Journal of Mathematics and Informatics
Vol. 10, 2017, 167-171
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 11 December 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v10a22

Journal of
Mathematics and Informatics

On Ternary Quadratic Diophantine Equation $6x^2+6y^2-11xy = 32z^2$

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Received 1 November 2017; accepted 7 December 2017

Abstract. The homogenous ternary quadratic Diophantine equation is given by $6x^2+6y^2-11xy=32z^2$ is considered and analyzed for its patterns of non zero distinct integer solutions. Introducing the linear transformation x=u+ v, y=u-v and employing the method of factorization, different patterns of non zero distinct integer solutions to the above equation are obtained. A few interesting the relation between the solution and polygonal numbers are obtained.

Keywords: Homogenous Quadratic, Ternary Quadratic, Integer solutions

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The ternary quadratic Diophantine equation offers an unlimited field for research because of their variety [1-2]. In particular, one may refer [3-9] for finding integer points on the some specific three dimensional surface. This communication concern with yet another ternary quadratic equation $6x^2+6y^2-11xy=32z^2$ representing cone for determining its infinitely many integral solutions. Employing the integral solutions on the given cone, a few interesting relations among the special polygonal numbers are given.

2. Notation used

$$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$ =Polygonal number of rank n with sides m.

2.1. Method of analysis

Consider the equation

$$6x^2+6y^2-11xy=32z^2 \tag{1}$$

The substitution of linear transformations

$$x=u+v \text{ and } y=u-v \qquad (u\neq v\neq 0) \tag{2}$$

In (1) leads to

$$u^2 + 23v^2 = 32z^2 \quad (3)$$

The above equation is solved through different methods and using (2), different patterns of integer solution to (1) are obtained.

2.2. Pattern I

Write 32 as

$$32 = (3 + i\sqrt{23})(3 - i\sqrt{23}) \quad (4)$$

Assume $z = a^2 + 23b^2$ where a, b>0 (5)

Using (4) and (5) in (3), and applying the method of factorization, define.

$$(u + i\sqrt{23}v) = (3 + i\sqrt{23})(a + i\sqrt{23}b)^2 \quad (6)$$

Equating the real and imaginary parts, we have

$$u = u(a,b) = 3a^2 - 69b^2 - 46ab$$

$$v = v(a,b) = a^2 - 23b^2 + 6ab$$

Substituting the above u and v in equation (2), the value of x and y are given by

$$\left.\begin{aligned} x &= x(a,b) = 4a^2 - 92b^2 - 40ab \\ y &= y(a,b) = 2a^2 - 46b^2 - 52ab \end{aligned}\right. \quad (7)$$

Thus (5) and (7) represent non-zero distinct integral solution of (1) in two parameters.

Properties

1. $x(a,1) - t_{6,a} - t_{6,a} \equiv 0 \pmod 4$
2. $z(a, a+1) - t_{26,a} - t_{26,a} \equiv 0 \pmod{23}$
3. $y(2,b) + t_{82,b} + t_{14,b} \equiv 0 \pmod 2$

2.3. Pattern II

Consider (3) as

$$u^2 - 9z^2 = 23(z^2 - v^2) \quad (8)$$

Write (8) in the form of ratio as

$$\frac{u + 3z}{z - v} = \frac{23(z + v)}{u - 3z} = \frac{\alpha}{\beta}, \beta \neq 0$$

This is equivalent to the following two equations

$$-\alpha u + 23\beta v + z(23\beta + 3\alpha) = 0$$

$$\beta u + \alpha v + z(3\beta - \alpha) = 0$$

On employing the method of cross multiplication, we get

$$\left.\begin{aligned} u &= -3\alpha^2 + 69\beta^2 - 46\alpha\beta \\ v &= -\alpha^2 + 23\beta^2 + 6\alpha\beta \end{aligned}\right\} \quad (9)$$

$$z = -\alpha^2 - 23\beta^2 \quad (10)$$

Substituting the values of u and v from (9) in (2) , the non-zero distinct integer values of x,y are given by

$$\left.\begin{aligned} x = x(\alpha,\beta) &= -4\alpha^2 + 92\beta^2 - 40\alpha\beta \\ y = y(\alpha,\beta) &= -2\alpha^2 + 46\beta^2 - 52\alpha\beta \end{aligned}\right\} \tag{11}$$

Thus (10) and (11) represent the non-zero distinct integer solution of equation (1) in two parameters.

Properties

1. $x(1,3\beta) - t_{402,\beta} - t_{154,\beta} \equiv 0 \pmod 4$
2. $x(1,\beta) - y(1,\beta) - t_{46,\beta} - t_{50,\beta} \equiv 0 \pmod 2$
3. $x(\alpha,\alpha+1) - t_{62,\alpha} - t_{38,\alpha} \equiv 0 \pmod 4$

Note: (8) Can also be expressed in the form of ratio in three different ways as follows

1. $\dfrac{u+3z}{23(z+v)} = \dfrac{z-v}{u-3z} = \dfrac{\alpha}{\beta}, \beta \neq 0$
2. $\dfrac{u+3z}{z+v} = \dfrac{23(z-v)}{u-3z} = \dfrac{\alpha}{\beta}, \beta \neq 0$
3. $\dfrac{u+3z}{23(z-v)} = \dfrac{z+v}{u-3z} = \dfrac{\alpha}{\beta}, \beta \neq 0$

Repeating the analysis as above, we get different pattern of solution to (1).

2.4. Pattern III

Rewrite (3) as

$$23v^2 = 32z^2 - u^2 \tag{12}$$

Write 23 as, $23 = (4\sqrt{2}+3)(4\sqrt{2}-3)$ (13)

Let $v = 32a^2 - b^2$ (14)

Using (13) and (14) in (12) and employing the method of factorization, we write

$$4\sqrt{2}z + u = (4\sqrt{2}+3)(4\sqrt{2}a+b)^2$$

Equating the rational and irrational parts, we have

$$z = z(a,b) = 32a^2 + b^2 + 6ab \tag{15}$$

$$u = u(a,b) = 96a^2 + 3b^2 + 64ab \tag{16}$$

Satisfying (14) and (16) in (2), the value of *x* and *y* are

$$\left.\begin{aligned} x = x(a,b) &= 128a^2 + 2b^2 + 64ab \\ y = y(a,b) &= 64a^2 + 4b^2 + 64ab \end{aligned}\right\} \tag{17}$$

Thus (17) and (15) represent the integer solution to (1).

Properties

1. $z(a,2)-t_{62,a}-t_{6,a}\equiv 0(\bmod 1)$
2. $x(a,3)-t_{202,a}-t_{58,a}\equiv 0(\bmod 6)$
3. $x(a,a+1)-y(a,a+1)-t_{62,a}-t_{66,a}\equiv 0(\bmod 2)$

2.5. Pattern IV

Equation (3) can be written as

$$u^2+23v^2=32z^2*1 \tag{18}$$

Note that $32=(3+i\sqrt{23})(3-i\sqrt{23})$; $1=\dfrac{(7+i3\sqrt{23})(7-i3\sqrt{23})}{16^2}$ (19)

Substituting (4), (19) in (18) and using the method of factorization; define

$$(u+i\sqrt{23}v)=(3+i\sqrt{23})(a+i\sqrt{23}b)^2\left(\frac{7+i3\sqrt{23}}{16}\right)$$

Equating real and imaginary parts, we get

$$u=u(a,b)=-3a^2+69b^2-46ab$$

$$v=v(a,b)=a^2-23b^2-6ab$$

In view if (2), note that

$$\left.\begin{array}{l} x=x(a,b)=-2a^2+46b^2-52ab \\ y=y(a,b)=-4a^2+92b^2-40ab \\ z=a^2+23b^2 \end{array}\right\} \tag{20}$$

Thus (20) represents non-zero distinct integer solutions to (1).

Properties

1. $z(a,2a+1)-t_{92,a}-t_{98,a}\equiv 0(\bmod 23)$
2. $x(a,1)-y(a,1)-t_{4,a}-t_{4,a}\equiv 0(\bmod 2)$

3. Conclusion

In this paper, we have presented different pattern of integer solutions to the ternary quadratic equation $6x^2+6y^2-11xy=32z^2$ representing the cone. As the Diophantine equations are rich in variety, one may attempt to find integer solutions to other choices of higher degree equations along with suitable properties.

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