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On the Non-Homogeneous Ternary Quadratic Equation $2(x^2+y^2)-3xy+(x+y)+1=z^2$

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Abstract. The non-homogeneous quadratic Diophantine equation represented by $2(x^2+y^2)-3xy+(x+y)+1=z^2$ is studied for its non-zero distinct integer solutions. Four different sets of distinct integer solutions to the above equation are obtained. A few interesting relations between the solutions and special polygonal numbers are presented.

Keywords: Non-homogeneous quadratic, ternary quadratic, integer solutions.

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1. Introduction

The Diophantine equation offer an unlimited field for due to their variety $[1-3]$. In particular, one may refer $[4-14]$for quadratic equations with three unknowns. This communication concerns with yet another interesting homogeneous quadratic equation with three unknowns given by $2(x^2+y^2)-3xy+(x+y)+1=z^2$ for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2. Notations

- $t_{m,n}$ = n[th] term of a regular polygon with m sides

$$= n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

3. Method of analysis

The ternary quadratic Diophantine equation to be solved for its non-zero distinct integral solution is

$$2(x^2+y^2)-3xy+(x+y)+1=z^2 \tag{1}$$

Introduction of the linear transformations $(u \neq v \neq 0)$

$$x = u + v,\ y = u - v \tag{2}$$

in (1) leads to

$$U^2 + 7v^2 = z^2 \tag{3}$$

where $U = u + 1$ (3a)

Different patterns of solutions of (1) are presented below.

3.1. PATTERN-1

Assume

$$z = a^2 + 7b^2 \quad (4)$$

where a and b are non-zero distinct integers.

Write 1 as

$$1 = \frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{16} \tag{5}$$

Using (4) and (5) in (3) and employing the method of factorization (3) is written as

$$(U + i\sqrt{7}v)(U - i\sqrt{7}v) = \frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{16}(a + i\sqrt{7}b)^2 (a - i\sqrt{7}b)^2$$

Equating the positive and negative factors, the resulting equations are

$$U + i\sqrt{7}v = \frac{(3 + i\sqrt{7})}{4}(a + i\sqrt{7}b)^2 \tag{6}$$

$$U - i\sqrt{7}v = \frac{(3 - i\sqrt{7})}{4}(a - i\sqrt{7}b)^2 \tag{7}$$

Equating the real and imaginary parts in (6),

$$U = \frac{1}{4}(3a^2 - 21b^2 - 14ab)$$

$$v = \frac{1}{4}(a^2 - 7b^2 + 6ab)$$

Replacing a, b by $2A, 2B$ respectively, we get

$$U = 3A^2 - 12B^2 - 14AB$$

$$v = A^2 - 7B^2 + 6AB$$

In view of (3a)

$u = 3A^2 - 21B^2 - 14AB - 1$

Substituting the values of u and v in (2), we get

$$x = x(A,B) = 4A^2 - 28B^2 - 8AB - 1 \tag{8}$$

$$y = y(A,B) = 2A^2 - 14B^2 - 20AB - 1 \tag{9}$$

Thus (8),(9) and(4) represents non-zero distinct integral solutions of (1) in two parameters.

Properties observed are as follows

1. $6\left(t_{18,\alpha^2} - x(\alpha^2,1) - z(\alpha^2,1) - 1\right)$ is a nasty number.
2. $2y(A,1) + z(A,1) - t_{18,A} + 2 \equiv 0(\mathrm{mod}\,3))$
3. $(9 - 3x(A, A+1))$ is a nasty number
4. $x(A,1) - y(A,1) + z(A,1) - t_{18,A} - 14 \equiv 0(\mathrm{mod}\,3)$
5. $x(A, A+1) + y(A, A+1) + z(A, A+1) + t_{66,A} + 16 \equiv 2(\mathrm{mod}\,5)$

3.2. PATTERN-2

Write 1 as

$$1 = \frac{(1 + i3\sqrt{7})(1 - i3\sqrt{7})}{64} \tag{10}$$

Using (4) and (10) in (3) and employing the method of factorization(3) is written as

$$(U + i\sqrt{7}v)(U - i\sqrt{7}v) = \frac{(1 + i3\sqrt{7})(1 - i3\sqrt{7})}{64}(a + i\sqrt{7}b)^2 (a - i\sqrt{7}b)^2$$

Equating the positive and negative factors, the resulting equations are

$$U + i\sqrt{7}v = \frac{(1 + i3\sqrt{7})}{8}(a + i\sqrt{7}b)^2 \tag{11}$$

$$U - i\sqrt{7}v = \frac{(1 - i3\sqrt{7})}{8}(a - i\sqrt{7}b)^2 \tag{12}$$

Equating the real and imaginary parts in (11),

$$U = \frac{1}{8}(a^2 - 7b^2 - 42ab)$$

$$v = \frac{1}{8}\left(3a^2 - 21b^2 + 2ab\right)$$

Replacing a,b by $4A,4B$ respectively, we get

$$U = 2A^2 - 14B^2 - 84AB$$

$$v = 6A^2 - 42B^2 + 4AB$$

In view of (3a)

$$u = 2A^2 - 14B^2 - 84AB - 1$$

Substituting the values of u and v in (2), we get

$$x = x(A,B) = 8A^2 - 56B^2 - 80AB - 1 \tag{13}$$

$$y = y(A,B) = -4A^2 + 28B^2 - 88AB - 1 \tag{14}$$

Thus (13),(14) and(4) represents non-zero distinct integral solutions of(1) in two parameters.

Properties observed are as follows

1. $x(A,1) - y(A,1) - t_{10,A} + 84 \equiv 1(\bmod 3)$
2. $y(2B-1,B) + z(2B-1,B) - t_{26,B} - 11 \equiv 0(\bmod 3))$
3. $x(A,1) - y(A,1) + z(A,1) - t_{50,A} - 28 \equiv 0(\bmod 3)$
4. $x(A,1) - y(A,1) + z(A,1) - t_{58,A} - 28 \equiv 0(\bmod 5)$
5. $y(A,1) + t_{10,A} + 27 \equiv 1(\bmod 3)$

3.3. PATTERN-3

(3) is written in the form of ratio as

$$\frac{z+U}{7v} = \frac{v}{z-U} = \frac{m}{n}, n \neq 0 \tag{15}$$

which is equivalent to the system of equations,

$$nz + nU - 7mv = 0 \tag{16}$$

$$mz - mU - nv = 0 \tag{17}$$

Applying the method of cross multiplication for solving (16) and (17), we have

$$U = 7m^2 - n^2$$

$$v = 2mn$$

$$z = 7m^2 + n^2 \tag{18}$$

In view of (3a) we get

$$u = 7m^2 - n^2 - 1$$

Substituting the values of u and v in (2)

$$x = x(m,n) = 7m^2 - n^2 + 2mn - 1$$

$$y = y(m,n) = 7m^2 - n^2 - 2mn - 1 \tag{19}$$

Thus (18) and (19) represent the integer solutions to (1)

Properties observed are as follows

1. $x(m,m+1) - y(m,m+1) = 8t_{3,m}$
2. $y(m,1) + z(m,1) - t_{30,m} + 1 \equiv 0 \pmod{11}$
3. $x(2n-1,n) - y(2n-1,n) = 4t_{6,n}$
4. $y(m,1) - t_{16,n} + 2 \equiv 0 \pmod{2}$
5. $x(m,m+1) + z(m,m+1) - t_{34,m} + 1 \equiv 2 \pmod{3}$

3.4. PATTERN-4

(3) is written in the form of ratio as

$$\frac{z+U}{v} = \frac{7v}{z-U} = \frac{m}{n}, n \neq 0 \tag{20}$$

which is equivalent to the system of equations

$$nz + nU - mv = 0 \tag{21}$$

$$mz - mU - 7nv = 0 \tag{22}$$

Applying the method of cross multiplication for solving (21) and (22), we have

$$U = m^2 - 7n^2$$

$$v = 2mn$$

$$z = m^2 + 7n^2 \tag{23}$$

In view of (3a) we get,

$$u = m^2 - 7n^2 - 1$$

Substituting the values of u and v in (2)

$x = x(m,n) = m^2 - 7n^2 + 2mn - 1$

$y = y(m,n) = m^2 - 7n^2 - 2mn - 1$ (24)

Thus (23) and (24) represent the integer solutions to (1)

Properties observed are as follows

1. $6\left(t_{6,\alpha^2} - y(\alpha^2,1) - z(\alpha^2,1) - 1\right)$ is a nasty number.
2. $6(6y(m,1) + 3z(m,1) + 31)$ is a nasty number
3. $x(m, m+1) + t_{10,m} + 8 \equiv 0 (\mathrm{mod}\, 5)$
4. $x(m,1) + y(m,1) + z(m,1) + 9 = 3t_{4,m}$
5. $x(m,1) + z(m,1) - t_{6,m} + 1 \equiv 0 (\mathrm{mod}\, 3)$

4. Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by

$$2(x^2 + y^2) - 3xy + (x + y) + 1 = z^2.$$

As quadratic equations are rich in variety, one may search for other choices quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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