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On Triples where the Sum of any Two Members of a Triple is a Perfect Square

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Abstract. This paper deals with the construction of families of integer triples where, in each triple, the sum of any two members is a perfect square. A few numerical examples are also given.

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1. Introduction

Every advanced under-graduate and graduate student of mathematics as well as any researcher in number theory is familiar with Pythagorean triple which provides the relation between three sides of a right-angled triangle in addition to the concept of three integers representing an arithmetic progression, geometric progression and harmonic progression respectively. In this context, one may refer [1] wherein the authors have given a collection of problems with solutions on integer triples in arithmetic progression.

Similar to a Pythagorean triple, we have a triple known as Heronian triple defined as follows: If a, b, c represent the sides of a triangle with integer area, then the triple (a, b, c) is known as Heronian triple. It is worth to note that not every Heronian triple is a Pythagorean triple. For example: (4, 13, 15) is a Heronian triple but not Pythagorean triple whereas (5, 12, 13) is both Heronian triple as well as Pythagorean triple. Also, we have a triple known as Eisenstein triple which is a set of integers which are the lengths of the sides of a triangle where one of the angle is 60^0. In other words, An Eisenstein triple (a, b, c) consists of three positive integers $a < c < b$ such that $a^2 - ab + b^2 = c^2$

No doubt that the triples in integers may be formulated in varieties of ways. For a review of various problems on triples, one may refer [2-6]. It is therefore towards this end, we are motivated to search for families of triples where, in each triple, the sum of any two of its members is a perfect square.

2. Construction of triples

Consider the Pythagorean equation given by

$$x^2 + y^2 = z^2 \qquad (1.1)$$

Observe that (1.1) is satisfied by

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2, \quad m > n > 0 \qquad (1.2)$$

We present below different families of triples satisfying the required conditions.

2.1. TRIPLE 1

Assume

$$x = 6p - 3 \qquad (1.3)$$

$$y = 6q + 2 \qquad (1.4)$$

From (1.2), (1.3) and (1.4), we have

$$p = \frac{1}{6}\left[m^2 - n^2 + 3\right], \quad q = \frac{1}{3}\left[mn - 1\right] \qquad (1.5)$$

The values of p and q are integers for the following choices:

i) $m = 6r - 2, \quad n = 6s - 5$

ii) $m = 6r - 1, \quad n = 6s - 4$

iii) $m = 6r + 1, \quad n = 6s - 2$

iv) $m = 6r + 2, \quad n = 6s - 1, \; r \geq s \geq 1$

<u>Choice : (i)</u> $(m = 6r - 2, \quad n = 6s - 5)$

Substituting the values $m = 6r - 2, \quad n = 6s - 5$ in (1.5), we have

$$p = 6r^2 - 6s^2 - 4r + 10s - 3 = f_1(r,s)$$

$$q = 12rs - 10r - 4s + 3 = g_1(r,s)$$

Let $a_1(r,s) = (6 f_1(r,s) - 3)^2$, $\quad b_1(r,s) = (6g_1(r,s) + 2)^2$

$\Rightarrow \quad a_1(r,s) + b_1(r,s)$ is a perfect square.

Let $c_1(r,s)$ be any non-zero integer distinct from $a_1(r,s), b_1(r,s)$ such that

$$a_1(r,s) + c_1(r,s) = \alpha^2 \qquad (1.6)$$

$$b_1(r,s) + c_1(r,s) = \beta^2 \qquad (1.7)$$

Subtraction of (1.7) from (1.6) gives

$$\alpha^2 - \beta^2 = a_1(r,s) - b_1(r,s) \qquad (1.8)$$

Employing the identity

$$(A+1)^2 - A^2 = 2A + 1$$

we have, from (1.8)

$$A = \frac{1}{2}\left[a_1(r,s) - b_1(r,s) - 1\right]$$

where $\alpha = A + 1, \quad \beta = A$ (1.9)

From (1.9) and (1.7), we have

$$c_1(r,s) = \frac{1}{4}\left(\left[a_1(r,s) - b_1(r,s) - 1\right]^2 - 4b_1(r,s)\right)$$

which is an integer for suitable values of r and s.

Hence, for those choices $(a_1(r,s), b_1(r,s), c_1(r,s))$ is the required triple where the sum of any two of them is a perfect square.

Table 1: Numerical examples

r	s	a_1	b_1	c_1	a_1+c_1	b_1+c_1	a_1+b_1
2	1	9801	400	22089600	4701^2	4700^2	101^2
3	2	42849	50176	13374720	3663^2	3664^2	305^2
5	3	378225	529984	5757244416	75879^2	75880^2	953^2
4	2	189225	94864	2225857536	47181^2	47180^2	533^2
3	3	7569	173056	6846396480	82743^2	82744^2	425^2

In a similar manner, the triples for choices (ii), (iii), (iv) are respectively given below in Tables 2, 3, 4.

<u>Choice : (ii)</u> $m = 6r - 1$, $n = 6s - 4$

Table 2: Numerical examples

r	s	a_1	b_1	c_1	a_1+c_1	b_1+c_1	a_1+b_1
3	2	50625	73984	136348416	11679^2	11680^2	353^2
5	3	416025	659344	14800496256	121659^2	121660^2	1037^2
4	1	275625	8464	17843607936	133581^2	133580^2	533^2
4	2	216225	135424	1632024576	40401^2	40400^2	593^2
3	3	8649	226576	11872926720	108963^2	108964^2	485^2

<u>Choice : (iii)</u> $m = 6r + 1$, $n = 6s - 2$

Table 3: Numerical examples

r	s	a_1	b_1	c_1	a_1+c_1	b_1+c_1	a_1+b_1
2	1	23409	10816	39628800	6297^2	6296^2	185^2
3	2	68121	144400	1454515200	38139^2	38140^2	461^2
5	3	497025	984064	59301006336	243519^2	243520^2	1217^2
4	2	275625	250000	163897344	12813^2	12812^2	725^2
3	3	11025	369664	32155292736	179319^2	179320^2	617^2

<u>Choice : (iv)</u> $m = 6r + 2$, $n = 6s - 1$

Table 4: Numerical examples

r	s	a_1	b_1	c_1	a_1+c_1	b_1+c_1	a_1+b_1
2	1	29241	19600	23212800	4821^2	4820^2	221^2
3	2	77841	193600	3349900800	57879^2	57880^2	521^2
5	3	540225	1183744	103528313856	321759^2	321760^2	1313^2
4	2	308025	327184	91449216	9579^2	9580^2	797^2
3	3	12321	462400	50642539200	225039^2	225040^2	689^2

2.2. TRIPLE 2

Assume

$$x = 6p - 3 \quad (1.10)$$

$$y = 4q + 4 \quad (1.11)$$

From (1.2), (1.10) and (1.11), we have

$$p = \frac{1}{6}\left[m^2 - n^2 + 3\right], \quad q = \frac{1}{2}\left[mn - 2\right] \quad (1.12)$$

The values of p and q are integers when

$$m = 6r + s - 3, \quad n = s \quad (1.13)$$

where $r, s \in Z - \{0\}$

Substituting (1.13) in (1.12), we have

$$p = 6r^2 - 6r + 2rs - s + 2 = f_2(r,s)$$

$$q = \frac{1}{2}\left(s^2 + 6rs - 3s - 2\right) = g_2(r,s)$$

Let $a_2(r,s) = (6f_2(r,s) - 3)^2$, $b_2(r,s) = (4g_2(r,s) + 4)^2$

$\Rightarrow$ $a_2(r,s) + b_2(r,s)$ is a perfect square.

Let $c_2(r,s)$ be any non-zero integer distinct from $a_2(r,s), b_2(r,s)$ such that

$$a_2(r,s) + c_2(r,s) = \alpha^2$$

$$b_2(r,s) + c_2(r,s) = \beta^2$$

Following the procedure as in triple: 1, it is seen that

$$c_2(r,s) = \frac{1}{4}\left(\left[a_2(r,s) - b_2(r,s) - 1\right]^2 - 4b_2(r,s)\right)$$

which is an integer for suitable values of r and s.

Hence, for those choices $(a_2(r,s), b_2(r,s), c_2(r,s))$ is the required triple such that the sum of any two of them is a perfect square.

Table 5: Numerical Examples

r	s	a_2	b_2	c_2	$a_2 + c_2$	$b_2 + c_2$	$a_2 + b_2$
2	3	18225	5184	42505216	6521^2	6520^2	153^2
3	5	140625	40000	2531257344	50313^2	50312^2	425^2
4	2	275625	8464	17843607936	133581^2	133580^2	533^2
5	2	700569	13456	118030711680	343557^2	343556^2	845^2
2	2	13689	1936	34525440	5877^2	5876^2	125^2

2.3. TRIPLE 3

Assume

$$x = 3p - 3 \quad (1.14)$$

$$y = 6q + 2 \quad (1.15)$$

From (1.2), (1.14) and (1.15), we have

$$p = \frac{1}{3}\left[m^2 - n^2 + 3\right], \quad q = \frac{1}{3}\left[mn - 1\right] \qquad (1.16)$$

The values of p and q are integers for the following choices:

i) $m = 3r + 3s - 2\,, \quad n = 3r - 2$

ii) $m = 3r + 3s - 1\,, \quad n = 3r - 1$

where $r, s \in Z - \{0\}$.

Choice : (i) $(m = 3r + 3s - 2\,, \quad n = 3r - 2)$

Substituting the values $m = 3r + 3s - 2\,, \quad n = 3r - 2$ in (1.16), we have

$$p = 3s^2 + 6rs - 4s + 1 = f_3(r,s)$$

$$q = 3r^2 - 4r - 2s + 3rs + 1 = g_3(r,s)$$

Let $a_3(r,s) = 9\left(f_3(r,s) - 1\right)^2\,, \quad b_3(r,s) = 4\left(3g_3(r,s) + 1\right)^2$

$\Rightarrow \quad a_3(r,s) + b_3(r,s)$ is a perfect square.

Let $c_3(r,s)$ be any non-zero integer distinct from $a_3(r,s), b_3(r,s)$ such that

$$a_3(r,s) + c_3(r,s) = \alpha^2$$

$$b_3(r,s) + c_3(r,s) = \beta^2$$

After performing a few calculations, it is seen that

$$c_3(r,s) = \frac{1}{4}\left(\left[a_3(r,s) - b_3(r,s) - 1\right]^2 - 4b_3(r,s) \right)$$

which is an integer for suitable values of r and s.

Hence for those choices $(a_3(r,s), b_3(r,s), c_3(r,s))$ is the required triple such that the sum of any two of them is a perfect square.

Table 6: Numerical examples

r	s	a_3	b_3	c_3	$b_3 + c_3$	$a_3 + c_3$	$a_3 + b_3$
2	1	1089	3136	1045440	1024^2	1023^2	652^2
5	3	99225	327184	12991113216	113980^2	113979^2	653^2
3	3	42849	50176	13374720	3664^2	3663^2	305^2
3	1	2601	19600	72230400	8500^2	8499^2	149^2

In a similar manner, the triple for choice (ii) is given in Table: 7

Choice : (ii) $m = 3r + 3s - 1\,, \quad n = 3r - 1$.

Table 7: Numerical examples

r	s	a_3	b_3	c_3	a_3+c_3	b_3+c_3	a_3+b_3
2	1	1521	6400	5947200	2439^2	2440^2	89^2
5	3	110889	414736	23080487040	151923^2	151924^2	725^2
3	3	50625	73984	136348416	11679^2	11680^2	353^2
3	1	3249	30976	192179520	13863^2	13864^2	185^2

2.4. TRIPLE 4

Assume

$$x = 5p - 5 \tag{1.17}$$

$$y = 2q + 2 \tag{1.18}$$

From (1.2), (1.17) and (1.18), we have

$$p = \frac{1}{5}\left[m^2 - n^2 + 5\right], \quad q = mn - 1 \tag{1.19}$$

The values of p and q are integers for the following choices:

i) $m = 2r + 5s - 1, \quad n = 2r - 1$
ii) $m = 3r + 5s - 4, \quad n = 2r - 1$
iii) $m = 3r + 5s - 5, \quad n = 2r$
iv) $m = 2r + 5s, \quad n = 2r, \; r \geq s \geq 1$

<u>Choice : (i)</u> $m = 2r + 5s - 1, \quad n = 2r - 1$

Substituting the values $m = 2r + 5s - 1, \quad n = 2r - 1$ in (1.19), we have

$$p = 5s^2 + 4rs - 2s + 1 = f_4(r,s)$$

$$q = 4r^2 - 4r - 5s + 10rs = g_4(r,s)$$

Let $a_4(r,s) = (5f_4(r,s) - 5)^2, \quad b_4(r,s) = (2g_4(r,s) + 2)^2$

$\Rightarrow \quad a_4(r,s) + b_4(r,s)$ is a perfect square.

Let $c_4(r,s)$ be any non-zero integer distinct from $a_4(r,s), b_4(r,s)$ such that

$$a_4(r,s) + c_4(r,s) = \alpha^2 \tag{1.20}$$

$$b_4(r,s) + c_4(r,s) = \beta^2 \tag{1.21}$$

Subtraction of (1.21) from (1.20) gives

$$\alpha^2 - \beta^2 = a_4(r,s) - b_4(r,s) \tag{1.22}$$

Employing the identity

$$(A+1)^2 - A^2 = 2A + 1$$

we have, from (1.22)

$$A=\frac{1}{2}\left[a_4(r,s)-b_4(r,s)-1\right]$$

where $\alpha = A+1, \quad \beta = A$ (1.23)

From (1.23) and (1.21), we have

$$c_4(r,s)=\frac{1}{4}\left(\left[a_4(r,s)-b_4(r,s)-1\right]^2-4b_4(r,s)\right)$$

which is an integer for suitable values of r and s.

Hence, for those choices $\left(a_4(r,s), b_4(r,s), c_4(r,s)\right)$ is the required triple where the sum of any two of them is a perfect square.

Table 8: Numerical examples

r	s	a_4	b_4	c_4	a_4+c_4	b_4+c_4	a_4+b_4
2	1	3025	2304	127296	361^2	360^2	73^2
5	3	245025	186624	852453376	29201^2	29200^2	657^2
3	3	140625	40000	2531257344	50313^2	50312^2	425^2
3	1	5625	10000	4777344	2187^2	2188^2	125^2

In a similar manner, the triples for choices (ii), (iii), (iv) are respectively given below in Tables 9, 10, 11.

<u>**Choice : (ii)**</u> $m=3r+5s-4\,, \quad n=2r-1$

Table 9: Numerical examples

r	s	a_4	b_4	c_4	a_4+c_4	b_4+c_4	a_4+b_4
5	3	354025	219024	4556030976	67501^2	67500^2	757^2
4	2	75625	63504	36660096	6061^2	6060^2	373^2
3	3	140625	40000	2531257344	50313^2	50312^2	425^2
4	4	540225	153664	37357004736	193281^2	193280^2	833^2

<u>**Choice : (iii)**</u> $m=3r+5s-5\,, \quad n=2r$

Table 10: Numerical examples

r	s	a_4	b_4	c_4	a_4+c_4	b_4+c_4	a_4+b_4
5	3	275625	250000	163897344	12813^2	12812^2	725^2
4	2	50625	73984	136348416	11679^2	11680^2	353^2
3	3	105625	51984	719260416	26821^2	26820^2	397^2
4	4	442225	186624	16332653376	127801^2	127800^2	793^2
3	1	2025	11664	23220736	4819^2	4820^2	117^2

<u>**Choice : (iv)**</u> $m=2r+5s\,, \quad n=2r$

Table 11: Numerical examples

r	s	a_4	b_4	c_4	a_4+c_4	b_4+c_4	a_4+b_4
2	1	4225	5184	225216	479^2	480^2	97^2
5	3	275625	250000	163897344	12813^2	12812^2	725^2
3	3	164025	63504	2526004096	50261^2	50260^2	477^2
3	1	7225	17424	25992576	5099^2	5100^2	157^2

2.5. TRIPLE 5

Assume

$$x = 7p - 7 \tag{1.24}$$

$$y = 2q + 2 \tag{1.25}$$

From (1.2), (1.24) and (1.25), we have

$$p = \frac{1}{7}\left[m^2 - n^2 + 7\right], \quad q = mn - 1 \tag{1.26}$$

The values of p and q are integers when

$$m = r + 7s - 7\,, \quad n = r \quad , \ r \geq s \geq 1 \tag{1.27}$$

Substituting (1.27) in (1.26), we have

$$p = 1 + (s-1)(2r + 7s - 7) = f_5(r,s)\,, \ q = r(r + 7s - 7) - 1 = g_5(r,s)$$

Let $a_5(r,s) = (7f_5(r,s) - 7)^2$, $b_5(r,s) = (2g_5(r,s) + 2)^2$

$\Rightarrow$ $a_5(r,s) + b_5(r,s)$ is a perfect square.

Let $c_5(r,s)$ be any non-zero integer distinct from $a_5(r,s), b_5(r,s)$ such that

$$a_5(r,s) + c_5(r,s) = \alpha^2 \tag{1.28}$$

$$b_5(r,s) + c_5(r,s) = \beta^2 \tag{1.29}$$

Subtraction of (1.29) from (1.28) gives

$$\alpha^2 - \beta^2 = a_5(r,s) - b_5(r,s) \tag{1.30}$$

Employing the identity

$$(A+1)^2 - A^2 = 2A + 1$$

we have, from (1.30)

$$A = \frac{1}{2}\left[a_5(r,s) - b_5(r,s) - 1\right]$$

where $\alpha = A + 1, \quad \beta = A$ (1.31)

From (1.29) and (1.31), we have

$$c_5(r,s) = \frac{1}{4}\left(\left[a_5(r,s) - b_5(r,s) - 1\right]^2 - 4b_5(r,s)\right)$$

which is an integer for suitable values of r and s.

Hence, for those choices $\left(a_5(r,s), b_5(r,s), c_5(r,s)\right)$ is the required triple where the sum of any two of them is a perfect square.

Table 12: Numerical examples

r	s	a_5	b_5	c_5	a_5+c_5	b_5+c_5	a_5+b_5
1	2	3969	256	3444480	1857^2	1856^2	65^2
5	4	423801	67600	31719542400	178101^2	178100^2	701^2
3	2	8281	3600	5472000	2341^2	2340^2	109^2
7	4	540225	153664	37357004736	193281^2	193280^2	833^2

3. Conclusion

In this paper, we have made an attempt to construct family of triples where the sum of any members of a triple is a perfect square. To conclude, one may search for families of triples with different relations among its members.

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