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On the Positive Integer Solutions for a Diophantine Equation

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Abstract. **L**et $\mathrm{P} \coloneqq \mathrm{P(t)}$ be a polynomial in Z[X].In this paper, we consider the polynomial solutions of Diophantine equation $\mathrm{D}: \mathrm{M}^2 - 42S^2 - 8\mathrm{M} - 252\mathrm{S} - 378 = 0$. We also obtain some formulas and recurrence relations on the polynomial solution $(\mathrm{M_n}, \mathrm{S_n})$ of D.

Keywords: Diophantine equation, polynomial solution, pell's equation, continued fraction expansion.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

A Diophantine equation is a polynomial equation $P(x_1, x_2, \cdots, x_n) = 0$ where the polynomial P has integral coefficients and one is interested in solutions for which all the unknowns take integer values. For example, $x^2 + y^2 = z^2$ and $x = 3, y = 4, z = 5$ is one of its infinitely many solutions. Another example is $x + y = 1$ and all its solutions are given by $x = t, y = 1 - t$ where t passes through all integers. A third example is $x^2 + 4y = 3$. This Diophantine equation has no solutions, although note that $x = 0, y = \frac{3}{4}$ is a solution with rational values for the unknowns. Diophantine equations are rich in variety. Two – variable Diophantine equation have been a subject to extensive research, and their theory constitutes one of the most beautiful, most elaborate part of mathematics, which nevertheless still keeps some of its secrets for the next generation of researchers.

In this paper, we investigate positive integral solutions of the Diophantine equation $M^2 - 42S^2 - 8M - 252S = 378$ which is transformed into a Pell's equation and is solved by various methods.

2. Preliminaries

Consider the Diophantine equation

$$D: M^2 - 42S^2 - 8M - 252S = 378 \tag{1}$$

to be solved over Z. It is not easy to solve and find the nature and properties of the solutions of (1). So we apply a linear transformation D to (1) to transfer to a simpler form for which we can determine the integral solutions.

Let $$T: \begin{cases} w = x + h \\ z = y + k \end{cases} \tag{2}$$

be the transformation where $h, k \in Z$.

Applying $T$to$D$, we get

$T(D) = \widetilde{D}: (x+h)^2 - 42(y+k)^2 - 8(x+h) - 252(y+k) = 378$ (3)

Equating the coefficients of x and y to zero, we get $h = 4$ and $k = -3$. Hence for $M = x + 4$ and $S = y - 3$, we have the Diophantine equation

$\widetilde{D}$: $x^2 - 42y^2 = 16$ (4)

which is a Pell equation. Now we try to find all integer solutions (x_n, y_n) of $\widetilde{D}$ and then we can retransfer all results from $\widetilde{D}$ to D by using the inverse of T.

Theorem 2.1. Let $\widetilde{D}$ be the Diophantine equation in (4). Then

(i) The continued fraction expansion of $\sqrt{42}$ is

$$\sqrt{42} = [6; \overline{2, 12}]$$

(ii) The fundamental solution of $x^2 - 42y^2 = 1$ is $(u_1, v_1) = (13, 2)$

(iii) For $n \geq 4$,

$$u_n = 27(u_{n-1} - u_{n-2}) + u_{n-3}$$
$$v_n = 27(v_{n-1} - v_{n-2}) + v_{n-3}$$

Proof:

(i) The continued fraction expansion of $\sqrt{42} = 6 + (\sqrt{42} - 6)$

$$= 6 + \frac{1}{\frac{1}{\sqrt{42}-6}}$$

$$= 6 + \frac{1}{\frac{\sqrt{42}+6}{6}}$$

$$= 6 + \frac{1}{2 + \frac{\sqrt{42}-6}{6}}$$

$$= 6 + \frac{1}{2 + \frac{1}{\sqrt{42}+6}}$$

$$= 6 + \frac{1}{2 + \frac{1}{12 + (\sqrt{42}-6)}}$$

Therefore the continued fraction expansion of $\sqrt{42}$ is

$$[6; \overline{2, 12}]$$

(ii) Note that by (3), if $(u_1, v_1) = (13, 2)$ is the fundamental solution of $x^2 - 42y^2 = 1$, then the other solutions (u_n, v_n) of $x^2 - 42y^2 = 1$ can be derived by using the equalities

$(u_n + v_n\sqrt{42}) = (u_1 + \sqrt{42}v_1)^n$ for $n \geq 2$, in other words,

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 & 42v_1 \\ 2 & u_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For $n \geq 2$. Therefore it can be shown by induction on n that

$$u_n = 27(u_{n-1} - u_{n-2}) + u_{n-3}$$

and also

$$v_n = 27(v_{n-1} - v_{n-2}) + v_{n-3}$$

for $n \geq 4$.

Now we consider the problem

Note that we denote the integer solutions of $x^2 - 42y^2 = 16$ by (x_n, y_n), and denote the integer solutions of $x^2 - 42y^2 = 1$ by (u_n, v_n). Then we have the following theorem.

Theorem 2.2. Define a sequence $\{(x_n, y_n)\}$ of positive integers by

$$(x_1, y_1) = (52, 8)$$

and $x_n = 52u_{n-1} + 336v_{n-1}$

$$y_n = 8u_{n-1} + 52v_{n-1},$$

where$\{(u_n, v_n)\}$ is a sequence of positive solutions of $x^2 - 30y^2 = 1$. Then

(1) (x_n, y_n) is a solution of $x^2 - 42y^2 = 16$ for any integer $n \geq 1$.

(2) For $n \geq 2$,

$$x_{n+1} = 13x_n + 84y_n$$
$$y_{n+1} = 2x_n + 13y_{n.}$$

(3) For $n \geq 4$

$$x_n = 27(x_{n-1} - x_{n-2}) + x_{n-3}$$
$$y_n = 27(y_{n-1} - y_{n-2}) + y_{n-3}$$

Proof:

(1) It is easily seen that

$$(x_1, y_1) = (52, 8)$$

is a solution of $x^2 - 42y^2 = 16$ since

$$\begin{aligned} x_1^2 - 42y_1^2 &= (52)^2 - 42(8)^2 \\ &= 16(13^2 - 42(2^2)) \\ &= 16(1) \\ &= 16 \end{aligned}$$

Note that by definition,(u_{n-1}, v_{n-1}) is a solution of $x^2 - 42y^2 = 1$, that is,

$$u_{n-1}^2 - 42v_{n-1}^2 = 1. \tag{7}$$

Also we see as above that (x_1, y_1) is a solution of $x^2 - 42y^2 = 16$, that is,

$$x_1^2 - 42y_1^2 = 16. \tag{8}$$

Applying, (7) and (8), we get

$$\begin{aligned} x_n^2 - 42y_n^2 &= (52u_{n-1} + 336v_{n-1})^2 - 42(8u_{n-1} + 52v_{n-1})^2 \\ &= u_{n-1}^2(2^4) - v_{n-1}^2(2^4(42)) \\ &= 2^4(u_{n-1}^2 - 42v_{n-1}^2) \\ &= 2^4 \end{aligned}$$

Therefore $(x_n\,, y_n)$ is a solution of $x^2 - 16\,y^2 = 2^4$.

(2) Recall that $x_{n+1} + y_{n+1}\sqrt{d} = \left(u_1 + v_1\sqrt{d}\right)(x_n + y_n\sqrt{d}\,)$

Therefore $x_{n+1} = u_1x_n + v_1y_nd$ and $y_{n+1} = v_1x_n + u_1y_n$

So $x_{n+1} = 13x_n + 84y_n$ and $y_{n+1} = 2x_n + 13y_n$ (*)

Since $u_1 = 13$ and$v_1 = 2$.

(3) Applying the equalities

$x_n = 2^2(13)u_{n-1} + 2^3(42)v_{n-1}andx_{n+1} = 13x_n + 84y_n$

We find by induction on n that

$$x_n = 27(x_{n-1} - x_{n-2}) + x_{n-3}$$

for $n \geq 4$.

Similarly it can be shown that
$y_n = 27(y_{n-1} - y_{n-2}) + y_{n-3}$.

We saw as above that the Diophantine equation D could be transformed into the Diophantine equation $\widetilde{D}$ via the transformation T. Also we showed that $M = x + 4$ and $S = y - 3$. So we can retransfer all results from $\widetilde{D}$ to D by using the inverse of T. Thus we can give the following main theorem**.**

3. Main results

Theorem 3.1. Let D be the Diophantine equation in (1), then

(1) The fundamental solution of D is $(M_1, S_1) = (56,\ 5)$.

(2) Define the sequence $\{(M_n, S_n)\}_{n\geq 1} = \{(x_n + 4, y_n - 3)\}$, where $\{(x_n, y_n)\}$ defined in (*). Then (M_n, S_n) is a solution of D. So it has infinitely many solutions $(M_n, S_n) \in Z \times Z$.

(3) The solution (M_n, S_n) satisfy

$$M_n = 13M_{n-1} + 84S_{n-1} + 204$$
$$S_n = 2M_{n-1} + 13S_{n-1} + 28$$

(4) The solutions (M_n, S_n) satisfy the recurrence relations

$$M_n = 27(M_{n-1} - M_{n-2}) + M_{n-3}$$
$$S_n = 27(S_{n-1} - S_{n-2}) + S_{n-3}$$

Proof:

(1) It is easily seen that $(M_1, S_1) = (56, 5)$ is the fundamental solution of D since $56^2 - 42(5)^2 - 8(56) - 262(5) - 378 = 0$.

(2) We prove it by induction. Let $n = 1$.

Then $(M_1, S_1) = (x_1 + 4, y_1 - 3) = (56, 5)$ which is the fundamental solution and so is a solution of D. Let us assume that the Diophantine equation in (1) is satisfied for $n - 1$, that is, $(x_{n-1} + 4)^2 - 42(y_{n-1} - 3)^2 - 8(x_{n-1} + 4) - 252(y_{n-1} - 3) - 378 = 0$. We want to show that this equation is also satisfied for n.

$$M^2 - 42S^2 - 8M - 252S - 378$$
$$= (x_n + 4)^2 - 42(y_n - 3)^2 - 8(x_n + 4) - 252(y_n - 3) - 378$$
$$= x_n^2 - 42y_n^2 - 16$$
$$= 0\text{(} x_n \text{and } y_n \text{ solutions of } \widetilde{D}\text{)}.$$

So $(M_n, S_n) = (x_n + 4, y_n - 3)$ is also a solution D.

(3) From (*) $\quad x_n = 13x_{n-1} + 84y_{n-1}$.

Adding 4 on both sides,

$$x_n + 4 = 13x_{n-1} + 84y_{n-1} + 4$$

We know that $\quad M_{n-1} = x_{n-1} + 4 \quad$ and $\quad S_{n-1} = y_{n-1} - 3$

Therefore, $x_{n-1} = M_{n-1} - 4$ and $y_{n-1} = S_{n-1} + 3$

$$x_n + 4 = 13x_{n-1} + 84y_{n-1} + 4$$

$$(M_n - 4) + 4 = 13(M_{n-1} - 4) + 84(S_{n-1} + 3) + 4$$

We get, $M_n = 13M_{n-1} + 84S_{n-1} + 204$ (9)

Similarly, $S_n = 2M_{n-1} + 13S_{n-1} + 28$ (10)

(4) We prove that M_n satisfy the recurrence relation. For $n = 4$, we get $M_1 = 56$, $M_2 = 1352, M_3 = 35000, M_4 = 908552$.
Hence

$$M_4 = 27(M_3 - M_2) + M_1$$
$$= 27(35000 - 1352) + 56$$

So $M_4 = 27(M_3 - M_2) + M_1$ is satisfied for $n = 4$. Let us assume that this relation is satisfied for $n - 1$, that is,

$M_{n-1} = 27(M_{n-2} - M_{n-3}) + M_{n-4}$ (11)

Then applying the previous assertion, (9) and (11), we conclude that
$M_n = 27(M_{n-1} - M_{n-2}) + M_{n-3}$, for $n \geq 4$.

Now prove that y_n satisfy the recurrence relation. For $n = 4$, we get $S_1 = 5$, $S_2 = 205$, $S_3 = 5397, S_4 = 140189$. Hence

$$S_4 = 27(S_3 - S_2) + S_1$$
$$= 27(5397 - 205) + 5$$

So $S_n = 27(S_{n-1} + S_{n-2}) - S_{n-3}$ is satisfied for $n = 4$. Let us assume that this relation is satisfied for $n - 1$, that is,

$S_{n-1} = 27(S_{n-2} - S_{n-3}) + S_{n-4}$ (12)

Then applying the previous assertion, (10) and (12), we conclude that
$S_n = 27(S_{n-1} - S_{n-2}) + S_{n-3}$, for $n \geq 4$.

4. Conclusion

Diophantine equations are rich in variety. There is no universal method for finding all possible solutions (if it exists) for Diophantine equations. The method looks to be simple but it is very difficult for reaching the solutions.

REFERENCES

1. S.P.Arya, On the Brahmagupta-Bhaskara equation, *Math. Ed.*, 8(1) (1991) 23-27.
2. C.Baltus, Continued fraction and the Pell equations: The work of Euler and Lagrange, *Comm. Anal. Theory Continued Fraction,* 3 (1994) 4-31.
3. E. Barbeau, *Pell's Equation,* Springer Verleg, 2003.
4. H.P.Edward, Fermat's Last Theorem, A Graduate Texts in Mathematics, 50. Springer – Verleg, New York, 1996.
5. D. Hensley, *Continued Fractions,* World Scientific Publishing, Hackensack, N. J, 2006.
6. P.Kaplan and K.S.Williams, Pell's equations $x^2 - my^2 = -1, -4$ and continued fractions, *Journal of Number Theory,* 23 (1986) 169-182.
7. H.W.Lenstra, Solving the pell's equation, *Notice of the AMS,* 49(2) (2002) 182-192.