Journal of Mathematics and Informatics
Vol. 10, 2017, 37-40
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 11 December 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v10a5

Journal of Mathematics and Informatics

On the Integral Solutions of System of Equations "b+T=x^2, b/3-T=y^2, T≠ a Square"

A. Hari Ganesh*[1], *K. Mahalakshmi* and *K. Prabhakaran

Poompuhar College (Autonomous), Melaiyur – 609 107
Nagai (Dt.), Tamilnadu, India.
[1]Corresponding Author

Received 2 November 2017; accepted 4 December 2017

Abstract. In this paper, the system of equations with three unknowns represented by "$b+T=x^2, \frac{b}{3}-T=y^2, T \neq$ a square" is analyzed with the help of Pell's equation for its non – trivial integral solutions. Moreover, the paper also discussed and analyzed the few interesting properties of the solutions of the system.

Keywords: system of equations, pell's equation, integral solutions

AMS Mathematics Subject Classification (2010): 11A11D

1. Introduction

A system of equations is a finite set of equations for which common solution namely the values of set of unknowns are sought. In solving a system of equations, we try to find values for each of the unknowns that will satisfy every equation in the system. System of equations is usually classified as system of linear equations and system of nonlinear equations. Among these two, system of non - linear equations are difficult to solve. But systems of non – linear equations with two unknowns have been solved by many authors [3, 4] with the help of pell's equation, which is the familiar second order equation with infinite solutions. In this paper, the system of equations with three unknowns represented by "$b+T=x^2, \frac{b}{3}-T=y^2, T \neq$ a square" is proposed and analyzed for its non – trivial integral solutions with the help of pell's equation. The proposed system of equations is described as follows:

The number 15 has the peculiar property that if unity is added to it, the sum is a perfect square, 16 and if unity is subtracted to its one third, 5, the result, 4, is also a perfect square. There is infinity of numbers satisfying the above pattern and they are obtained by solving the system of equations. This property has motivated us to search for non – zero integers b and T ($\neq$ a square) such that "$b+T=x^2, \frac{b}{3}-T=y^2$. Finally, the recurrence relations satisfied by the solutions of the system are also given in this paper.

2. Analytic Method for solving the system

Let b and T ($\neq$ a square) be any two non – zero integer such that

$$b + T = x^2 \qquad \text{……….(1.1)}$$

$$\frac{b}{3} - T = y^2 \qquad \text{……….(1.2)}$$

Eliminating b, we get the Pell equation

$$x^2 - 3y^2 = 4T \qquad \text{……….(1.3)}$$

The general solution [2] of (1.1) and (1.2) are given by

$$x_n = \frac{1}{2}\left[\left(2+\sqrt{3}\right)^{n+1}\left(x_0+\sqrt{3}y_0\right)+\left(2-\sqrt{3}\right)^{n+1}\left(x_0-\sqrt{3}y_0\right)\right] \qquad \text{……….(1.4)}$$

$$y_n = \frac{1}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1}\left(x_0+\sqrt{3}y_0\right)-\left(2-\sqrt{3}\right)^{n+1}\left(x_0-\sqrt{3}y_0\right)\right] \qquad \text{……….(1.5)}$$

where $\left(x_0+\sqrt{3}y_0\right)$ is the fundamental solution of (1.3)

Thus, knowing the values of x_n, y_n in (1.1) and (1.2), the sequences of values of b are obtained. In particular from (1.1) and (1.4), we get

$$b = \left\{\frac{1}{2}\left[\left(2+\sqrt{3}\right)^{n+1}\left(x_0+\sqrt{3}y_0\right)+\left(2-\sqrt{3}\right)^{n+1}\left(x_0-\sqrt{3}y_0\right)\right]\right\}^2 - T \qquad \text{……….(1.6)}$$

n = 0, 1, 2, 3, ………

When $T = \alpha^2 + 2\alpha - 2$, the equation (1.6) becomes

$$b = \left\{\frac{1}{2}\left[\left(2+\sqrt{3}\right)^{n+1}\left(2(\alpha+1)+2\sqrt{3}\right)+\left(2-\sqrt{3}\right)^{n+1}\left(2(\alpha+1)-2\sqrt{3}\right)\right]\right\}^2 - \left(\alpha^2+2\alpha-2\right)$$

……….(1.7)

where $x_0+\sqrt{3}y_0 = 2(\alpha+1)+2\sqrt{3}$ is the fundamental solution of $x_2 - 3y_2 = 4(\alpha^2+2\alpha-2)$

For the sake of simplicity a few solution of (1.7) for T = 1, 6, 13,…… are presented in Table 1

Table 1:

Serial No.	The Values of n	The solutions b for T = 1	The solutions b for T = 6	The solutions b for T = 13
1	0	15	30	51
2	1	195	318	471
3	2	2703	4350	6387
4	3	37635	60510	88791
5	4	524175	842718	1236531
6	5	7300803	11737470	17222487

It is interesting to note that all the solutions obtained in this case are odd. When T = 1, 6, 13, the solutions of last digits from the following patterns respectively:

5 5 3; 0 8 0; 1 1 7

as seen in the above table.

Further the solutions satisfy the following recurrence relation:

(a) Recurrence relations for solution (b_α) among different values of T:

$$[b_{\alpha+2} + c(\alpha+2)]^{1/2} - 2[b_{\alpha+1} + c(\alpha+1)]^{1/2} + [b_\alpha + c(\alpha)]^{1/2} = 0 \quad \text{where} \quad c(\alpha) = \alpha^2 + 2\alpha - 2$$

In particular for c(3) = 13, c(2) = 6 and c(1) = 1, when $\alpha = 1$, we have

$$[b_3 + 13]^{1/2} - 2[b_2 + 6]^{1/2} + [b_1 + 1]^{1/2} = 0 \quad \text{where} \quad n = 0$$

(b) Recurrence relations for solutions ($b_n^{(\alpha)}$) among the particular value of T:

$$[b_{n+2}^{(\alpha)} + C]^{1/2} - 4[b_{n+1}^{(\alpha)} + C]^{1/2} + [b_n^{(\alpha)} + C]^{1/2} = 0 \quad \text{where} \quad C = \alpha^2 + 2\alpha - 2$$

In particular for C = 1 when $\alpha = 1$ and C = 6 when $\alpha = 2$, we have

$$[b_3^{(1)} + 1]^{1/2} - 4[b_2^{(1)} + 1]^{1/2} + [b_1^{(1)} + 1]^{1/2} = 0 \quad \text{where} \quad n = 1$$

$$[b_3^{(2)} + 6]^{1/2} - 4[b_2^{(2)} + 6]^{1/2} + [b_1^{(2)} + 6]^{1/2} = 0 \quad \text{where} \quad n = 1$$

REFERENCES

1. H.Beiler Albert, Recreation in the numbers, Dover Publication, 1963.
2. I.E.Dickson, History of Numbers, Vol. II, Chelsea Publication Company, New York.

3. M.A.Gopalan and P.Jayakumar, On the system of double equations: “$b + T = x^2, \frac{b}{2} + T = y^2, T \neq$ a square, *International Journal of Acta Ciencia Indica*, 32M (4) (2006) 1465 – 1468.
4. T.Ramaraj and P.Jayakumar, On the system of double equations: $b + T = x^2, \frac{b}{N} + T = y^2, T \neq$ a square”, *Varahmihir Journal of Mathematical Sciences*, 6 (2) (2006) 457 – 463.