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Observation on the Cubic Diophantine Equation with Four Unknowns $(x^3+y^3)+(x+y)(x+y+1)=zw^2$

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Abstract. The non-homogeneous cubic Diophantine equation represented by $\left(x^3+y^3\right)+(x+y)(x+y+1)=zw^2$ is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and Polygonal numbers, Pyramidal numbers are also presented.

Keywords: Integer solutions, non-homogeneous cubic, Polygonal numbers, Pyramidal numbers.

AMS Mathematics Subject Classification (2010): 11D25

1. Introduction

Integral solution for the Homogeneous or Non- homogeneous Diophantine cubic equation is an interesting concept as it can be seen from $[1,2,3]$. In $[4-11]$ a few special cases of cubic Diophantine equation with four unknowns are studied. In this communication we present the integral solutions of an interesting cubic equations with four unknowns $(x^3+y^3)+(x+y)(x+y+1)=zw^2$. A few remarkable relations between the solutions are also presented.

2. Notations

$t_{m,n}$ = nth term of a regular polygon with m sides

$$=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

$\Pr_n$ = Pronic number of rank n.

$$=n(n+1)$$

S_n = Star number of rank n.

$$=6n(n-1)+1$$

$CP_{n,6}$ = Centered Pyramidal Number of rank n with sides m

$$= n^3$$

3. Method of analysis

The cubic Diophantine equation with four unknowns to be solved for its non-zero distinct integral solution is

$$\left(x^3 + y^3\right) + (x+y)(x+y+1) = zw^2 \tag{1}$$

The substuition of the linear transformations

$$x = u + v\,,\ y = u - v \quad \left(u \neq v \neq 0\right) \tag{2}$$

in (1) leads to

$$U^2 + 3v^2 = w^2 \tag{3}$$

where $U = u + 1$ (4)

Different patterns of solutions of (1) are presented below

3.1. PATTERN-1

Assume

$$w = a^2 + 3b^2 \tag{5}$$

where a and b are non-zero distinct integers.

Substituting (5) in (3), we get

$$U^2 + 3v^2 = (a^2 + 3b^2)^2 \tag{6}$$

Employing the method of factorization, we get

$$(U + i\sqrt{3}v)(U - i\sqrt{3}v) = (a + i\sqrt{3}b)^2 (a - i\sqrt{3}b)^2$$

Equating the positive and negative factors, we get

$$(U + i\sqrt{3}v) = (a + i\sqrt{3}b)^2$$

$$(U - i\sqrt{3}v) = (a - i\sqrt{3}b)^2$$

Equating the real and imaginary part, we get

$$U = a^2 - 3b^2 \tag{7}$$

$$v = 2ab \tag{8}$$

In view of (4)

$$u = a^2 - 3b^2 - 1 \tag{9}$$

Substituting (8) and (9) in (2), we get the distinct non-zero integer solutions of (1) are given below

$$x = x\left(a,b\right) = a^2 - 3b^2 + 2ab - 1$$

$$y = y\left(a,b\right) = a^2 - 3b^2 - 2ab - 1$$

$$z = z(a,b) = 2a^2 - 6b^2 - 2$$

PROPERTIES

1. $x(a,1)+y(a,1)+z(a,1)-4t_{4,a}+16=0$
2. $y(a,2a-1)+t_{32,a}+4=0$
3. $x(a,a)+y(a,a)+z(a,a)+w(a,a)+4t_{4,a}+4=0$
4. $z(a,1)-w(a,1)-t_{4,a}+11=0$

3.2. PATTERN-2

Write (5) as $a^2+3b^2=w*1$ (10)

Now write 1 as

$$1=\frac{\left(1+i\sqrt{3}\right)\left(1-i\sqrt{3}\right)}{4} \tag{11}$$

Using (10) and (11) in (3) and employing the method of factorization, the above equation (3) is written as

$$\left(U+i\sqrt{3}v\right)\left(U-i\sqrt{3}v\right)=\frac{\left(1+i\sqrt{3}\right)\left(1-i\sqrt{3}\right)}{4}\left(a+i\sqrt{3}b\right)^2\left(a-i\sqrt{3}b\right)^2$$

Equating the positive and negative factors, we get

$$U+i\sqrt{3}v=\frac{\left(1+i\sqrt{3}\right)}{2}\left(a+i\sqrt{3}b\right)^2 \tag{12}$$

$$U-i\sqrt{3}v=\frac{\left(1-i\sqrt{3}\right)}{2}\left(a-i\sqrt{3}b\right)^2 \tag{13}$$

Equating the real and imaginary parts, we get

$$U=\frac{1}{2}\left(a^2-3b^2-6ab\right) \tag{14}$$

$$v=\frac{1}{2}\left(a^2-3b^2+2ab\right) \tag{15}$$

Replacing *a,b by 2A,2B* respectively, we get

$$U=2A^2-6B^2-12AB \tag{16}$$

$$v=2A^2-6B^2+4AB \tag{17}$$

In view of (4),

$$u=2A^2-6B^2-12AB-1 \tag{18}$$

Substituting the above values of*u* and *v* in (2), we get

$$x=x(A,B)=4A^2-12B^2-8AB-1$$

$$y=y(A,B)=-16AB-1$$

$z = z(A, B) = 4A^2 - 12B^2 - 24AB - 2$

$w = w(A, B) = 4A^2 + 12B^2$

represents the non-zero distinct integral solutions of (1)

PROPERTIES

1. $6[w(1,1)] = 6(4^2)$ is a Nasty Number
2. $y(n(n-1,1)) + S_n + t_{22,n} \equiv 0 \pmod{n}$
3. $w(1, B) - z(1, B) - t_{50,B} - 2 \equiv 1 \pmod 2$
4. $w(A, A+1) - y(A, A+1) - 16PR_A - t_{34,A} - 13 \equiv 0 \pmod{13}$

NOTE:

We note that, if we replace a, b by $2A+1, 2B+1$ respectively in (14) and (15) we get the different set of distinct non-zero integer solutions of (1) given by

$x = x(A, B) = 4A^2 - 12B^2 - 16B - 8AB - 5$

$y = y(A, B) = -8A - 8B - 16AB - 5$

$z = z(A, B) = 4A^2 - 12B^2 - 8A - 24B - 24AB - 10$

$w = w(A, B) = 4A^2 + 12B^2 + 4A + 12B + 4$

PROPERTIES

1. $6[w(1,1)] = 6(6^2)$ is a Nasty Number
2. $w(A,1) - y(A,1) - t_{6,A} - 41 \equiv 1 \pmod 3$
3. $x(A,1) - w(A,1) + 61 \equiv 0 \pmod 3$
4. $x(A,1) + z(A,1) + w(A,1) - t_{26,A} + 51 \equiv 0 \pmod 5$

3.3. PATTERN-3

We can also write 1 as,

$$1 = \frac{(1 + i4\sqrt{3})(1 - i4\sqrt{3})}{49} \qquad (19)$$

Using (5) and (19) in (3) and employing the method of factorization, we get

$$(U + i\sqrt{3}v)(U - i\sqrt{3}v) = \frac{(1 + i4\sqrt{3})(1 - i4\sqrt{3})}{49}(a + i\sqrt{3}b)^2 (a - i\sqrt{3}b)^2$$

Equating the positive and negative factors, we have

$$U + i\sqrt{3}v = \frac{(1 + i4\sqrt{3})}{7}(a + i\sqrt{3}b)^2$$

$$U-i\sqrt{3}v=\frac{\left(1-i4\sqrt{3}\right)}{7}\left(a-i\sqrt{3}b\right)^2$$

Equating the real and imaginary parts ,

$$U=\frac{1}{7}\left(a^2-3b^2-24ab\right) \quad (20)$$

$$v=\frac{1}{7}\left(4a^2-12b^2+2ab\right) \quad (21)$$

Replacing *a,b by 7A ,7B* respectively, we get

$$U=7A^2-21B^2-168AB \quad (22)$$

$$v=28A^2-84B^2+14AB \quad (23)$$

In view of (4)

$$u=7A^2-21B^2-168AB-1 \quad (24)$$

Substituting (23) and (24) in (2), we get

$$x=x(A,B)=35A^2-105B^2-154AB-1$$
$$y=y(A,B)=-21A^2+63B^2-182AB-1$$
$$z=z(A,B)=14A^2-42B^2-336AB-2$$
$$w=w(A,B)=49A^2+147B^2$$

represents the non-zero distinct integral solutions of (1).

PROPERTIES

1. $6[w(1,1)]=6(14^2)$ is a Nasty Number
2. $z(A,1)+y(A,1)+w(A,1)-t_{86,A}-165\equiv 0(\mathrm{mod}\,159)$
3. $z(AA)+364t_{4,A}+2=0$
4. $x(A,1)-t_{72,A}+106\equiv 0(\mathrm{mod}\,2)$

3.4. PATTERN-4

(3) is written in the form of ratio as

$$\frac{w+U}{3v}=\frac{v}{w-U}=\frac{m}{n},n\neq 0 \quad (25)$$

which is equivalent to the system of equations,

$$nw+nU-3mv=0 \quad (26)$$
$$mw-mU-nv=0 \quad (27)$$

Applying the method of cross multiplication, for solving (26) and (27) we have

$U = 3m^2 - n^2$
$v = 2mn$
$w = 3m^2 + n^2$
In view of (4),
$u = 3m^2 - n^2 - 1$
Substituting the values of u and v in (2), we get the non zero distinct integer solutions of (1) is given by

$$x = x(m,n) = 3m^2 - n^2 + 2mn - 1$$
$$y = y(m,n) = 3m^2 - n^2 - 2mn - 1$$
$$z = z(m,n) = 6m^2 - 2n^2 - 2$$
$$w = w(m,n) = 3m^2 + n^2$$

PROPERTIES

1. $x(m,m+1) - y(m,m+1) - 4PR_m = 0$
2. $x(m,m+1) + y(m,m+1) - t_{12,m} + PR_m + 4 \equiv 0 \pmod m$
3. $x(m,1) + y(m,1) + z(m,1) + w(m,1) - 15t_{4,m} + 7 = 0$
4. $x(m,m+1) + y(m,m+1) + z(m,m+1) - t_{20,m} + PR_m + 8 \equiv 0 \pmod m$

3.5. PATTERN-5

One may write (3) in the form of ratio as

$$\frac{w+U}{v} = \frac{3v}{w-U} = \frac{m}{n}, n \neq 0 \qquad (28)$$

which is equivalent to the system of equations

$$nw + nU - mv = 0 \qquad (29)$$
$$mw - mU - 3nv = 0 \qquad (30)$$

Applying the method of cross multiplication for solving (29) and (30), we have
$U = m^2 + 3n^2$
$v = 2mn$
$w = m^2 + 3n^2$
In view of (4),
$u = m^2 - 3n^2 - 1$
Substituting the values of u and v in (2), we have

$$x = x(m,n) = m^2 - 3n^2 + 2mn - 1$$
$$y = y(m,n) = m^2 - 3n^2 - 2mn - 1$$
$$z = z(m,n) = 2m^2 - 6n^2 - 2$$
$$w = w(m,n) = 3n^2 + m^2$$

represents the distinct non zero integral solutions of (1)

PROPERTIES

1. $x(m,1)-w(m,1)+7\equiv 0(\mathrm{mod}\,2)$
2. $z(m,m+1)-y(m,m+1)-2PR_m+t_{6,m}+4\equiv 0(\mathrm{mod}\,7)$
3. $x(1,n+1)+y(1,n+1)+z(1,n+1)+w(1,n+1)-t_{12,m}+36=0$
4. $x(m,1)+y(m,1)+w(m,1)-3t_{4,m}+5=0$

3.6. PATTERN-6

Introduction of the linear transformations

$U=X+3T\;\; v=X-T$ (31)

in (3) gives,

$4X^2+12T^2=w^2$ (32)

which is satisfied by,

$X=2a^2-6b^2$

$T=4ab$

$w=4a^2+12b$

Substituting the values of X,T *in* (31), we get

$U=2a^2-6b^2+12ab$ (33)

$v=2a^2-6b^2-4ab$ (34)

In view of (4),

$u=2a^2-6b^2+12ab-1$ (35)

Substituting (34) and (35) in (2), we get the distinct non- zero integer solutions of (1) is given by

$x=x(a,b)=4a^2-12b^2+8ab-1$

$y=y(a,b)=16ab-1$

$z=z(a,b)=4a^2-12b^2+24ab-2$

$w=w(a,b)=4a^2+12b^2$

PROPERTIES

1. $x(a,a)+z(a,a)-16t_{4,a}+3=0$
2. $x(a,1)+y(a,1)+z(a,1)+w(a,1)-t_{26,a}+16\equiv 1(\mathrm{mod}\,2)$
3. $x(a,1)+y(a,1)+z(a,1)-8t_{4,a}+28\equiv 0(\mathrm{mod}\,2)$

4. $6[w(1,1)] = 6(4^2)$ is a Nasty Number

3.7. PATTERN-7

Introduction of the linear transformations

$U = X + 3T \ v = X - T$

in (3) gives,

$4X^2 + 12T^2 = w^2$

which can be written as

$$4X^2 + 12T^2 = w^2 * 1 \tag{36}$$

Write 1 as

$$1 = \frac{(2 + i\sqrt{12})(2 - i\sqrt{12})}{16} \tag{37}$$

Substituting (37) in (36) and employing the method of factorization, we get

$X = a^2 - 3b^2 - 6ab$

$T = a^2 - 3b^2 + 2ab$

Substituting the values of X,T *in* (31),we get

$$U = 4a^2 - 12b^2 \tag{38}$$

$$v = -8ab \tag{39}$$

In view of (4),

$$u = 4a^2 - 12b^2 - 1 \tag{40}$$

Substituting (39) and (40) in (2), we get the distinct non- zero integer solutions of (1) is given by,

$x = x(a,b) = 4a^2 - 12b^2 - 8ab - 1$

$y = y(a,b) = 4a^2 - 12b^2 + 8ab - 1$

$z = z(a,b) = 8a^2 - 24b^2 - 2$

$w = w(a,b) = 4a^2 + 12b^2$

PROPERTIES

1. $x(a, a(a+1)) - y(a, a(a+1)) + 16CP_{n,6} + 16t_{4,a} = 0$
2. $w(2a+1, b) - 28t_{4,b} - 4 \equiv 0 \pmod 2$
3. $6[x(1,1) + y(1,1) + z(1,1) - 4] = 6(8^2)$is a Nasty Number
4. $x(a, a+1) + y(a, a+1) + z(a, a+1) + w(a, a+1) - t_{42,a} + t_{74,a} + 40 \equiv 0 \pmod 4$

4. Conclusion

To conclude one may search for other choices of solutions to (1) along with the corresponding properties.

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