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On the Ternary Quadratic Diophantine Equation $5y^2 = 3x^2+2z^2$

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Abstract. The ternary homogeneous quadratic equation given by $5y^2 = 3x^2 + 2z^2$ is analysed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramidal numbers are presented. Also, given a solution, a generation of sequence of solution based on the given solutions are presented.

Keywords: ternary quadratic, integer solutions, homogeneous quadratic, polygonal numbers, pyramidal numbers.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-9] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $5y^2 = 3x^2 + 2z^2$ representing homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2. Notations

$T_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$ - Polygonal number of rank n with sides*m*

$Ct_{m,n} = \frac{mn(n-1)+2}{2}$ - Centered Polygonal number of rank *n* with sides *m*

$S_n = 6n(n-1)+1$ - Star number of rank *n*

$PR_n = n(n+1)$ - Pronic number of rank *n*

$G_n = 2n-1$ - Gnomonic number of rank *n*

$So_n = n(2n^2-1)$ - Stella octangular number of rank *n*

$CP_{n,6} = n^3$ - Centered hexagonal pyramidal number of rank *n*

3. Method of analysis

The ternary quadratic equation under consideration is

$$5y^2 = 3x^2 + 2z^2 \quad (1)$$

It is observed that (1) is satisfied by

$(5k,5k,-5k),(-3k,5k,7k),(3k,11k,17k),(3k,3k,-3k),(18k,14k,2k)$ and $(-2k,2k,2k)$

However, we have other patterns of solutions of (1) which we present below.

3.1. PATTERN I

The substitution of linear transformation

$$x = X + 2T \text{ and } z = X - 3T \quad (2)$$

in (1) leads to

$$y^2 - X^2 = 6T^2 \quad (3)$$

Write (3) as

$$(y + X)(y - X) = 6T^2 \quad (4)$$

The equation (4) is written as the system of 2 equations as follows

Cases	$y+X$	$y-X$
1	T^2	6
2	$3T^2$	2

CASE 1:

Consider,

$$y + X = T^2 \quad y - X = 6$$

and solving we get

$$\left.\begin{aligned} y &= 2k^2 + 3 \\ X &= 2k^2 - 3 \\ T &= 2k \end{aligned}\right\} \quad (5)$$

Substituting (5) in (2), we get the corresponding non zero distinct integer solution to (1) as follows.

$$x(k) = 2k^2 + 4k - 3 \quad y(k) = 2k^2 + 3 \quad z(k) = 2k^2 - 6k - 3$$

PROPERTIES:

1. $x(n) + y(n) - 4PR_n = 0$
2. $x(1) + y(1) - CP_{2,6} = 0$
3. $x(a) - z(a) - 10PR_a + 10t_{4,n} = 0$
4. $z(n) + y(n) - S_n + PR_n + 1 \equiv 0 \pmod{n}$
5. $3[y(n) - 3]$ is a nasty number

Note: In addition to (2), one may also consider the linear transformation $x = X - 2T$ and $z = X + 3T$ and get the different set of non zero distinct integer solution of (1) as

$$x = 2k^2 - 4k - 3 \; y = 2k^2 + 3 \; z = 2k^2 + 6k - 3$$

CASE 2:
Consider,

$$y + X = 3T^2 \; y - X = 2$$

and solving we get

$$\left.\begin{aligned} y &= 6k^2 + 1 \\ X &= 6k^2 - 1 \\ T &= 2k \end{aligned}\right\} \qquad (6)$$

Substituting (6) in (2), we get the corresponding non zero distinct integer solution to (1) as follows.

$$x(k) = 6k^2 + 4k - 1 \; y(k) = 6k^2 + 1 \; z(k) = 6k^2 - 6k - 1$$

PROPERTIES

1. $y(n) + z(n) - S_n - 6t_{4,n} + 1 = 0$
2. $x(n) - z(n) - G_n - 1 \equiv 0 (\mathrm{mod}\, 8)$
3. $x(b) + y(b) - 2ct_{4,b} - 2PR_n + 2 \equiv 0 (\mathrm{mod}\, 6)$
4. $x(n) - 2t_{4,n} - 4PR_n + 1 = 0$
5. $[y(n) - 1]$ is a nasty number

Note: In addition to (2), one may also consider the linear transformation $x = X - 2T$ and $z = X + 3T$ and get different set of non zero distinct integer solution of (1) as

$$x = 6k^2 - 4k - 1 \; y = 6k^2 + 1 \; z = 6k^2 + 6k - 1$$

3.2. PATTERN II

Equation (1) can be written as

$$3x^2 = 5y^2 - 2z^2 \qquad (7)$$

The substitution of linear transformation

$$y = X + 2T \text{ and } z = X + 5T \qquad (8)$$

in (7) leads to

$$X^2 - x^2 = 10T^2 \qquad (9)$$

Write (9) as

$$(X + x)(X - x) = 10T^2 \qquad (10)$$

The equation (10) is written as the system of 2 equations as follows

Cases	$X + x$	$X - x$
1	T^2	10
2	$5T^2$	2

CASE 1:
Consider,

$$X + x = T^2 \quad X - x = 10$$

and solving we get

$$\left.\begin{aligned} X &= 2k^2 + 5 \\ x &= 2k^2 - 5 \\ T &= 2k \end{aligned}\right\} \qquad (11)$$

Substituting (11) in (8), we get the corresponding non zero distinct integer solution to (7) as follows.

$$x(k) = 2k^2 - 5 \quad y(k) = 2k^2 + 4k + 5 \quad z(k) = 2k^2 + 10k + 5$$

PROPERTIES

1. $z(a) - y(a) - G_a - 1 \equiv 0 \pmod 4$
2. $y(n) - x(n) - 4PR_n + 4t_{4,n} - 10 = 0$
3. $x(1) + z(1) - So_2 = 0$
4. $y(n) + z(n) - 2t_{6,n} - 10 \equiv 0 \pmod{16}$
5. $[3(x(n)+5)]$ is a nasty number

Note: In addition to (8), one may also consider the linear transformation $y = X - 2T$ and $z = X - 5T$ and get different set of non zero distinct integer solution of (7) as

$$x = 2k^2 - 5 \quad y = 2k^2 - 4k + 5 \quad z = 2k^2 - 10k + 5$$

CASE 2:

Consider,

$$X + x = 5T^2 \quad X - x = 2$$

and solving we get

$$\left.\begin{aligned} X &= 10k^2 + 1 \\ x &= 10k^2 - 1 \\ T &= 2k \end{aligned}\right\} \qquad (12)$$

Substituting (12) in (8), we get the corresponding non zero distinct integer solution to (7) as follows.

$$x(k) = 10k^2 - 1 \quad y(k) = 10k^2 + 4k + 1 \quad z(k) = 10k^2 + 10k + 1$$

PROPERTIES

1. $x(n) + y(n) - 2t_{22,n} \equiv 0 \pmod{22}$
2. $z(n) - x(n) - G_n - 3 \equiv 0 \pmod 8$ 3. $z(b) - 10PR_n - 1 = 0$
4. $y(a) - x(a) - S_a + 6t_{4,a} - 1 \equiv 0 \pmod{10}$
5. $[y(a) - z(a) + 6PR_a]$ is a nasty number

Note: In addition to (8), one may also consider the linear transformation $y = X - 2T$ and $z = X - 5T$ and get different set of non zero distinct integer solution of (7) as

$x = 10k^2 - 1 \; y = 10k^2 - 4k + 1 \; z = 10k^2 - 10k + 1$

3.3. PATTERN III

Equation (1) can also be written as

$$2z^2 = 5y^2 - 3x^2 \tag{13}$$

The substitution of linear transformation

$$y = X + 3T \text{ and } x = X + 5T \tag{14}$$

in (13) leads to

$$X^2 - z^2 = 15T^2 \tag{15}$$

Write (15) as

$(X + z)(X - z) = 15T^2$ (16) The equation (16) is written as the system of 2 equations as follows.

Cases	$X + z$	$X - z$
1	T^2	15
2	$3T^2$	5
3	$5T^2$	3

CASE 1:

Consider,

$X + z = T^2 \; X - z = 15$

and solving we get

$$\left.\begin{aligned} X &= 2k^2 + 2k + 8 \\ z &= 2k^2 + 2k - 7 \\ T &= 2k + 1 \end{aligned}\right\} \tag{17}$$

Substituting (17) in (14), we get the corresponding non zero distinct integer solution to (13) as follows.

$x(k) = 2k^2 + 12k + 13 \; y(k) = 2k^2 + 8k + 11 \; z(k) = 2k^2 + 2k - 7$

PROPERTIES

1. $x(a) - y(a) - G_a - 3 \equiv 0 \pmod 2$
2. $y(n) + z(n) - 4PR_n + S_n - 6t_{4,n} - 5 = 0$
3. $z(b) - 2PR_b + 7 = 0$
4. $z(n) - x(n) - 2ct_{10,n} + 10PR_n + 22 \equiv 0 \pmod{10}$
5. $3[z(n) - 2n + 7]$ is a nasty number

Note: In addition to (14), one may also consider the linear transformation $y = X - 3T$ and $x = X - 5T$ and get different set of non zero distinct integer solution of (13) as

$x = 2k^2 - 8k + 3 \; y = 2k^2 - 4k + 5 \; z = 2k^2 + 2k - 7$

CASE 2:

Consider ,

$$X + z = 3T^2 \quad X - z = 5$$

and solving we get

$$\left.\begin{aligned} X &= 6k^2 + 6k + 4 \\ y &= 6k^2 + 6k - 1 \\ T &= 2k + 1 \end{aligned}\right\} \tag{18}$$

Substituting (18) in (14), we get the corresponding non zero distinct integer solution to (13) as follows.

$$x(k) = 6k^2 + 16k + 9 \quad y(k) = 6k^2 + 12 + 7 \quad z(k) = 6k^2 + 6k - 1$$

PROPERTIES

1. $z(a) - 6PR_a + 1 = 0$
2. $y(n) - z(n) + 2ct_{6,n} - S_n - 9 \equiv 0 \pmod 6$
3. $x(n) - z(n) - G_n - 11 \equiv 0 \pmod 8$
4. $x(n) + y(n) - 12PR_n - 2t_{18,n} + 2ct_{16,n} - 18 \equiv 0 \pmod{14}$
5. $[y(n) - 12n - 7]$ is a nasty number

Note: In addition to (14), one may also consider the linear transformation $y = X - 3T$ and $x = X - 5T$ and get different set of non zero distinct integer solution of (13) as

$$x = 6k^2 - 4k - 1 \quad y = 6k^2 + 1 \quad z = 6k^2 + 6k - 1$$

CASE 3:

Consider,

$$X + z = 5T^2 \quad X - z = 3$$

and solving we get

$$\left.\begin{aligned} X &= 10k^2 + 10k + 4 \\ z &= 10k^2 + 10k + 1 \\ T &= 2k + 1 \end{aligned}\right\} \tag{19}$$

Substituting (19) in (14), we get the corresponding non zero distinct integer solution to (13) as follows.

$$x(k) = 10k^2 + 20k + 9 \quad y(k) = 10k^2 + 16k + 7 \quad z(k) = 10k^2 + 10k + 1$$

PROPERTIES

1. $z(b) - 10PR_b - 1 = 0$
2. $x(n) - z(n) + 2ct_{10,n} - 10t_{4,n} - 10 = 0$
3. $x(a) + y(a) - S_a - 14t_{4,a} - 15 \equiv 0 \pmod{42}$
4. $y(n) + z(n) - 2t_{22,n} - 8 \equiv 0 \pmod{44}$
5. $y(a) - 10t_{4,a} - G_a - 8 \equiv 0 \pmod{14}$

Note: In addition to (14), one may also consider the linear transformation $y = X - 3T$ and $x = X - 5T$ and get different set of non zero distinct integer solution of (13) as

$$x = 10k^2 - 1 \quad y = 10k^2 + 4k + 1 \quad z = 10k^2 + 10k + 1$$

4. Remarkable observations

Let (x_0, y_0, z_0) be the given initial integer solution of (1).

4.1. TRIPLE 1

Let $x_1 = 3x_0, y_1 = 3y_0 + h, z_1 = 3z_0 + h$ (20)

be the second solution of (1), where h is a non zero integer to be determined.
Substituting (20) in (1) and simplifying, we get

$$h = 4z_0 - 10y_0$$

Therefore, the second solution of (1) expressed in the matrix form is,

$$(y_1, z_1)^t = M(y_0, z_0)^t \ , \ x_1 = 3x_0$$

where, $M = \begin{bmatrix} -7 & 4 \\ -10 & 7 \end{bmatrix}$

Repeating the above process, we have, in general

$$(y_n, z_n)^t = M(y_0, z_0)^t, x_n = 3^n x_0 \quad (21)$$

where, $M^n = \frac{1}{6}\begin{bmatrix} -4(3^n)+10(-3)^n & 4(3)^n - 4(-3)^n \\ -10(3^n)+10(-3)^n & 10(3^n)-4(-3)^n \end{bmatrix}$

Giving $n = 1,2,3,...$ in turn in (21), one obtains sequence of integer solutions to (1) based on the given solution (x_0, y_0, z_0).

4.2.TRIPLE 2

Let $x_1 = 5x_0 + h, y_1 = 5y_0, z_1 = 5z_0 + h$ (22)

be the second solution of (1), where h is a non zero integer to be determined.
Substituting (22) in (1) and simplifying, we get

$$h = -6x_0 - 4z_0$$

Therefore, the second solution of (1) expressed in the matrix form is,

$$(x_1, z_1)^t = M(x_0, z_0)^t \ , \ y_1 = 5y_0$$

where, $M = \begin{bmatrix} -1 & -4 \\ -6 & 1 \end{bmatrix}$

Repeating the above process, we have, in general

$$(x_n, z_n)^t = M(x_0, z_0)^t, y_n = 5^n y_0 \quad (23)$$

where, $M^n = \frac{1}{10}\begin{bmatrix} 4(5^n)+6(-5)^n & -4(5^n)+4(-5)^n \\ -6(5^n)+6(-5)^n & 6(5^n)+4(-5)^n \end{bmatrix}$

Giving $n = 1,2,3,...$ inturn in (23), one obtains sequence of integer solutions to (1) based on the given solution (x_0, y_0, z_0).

4.3. TRIPLE 3

Let $x_1 = x_0 + h$, $y_1 = y_0 + h$, $z_1 = z_0$ (24)

be the second solution of (1), where h is a non zero integer to be determined.
Substituting (24) in (1) and simplifying, we get

$$h = 3x_0 - 5y_0$$

Therefore, the second solution of (1) expressed in the matrix form is,

$$(x_1, y_1)^t = M(x_0, y_0)^t \ , \ z_1 = z_0$$

where, $M = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix}$

Repeating the above process, we have, in general

$$(x_n, y_n)^t = M(x_0, y_0)^t , z_n = z_0 \tag{25}$$

where, $M^n = \frac{1}{2}\begin{bmatrix} 5-3(-1)^n & -5+5(-1)^n \\ 3-3(-1)^n & -3+5(-1)^n \end{bmatrix}$

Giving $n = 1,2,3,...$ inturn in (25), one obtains sequence of integer solutions to (1) based on the given solution (x_0, y_0, z_0)

5. Conclusion

In this paper, we have made an attempt to obtain all integer solutions to the ternary quadratic equation $5y^2 = 3x^2 + 2z^2$. One may search for other patterns of solutions and their corresponding properties.

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