ab + 8a^2 - 24b^2 \end{aligned}\right\} \tag{21}$$

As our interest is to get the integral solutions. So we replace a by $7a$ and b by $7b$ we get

$$\left.\begin{aligned} u &= 14a^2 - 42b^2 - 336ab \\ v &= 28ab + 56a^2 - 168b^2 \end{aligned}\right\} \tag{22}$$

Using (2) in (22), the corresponding non-zero distinct integral solutions to (1) are obtained as

$$\left.\begin{aligned} x &= 70a^2 - 210b^2 - 308ab \\ y &= -42a^2 + 126b^2 - 364ab \\ z &= 42a^2 - 126b^2 - 1008a \\ p &= 49a^2 + 147b^2 \end{aligned}\right\} \tag{23}$$

Properties:

1. $x(a,1) + 616t_{3,a} - 378t_{4,1} + 210 = 0$
2. $y(a,a) + z(a,a) - 1372t_{4,a} = 0$
3. $y(a,1) + z(a,1) + p(a,1) - 7(203t_{4,a} - 392t_{3,a} + 21) = 0$

3.5. Pattern-5

Write (3) in form of ratio

$$\frac{u+p}{p-v} = \frac{3(p+v)}{u-p} = \frac{\alpha}{\beta}, \beta \neq 0 \tag{24}$$

This is equivalent to the system of double equations

$$\left.\begin{aligned} \beta u + \alpha v + p(\beta - \alpha) = 0 \\ \alpha u - 3v\beta - p(3\beta + \alpha) = 0 \end{aligned}\right\} \tag{25}$$

Solving (25) by applying the cross multiplication and using (2), the corresponding non-zero distinct integer solutions to (1) are obtained as

$$\left.\begin{aligned} x &= 6\beta^2 - 2\alpha^2 - 4\alpha\beta \\ y &= -8\alpha\beta \\ z &= 9\beta^2 - 3\alpha^2 - 18\alpha\beta \\ p &= -3\beta^2 - \alpha^2 \end{aligned}\right\} \tag{26}$$

Properties:

1. $x(\alpha,\beta) + y(\alpha,\beta) - 18t_{4,\beta} + 24t_{4,\alpha} = 0$
2. $x(\alpha,\alpha) - 8t_{4,\alpha} = 0$
3. $x(\alpha,1) + p(\alpha,1) - 12t_{3,\beta} + 5 = 0$

Note that (5) is represented as the system of double equations as shown in the table1 below

Table1:

System	1	2	3	4	5
$X+u$	T^2	$6T^2$	$3T^2$	$2T^2$	$12T$
$X-u$	12	2	4	6	T

Solving each of the above system in turn the corresponding non-zero values of X, u, T are obtained. In view of (2), the corresponding integer solutions to (1) are exhibited in Table 2 below.

Table 2:

System	x	y	z	p
1	$4k^2-8k$	$8k-12$	$6k^2-18$	$2k^2-6k+6$
2	$24k^2-8k$	$8k-4$	$36k^2-3$	$12k^2-6k+1$
3	$12k^2-8k$	$8k-4$	$18k^2-6$	$6k^2+6k+6$
4	$8k^2-8k$	$8k-6$	$12k^2-9$	$4k^2-6k+3$
5	$2k$	0	$3k$	k

In addition to the above set of solution we have some more set of different solution satisfying (1) and they illustrated.

3.6. Pattern -6

Consider

$$u = X \pm 3T, v = X \mp T \tag{27}$$

Substituting (27) in (3) we have

$$X^2 + 3T^2 = p^2 \tag{28}$$

Introducing the linear transformation

$$T = 2rs, X = 3r^2 - s^2, p = 3r^2 + s^2 \tag{29}$$

Substituting (29) in (27), we get

$$\begin{aligned} u &= 3r^2 - s^2 + 6rs \\ v &= 3r^2 - s^2 - 2rs \end{aligned} \tag{30}$$

In view of (2), we get

$$\left.\begin{aligned} x &= 6r^2 - s^2 \pm 4rs \\ y &= \pm 8rs \\ z &= 9r^2 - 3s^2 \pm 18rs \\ p &= 3r^2 + s^2 \end{aligned}\right\} \tag{31}$$

Thus (31) represent non-zero distinct integral solutions to (1)

Properties:

1. $x(r,1) + y(r,1) - 2(12t_{3,r} + 3t_{4,s}) + 1 = 0$
2. $z(r,1) + p(r,1) - 36t_{3,r} + 6t_{4,r} + 2 = 0$
3. $x(r,1) - y(r,1) + 8t_{3,s} - 10t_{4,r} + 1 = 0$

Note that (28) is represented as the system of double equations as shown in the table1 below.

System	1	2	3
$p+X$	T^2	$3T$	$3T^2$
$p-X$	3	T	1

Table 3:

Solving each of the above system in turn the corresponding non-zero values of X, u, T are obtained. In view of (2), the corresponding integer solutions to (1) are exhibited in Table4 below.

System	x	y	z	p
1	$(4k^2+8k, 4k^2-4)$	$(4s+2, -4s+2)$	$6k^2+21k+6, 6k^2-15k-11$	$2k^2+k+1$
2	$(4k, 0)$	$(4k, -4k)$	$(12k, -6k)$	$2k$
3	$(12k^2+16k+4, 12k^2+8k$	$(8k+4, -8k-4)$	$\begin{pmatrix}18k^2+36k+12,\\ 18k^2-6\end{pmatrix}$	$6k^2+6k+2$

Table 4:

In addition to the above set of solution we have some more set of different solution satisfying (1) and they illustrated.

3.6. Pattern -6

Write (28) as

$$X^2+3T^2=p^2*1 \tag{32}$$

Substituting (14) in (28) and applying the method of factorization we define

$$\left(X+i\sqrt{3}T\right)=\left(a+i\sqrt{3}b\right)^2 \tag{33}$$

Equating the real and imaginary part

$$\left.\begin{aligned} X &= a^2-3b^2 \\ T &= 2ab \end{aligned}\right\} \tag{34}$$

Using (3) in (27), we get

$$u = a^2-3b^2 \pm 6ab$$

$$v = a^2-3b^2 \mp 2ab \tag{35}$$

In view of (2), we get

$$\left.\begin{aligned} x &= 2a^2-6b^2 \pm 4ab \\ y &= \pm 8ab \\ z &= 3a^2-9b^2 \pm 18ab \end{aligned}\right\} \tag{36}$$

Thus (36) and (32) represent non-zero distinct integral solutions to (1).

Properties:

1. $x(a,a)+y(a,a)-8t_{4,a}=0$
2. $z(a,a)-p(a,a)-8t_{4,a}=0$
3. $z(a,a)-12t_{4,a}=0$

4. Conclusion

This paper, we have many non-zero distinct integral solutions to the homogeneous cubic equation given by $3(x^3+y^3)=8zp^2$. As Diophantine equations are rich in variety. One may search for integer solution to other choices of homogeneous cubic equation and determine their corresponding properties.

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