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On the Non-Homogeneous Quintic Equation with Five Unknowns X^4-Y^4=10 p^3 (Z^2-W^2)

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Abstract. The quintic non-homogeneous equation with five unknowns represented by the Diophantine equation $\left(x^4 - y^4\right) = 10\left(z^2 - w^2\right)P^3$ is analyzed for its patterns of non-zero distinct integral solutions. Various interesting relations between the solutions and special numbers, namely polygonal numbers, pyramidal numbers are exhibited.

Keywords: Non-homogenous quintic equation, quintic with five unknowns, integral solutions.

AMS Mathematics Subject Classification (2010): 11D41

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems [1-3]. Particularly, in [4-7] quintic equations with three unknowns are studied for their integral solutions. In [8,9] quintic equations with four unknowns for their non-zero integer solutions are analyzed. In [10-13] analyze quintic equations with five unknowns for their non-zero integer solutions. This communication concerns with yet another interesting non-homogeneous quintic equation with five unknowns given by $\left(x^4 - y^4\right) = 10\left(z^2 - w^2\right)P^3$ is analayzed for its infinitely many non-zero distinct integer solutions. Various interesting properties among the values of x, y, z, w, P are presented.

Notations

$t_{m,n}$: polygonal number of rank n with size m.

P_m^n : Pyramidal number of rank n with size m.

2. Method of analysis

The non-homogeneous quintic equation with five unknowns to be solved for its distinct non-zero integral solutions is

$$\left(x^4 - y^4\right) = 10\left(z^2 - w^2\right)P^3 \tag{1}$$

Introduction of the linear transformations

$$x = u + v, y = u - v, z = 2u + v, w = 2u - v \tag{2}$$

in (1) leads to

$$u^2 + v^2 = 10P^3 \tag{3}$$

Different methods of obtaining the patterns of integer solutions to (1) are illustrated below:

2.1. PATTERN

Let

$$P = a^2 + b^2 \tag{4}$$

where a and b are non-zero integers.

Write 10 as

$$10 = (1 + 3i)(1 - 3i) \tag{5}$$

Using (4), (5) in (3) and applying the method of factorization, define

$$(u + iv) = (1 + 3i)(a + ib)^3 \tag{6}$$

from which we have

$$\left.\begin{aligned} u &= a^3 + 3b^3 - 3ab^2 - 9a^2b \\ v &= 3a^3 - b^3 - 9ab^2 + 3a^2b \end{aligned}\right\} \tag{7}$$

Using (7) and (2), the values of x, y, z and w are given by

$$\left.\begin{aligned} x(a,b) &= 4a^3 + 2b^3 - 12ab^2 - 6a^2b \\ y(a,b) &= -2a^3 + 4b^3 + 6ab^2 - 12a^2b \\ z(a,b) &= 5a^3 + 5b^3 - 15ab^2 - 15a^2b \\ w(a,b) &= -a^3 + 7b^3 + 3ab^2 - 21a^2b \end{aligned}\right\} \tag{8}$$

Thus (4) and (8) represent the non-zero integer solutions to (1).

PROPERTIES

a) $x(1,b) - 4\left[P(1,b) + 3P_b^{\,3}\right] + t_{10,b} + t_{12.b} + t_{14,b} + t_{16,b} = -28b$

b) $x(a,1) - z(a,1) + w(a,1) = -4\left[3P_a^{\,3} + 5t_{3,a-1} - P(a,1)\right]$

c) $x(1,b) + y(1,b) + z(1,b) + w(1,b) + 6\left[26t_{3,b} - P(1,b) - 18P_b^{\,3}\right] = -12b$

d) $x(a,1) - z(a,1) + 3\left[P(a,1) + 2P_a^{\,3}\right] - t_{16,a} - t_{18,a} = 18a$

e) $y(1,b) + z(1,b) + 3\left[t_{18,b} + t_{12,b} - P(1,b) - 18P_b^{\,3}\right] = -78b$

2.2. PATTERN

Let

$$P = a^2 + b^2$$

where a and b are non-zero integers.

Write 10 as

$$10 = (3+i)(3-i) \qquad (9)$$

Using (4), (9) in (3) and applying the method of factorization, define

$$(u+iv) = (3+i)(a+ib)^3 \qquad (10)$$

from which we have

$$\left.\begin{aligned} u &= 3a^3 + b^3 - 9ab^2 - 3a^2b \\ v &= a^3 - 3b^3 - 3ab^2 + 9a^2b \end{aligned}\right\} \qquad (11)$$

Using (11) and (2), the values of x, y, z and w are given by

$$\left.\begin{aligned} x(a,b) &= 4a^3 - 2b^3 - 12ab^2 + 6a^2b \\ y(a,b) &= 2a^3 + 4b^3 - 6ab^2 - 12a^2b \\ z(a,b) &= 7a^3 - b^3 - 21ab^2 + 3a^2b \\ w(a,b) &= 5a^3 + 5b^3 - 15ab^2 - 15a^2b \end{aligned}\right\} \qquad (12)$$

Thus (4) and (12) represent the non-zero integer solutions to (1).

PROPERTIES

a) $y(a,1) + 2\left[t_{16,a} + t_{10,a} - 2P(a,1) - 6P_a^{\,3}\right] = -28a$

b) $x(1,b) + z(1,b) - y(1,b) - w(1,b) + 4\left[18P_b^{\,3} - P(1,b) - 10t_{3,b}\right] = 40b$

c) $x(a,1) + y(a,1) + w(a,1) - 7P(a,1) + 6\left[t_{20,a} - 11P_a^{\,3}\right] + t_{16,a} = -109a$

d) $x(1,b) + y(1,b) - z(1,b) + P(1,b) + t_{12,b} - 18P_b^{\,3} = -19b$

e) $y(1,b) - z(1,b) + 5\left[P(1,b) - 6P_b^{\,3} - 2t_{3,b-1}\right] = -20b$

2.3. PATTERN

In (3), taking $u = r + s, v = r - s$ (13)

it leads to

$$r^2 + s^2 = 5P^3 \qquad (14)$$

Let

$$P = A^2 + B^2 \qquad (15)$$

where A and B are non-zero integers.

Write 5 as

$$5 = (2+i)(2-i)$$

(16)

Using (16), (15) in (14) and applying the method of factorization, define

$$(r+is) = (2+i)(A+iB)^3 \qquad (17)$$

from which we have

$$\left.\begin{aligned} r &= 2A^3 + B^3 - 6AB^2 - 3A^2B \\ s &= A^3 - 2B^3 - 3AB^2 + 6A^2B \end{aligned}\right\} \qquad (18)$$

From (18) and (13), the values of u and v are given by

$$\left.\begin{array}{l} u = 3A^3 - B^3 - 9AB^2 + 3A^2B \\ v = A^3 + 3B^3 - 3AB^2 - 9A^2B \end{array}\right\} \quad (19)$$

Using (19) and (2), the values of x, y, z and w are given by

$$\left.\begin{array}{l} x(A,B) = 4A^3 + 2B^3 - 12AB^2 - 6A^2B \\ y(A,B) = 2A^3 - 4B^3 - 6AB^2 + 12A^2B \\ z(A,B) = 7A^3 + B^3 - 21AB^2 - 3A^2B \\ w(A,B) = 5A^3 - 5B^3 - 15AB^2 + 15A^2B \end{array}\right\} \quad (20)$$

Thus (20) and (15) represent the non-zero integer solutions to (1).

PROPERTIES

a) $x(A,1) + AP(A,1) + 6\left[7t_{3,A} - 5P_A^{\,3}\right] = 2$

b) $y(1,B) + BP(1,B) + 18P_B^{\,3} - t_{8,B} \equiv 1 \pmod 2$

c) $x(A,1) + y(A,1) + 2P(A,1) - 36P_A^{\,3} - 20t_{3,A} = -20A$ s

d) $z(1,B) + y(1,B) - x(1,B) + 5\left[6P_B^{\,3} - P(1,B) + 2t_{3,B-1}\right] = 20B$

e) $y(A,1) + 4z(A,1) + w(A,1) + 5\left[P(A,1) - 42P_A^{\,3} + t_{18,A} + t_{20,A}\right] = -250A$

2.4. PATTERN

Let

$$P = A^2 + B^2$$

where A and B are non-zero integers.

Write 5 as

$$5 = (1+2i)(1-2i) \quad (21)$$

Using (21), (15) in (14) and applying the method of factorization, define

$$(r + is) = (1+2i)(A+iB)^3 \quad (22)$$

from which we have

$$\left.\begin{array}{l} r = A^3 + 2B^3 - 3AB^2 - 6A^2B \\ s = 2A^3 - B^3 - 6AB^2 + 3A^2B \end{array}\right\} \quad (23)$$

From (23) and (13), the values of u and v are given by

$$\left.\begin{array}{l} u = 3A^3 + B^3 - 9AB^2 - 3A^2B \\ v = -A^3 + 3B^3 + 3AB^2 - 9A^2B \end{array}\right\} \quad (24)$$

Using (24) and (2), the values of x, y, z and w are given by

$$\left.\begin{aligned} x(A,B) &= 2A^3 + 4B^3 - 6AB^2 - 12A^2B \\ y(A,B) &= 4A^3 - 2B^3 - 12AB^2 + 6A^2B \\ z(A,B) &= 5A^3 + 5B^3 - 15AB^2 - 15A^2B \\ w(A,B) &= 7A^3 - B^3 - 21AB^2 + 3A^2B \end{aligned}\right\} \quad (25)$$

Thus (25) and (15) represent the non-zero integer solutions to (1).

PROPERTIES

a) $x(A,1) - 4P(A,1) = 4\left[3P_A{}^3 - 5t_{3,A}\right]$

b) $x(1,B) - y(1,B) + 2\left[6P_B{}^3 - P(1,B) + t_{16,B}\right] = 4 - 2B$

c) $y(A,1) + 2\left[P(A,1) - 12P_A{}^3 + 4t_{3,A}\right] = -16A$

d) $x(A,1) + z(A,1) + P(A,1) + 10w(A,1) - 462P_A{}^3 + 454t_{3,A} = -81A$

e) $y(A,1) - z(A,1) + 7P(A,1) + 2\left[3P_A{}^3 - t_{16,A} - t_{18,A} - t_{3,A}\right] = 17A$

3. Conclusion

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous quintic equations with five unknowns. As the quintic equations are rich in variety, one may search for other forms of quintic equation with variables greater than or equal to five and obtain their corresponding properties.

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